Yang-Mills Fields and the $\pi\pi$ Interaction

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Starting from a Lagrangian where the ρ meson is treated as a Yang-Mills field, we calculate $\pi\pi$ scattering amplitudes in the one-loop approximation. The summation of the strong-coupling perturbation series is performed with a unitary-rational-function approximation, and the ρ appears as a true resonance. We show that the interaction generates the d-wave f_0 resonance, and a very broad s-wave σ resonance. The model yields a version of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation, which agrees very well with experiment, and our s-wave amplitudes are close to the current-algebra predictions. We draw some conclusions on $\pi\pi$ dynamics.

I. INTRODUCTION

MASSIVE Yang-Mills fields[1] have raised considerable interest in recent years, in particular in connection with the question of renormalizability.[2,3] From a phenomenological point of view, we have the very attractive idea suggested by Sakurai[4] that strong interactions may be mediated by vector mesons universally coupled to various conserved currents $(\mathbf{I},Y,B)$. On the other hand, in recent years much progress has been done in the calculation of strong-interaction processes starting from field-theoretical considerations, owing to systematic use of rational-function summation procedures for divergent series.[5-7] The Padé approximation treats the renormalized strong-coupling perturbation series and, among other features, it leads to partial-wave amplitudes which are exactly unitary; furthermore, numerical results obtained up to now are very encouraging, in particular in $\pi\pi$ scattering.[7] Since the Yang-Mills theory behaves as a renormalizable one in the one-loop approximation, it is of interest to analyze its dynamical content along these lines.

In the present paper we study $\pi\pi$ scattering where the only vector field of interest is associated with the ρ meson. In subsequent papers we propose to study more complicated systems. Among its many interesting features, the model has the advantage of prohibiting exotic resonances since forces are repulsive in these channels. The $I=1$ p wave is elementary in the sense that it contains the ρ-meson pole which is present in the Lagrangian. We shall see how the unitarity properties of the Padé approximation allow us to handle unstable particles in Feynman graphs as far as direct-channel poles are concerned—i.e., the final ρ meson appears in the second Riemann sheet. Hence, the most interesting predictions of the model concern s and d waves, where we shall see that the f_0 is generated in good agreement with experiment, and also that a very broad s-wave σ resonance appears. Our Lagrangian contains two parameters which are immediately fixed in terms of the physical ρ mass and ρ width. The subtraction constant of the second-order $\pi\pi$ amplitude is chosen so that Adler's self-consistency condition[8] is satisfied at that order. In the course of this work, we obtain a version of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation,[9] which is slightly different from the usual one and which compares extremely well with experiment. In this sense, our amplitudes satisfy the useful constraints of current algebra, and we show that second-order corrections to scattering lengths bring these close to Weinberg's values,[10] thereby decreasing the "isospin-two σ term." Finally, since ρ exchange generates a σ resonance in the present model, and since it was shown that σ exchange can generate the ρ in the σ model,[7] we observe a "reciprocal-bootstrap" situation between the ρ and the σ. From the structure of Feynman graphs, one can consider the σ model as the superposition of two s-wave forces, the Φ^4 contribution and σ exchange. However, these two forces have to cancel each other at $s=t=u=m_\pi^2$ because of Adler's condition, leaving us with an important p-wave contribution. In the present model we start only with the p-wave force and we are able to generate the main features of s-wave amplitudes, owing to the KSRF relation.

The paper is organized as follows. In Sec. II we recall the basic properties of the Yang-Mills Lagrangian. In Sec. III we discuss the structure of Born terms, the

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relation with current algebra, and the KSRF relation. In Sec. IV we discuss second-order terms and we give the numerical results. Section V contains our concluding remarks. The explicit calculation of first- and second-order amplitudes is given in the Appendices, together with our explicit renormalization procedure.

II. LAGRANGIAN

Following Yang and Mills,[1] we construct a Lagrangian invariant under isospin gauge transformations of the second kind. In order to do this, we start from the free Lagrangian for the pion system,

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu \boldsymbol{\pi})^2-\tfrac{1}{2}\mu^2\boldsymbol{\pi}^2, \tag{2.1}$$

which is invariant under the group $SU(2)$. Under an infinitesimal gauge transformation of the second kind of $SU(2)$, the pion field and its derivative transform as

$$\boldsymbol{\pi}\to\boldsymbol{\pi}+\delta\boldsymbol{\omega}\times\boldsymbol{\pi}, \tag{2.2a}$$

$$\partial_\mu\boldsymbol{\pi}\to(\partial_\mu\boldsymbol{\pi}+\delta\boldsymbol{\omega}\times\partial_\mu\boldsymbol{\pi})+(\partial_\mu\delta\boldsymbol{\omega})\times\boldsymbol{\pi}. \tag{2.2b}$$

Since the Lagrangian (2.1) is invariant under $SU(2)$, its only variation under transformations (2.2a) and (2.2b) comes from the last term $(\partial_\mu\delta\boldsymbol{\omega})\times\boldsymbol{\pi}$. In order to transform the Lagrangian (2.1) into a gauge-invariant one, we must cancel these contributions. This is achieved by introducing the covariant derivative

$$\partial_\mu\boldsymbol{\pi}\to D_\mu\boldsymbol{\pi}=\partial_\mu\boldsymbol{\pi}+g\boldsymbol{\varrho}_\mu\times\boldsymbol{\pi}, \tag{2.3}$$

where $\boldsymbol{\varrho}_\mu$ is a vector field which belongs to the regular representation of $SU(2)$. It transforms as

$$\boldsymbol{\varrho}_\mu\to\boldsymbol{\varrho}_\mu+\delta\boldsymbol{\omega}\times\boldsymbol{\varrho}_\mu-(1/g)\partial_\mu\delta\boldsymbol{\omega}, \tag{2.4}$$

thereby canceling the term $\partial_\mu\delta\boldsymbol{\omega}$ of Eq. (2.2b). We now have to add in the Lagrangian a term which contains the kinematical part for the field $\boldsymbol{\varrho}_\mu$ and which is also gauge invariant:

$$\mathcal{L}(\boldsymbol{\varrho}_\mu)=-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}, \tag{2.5}$$

$$G_{\mu\nu}=\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu+g\boldsymbol{\varrho}_\mu\times\boldsymbol{\varrho}_\nu. \tag{2.6}$$

It is easy to see that terms involving derivatives of $\delta\boldsymbol{\omega}$ will cancel out. Using Eq. (2.4), we have

$$\partial_\mu\boldsymbol{\varrho}_\nu\to\partial_\mu\rho_\nu+\delta\boldsymbol{\omega}\times(\partial_\mu\boldsymbol{\varrho}_\nu)+(\partial_\mu\delta\boldsymbol{\omega})\times\boldsymbol{\varrho}_\nu-(1/g)\partial_\mu\partial_\nu\delta\boldsymbol{\omega}, \tag{2.7a}$$

$$\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu\to\partial_\mu\rho_\nu+\delta\boldsymbol{\omega}\times\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu-\delta\boldsymbol{\omega}\times\partial_\nu\boldsymbol{\varrho}_\mu+(\partial_\mu\delta\boldsymbol{\omega})\times\rho_\nu-(\partial_\nu\delta\boldsymbol{\omega})\times\boldsymbol{\varrho}_\mu, \tag{2.7b}$$

$$g\boldsymbol{\varrho}_\mu\times\boldsymbol{\varrho}_\nu\to g(\boldsymbol{\varrho}_\mu+\delta\boldsymbol{\omega}\times\boldsymbol{\varrho}_\mu)\times(\boldsymbol{\varrho}_\nu+\delta\boldsymbol{\omega}\times\boldsymbol{\varrho}_\nu)-\boldsymbol{\varrho}_\mu\times(\partial_\nu\delta\boldsymbol{\omega})-(\partial_\mu\delta\boldsymbol{\omega})\times\boldsymbol{\varrho}_\nu, \tag{2.7c}$$

so that $G_{\mu\nu}G^{\mu\nu}$ is gauge invariant.

Under the previous considerations, the resulting Lagrangian can be written as

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi}+g\boldsymbol{\varrho}_\mu\times\boldsymbol{\pi})^2-\tfrac{1}{2}\mu^2(\boldsymbol{\pi})^2-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}.$$

At this stage, the following points are worth keeping in mind.

(1) The same procedure gives the minimal coupling in quantum electrodynamics, when applied to phase transformations. However, in that case, since the group is Abelian, the vector field does not have the charge of the group and $G_{\mu\nu}$ is a free-field term. When the group is non-Abelian, the vector field is self-coupled and there are interaction terms, even when only the vector field is considered.

(2) Given a continuous internal-symmetry group, one can always build in this way a vector field universally coupled to the conserved current associated with the group.

(3) In electrodynamics, the minimal coupling with a vector field yields a renormalizable Lagrangian in the following sense: In the positive metric there exist highly divergent graphs, but at each order, the sum of too strongly divergent contributions cancels for on-shell S-matrix elements owing to the conservation of the current (Ward identities). Moreover, if the vector field is given a nonvanishing physical mass, thereby breaking the local gauge invariance, the previous property remains, i.e., the most divergent terms due to the $k_\mu k_\nu/m^2$ part of the vector propagator cancel. In the non-Abelian case, renormalizability has only been proven for the zero-mass.[2] In the massive case, nothing has been proven in general. In the one-loop approximation, the leading divergences from the $k_\mu k_\nu/m^2$ terms cancel. In a given gauge, the remainder can be understood as follows: With each vector-meson loop one associates a similar loop involving a scalar negative-metric particle with derivative couplings. This rule does not hold any longer in the two-loop case where more terms appear.[3]

In the present work we have calculated $\pi\pi$ amplitudes in the one-loop approximation.

III. BORN TERMS

In the zero-loop approximation, the $\pi\pi$ amplitudes depend on two parameters g and m. The quantity $|g|^{1/2}$ is essentially the $\rho\pi\pi$ coupling constant which is related to the physical ρ width. The parameter m is the real part of the physical ρ-resonance mass (as we shall see presently, the subtraction constants are chosen so that higher-order corrections alter only its imaginary part). In the three isospin channels, the Born terms for $\pi\pi\to\pi\pi$ are (on the mass shell)

$$2g\left(\frac{s-u}{m^2-t}+\frac{s-t}{m^2-u}\right)\quad\text{for}\quad I=0, \tag{3.1}$$

$$g\left(2\frac{t-u}{m^2-s}+\frac{s-u}{m^2-t}-\frac{s-t}{m^2-u}\right)\quad\text{for}\quad I=1, \tag{3.2}$$

$$-g\left(\frac{s-u}{m^2-t}+\frac{s-t}{m^2-u}\right)\quad\text{for}\quad I=2, \tag{3.3}$$

where s, t, and u are the usual Mandelstam variables. Our normalization is such that the unitarity relation for partial-wave amplitudes reads

$$\mathrm{Im}t_l(s)=\frac{\pi}{2}\left(\frac{s-4}{s}\right)^{1/2}|t_l(s)|^2, \tag{3.4}$$

where, as in all that follows, we have set the pion mass to be unity: $m_\pi^2=1$. Since m is the ρ mass, we have $m\simeq 760$ MeV. A first estimate for the value of g can be obtained by using Eq. (3.2) as a K matrix, once projected on the p wave. Near the ρ pole, $s\simeq m^2$, we have

$$t_1(s)\simeq\frac{\frac{2}{3}g(s-4)}{m^2-s-ig(\pi/3)[(s-4)/s]^{1/2}(s-4)} \tag{3.5}$$

so that, by identification with a Breit-Wigner formula, we obtain

$$g\simeq\frac{3}{\pi}\frac{m^2\Gamma}{(m^2-4)^{3/2}}\sim\frac{\Gamma}{m}. \tag{3.6}$$

Γ is the ρ width ($\Gamma\sim 100$–110 MeV), yielding $g\sim 0.16$, which is our effective expansion parameter near threshold.

The s- and p-wave scattering lengths are of interest in understanding the basic features of the model. Since second-order corrections are small near threshold (this is borne out by the calculation, as we shall see presently), first estimates for these scattering lengths may be evaluated readily from Eqs. (3.1)–(3.3). To order $(m_\pi/m_\rho)^2$, we have

$$\lim_{s\to 4}\left[\frac{2}{\pi}\left(\frac{s}{s-4}\right)^{1/2}e^{i\delta_l}\sin\delta_l\right]=\begin{cases}16g(m_\pi/m_\rho)^2, & I=l=0\\ g[(s-4m_\pi^2)/m_\rho^2], & I=l=1\\ -8g(m_\pi/m_\rho)^2, & I=2, l=0.\end{cases} \tag{3.7}$$

With $g=0.16$ and $m_\rho^2=30\ m_\pi^2$, we obtain for the scattering lengths $a_0\sim 0.134\ m_\pi^{-1}$, $a_2\simeq -0.067\ m_\pi^{-1}$, and $a_1\sim 0.034\ m_\pi^{-3}$.

To understand these numbers, it is interesting to study the soft-pion limits of our amplitudes. The completely off-mass-shell contribution of the direct ρ pole for $(p_1,\alpha;\, p_2,\beta)\to(p_3,\gamma;\, p_4,\delta)$ is

$$g(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})\frac{t-u+(p_1^2-p_2^2)(p_3^2-p_4^2)/m^2}{m^2-s}, \tag{3.8}$$

where $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $a=(p_1-p_4)^2$. With respect to current-algebra considerations, the following comments are in order.

(a) Since our amplitude (3.8) is antisymmetric in the invariants, Adler's self-consistency condition[8] is identically satisfied by the Born term; i.e., we have

$$\lim_{p_{1\mu}\to 0}T_{\alpha\beta\gamma\delta}(p_1,p_2;\, p_3,p_4)=0,\quad p_2^2=p_3^2=p_4^2=m_\pi^2. \tag{3.9}$$

Furthermore, as we shall see later on, our subtraction constants are chosen so that Eq. (3.9) is satisfied at each order of perturbation theory.

(b) In the limit where the two pion four-momenta vanish, the current-algebra constraints (Weinberg's condition[10]) state that we should have

$$\lim_{p_1\to 0,\,p_2\to 0}T_{\alpha\beta\gamma\delta}(p_1,p_2;\, p_3,p_4)\sim M_{\alpha\beta\gamma\delta}{}^0-\frac{2p_1\cdot p_3}{8\pi^2 f_\pi^2}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})+O(p_1^2,p_2^2,p_1\cdot p_2), \tag{3.10}$$

where f_π is the pion decay constant ($f_\pi\sim 94$ MeV experimentally). Taking the same limit with our expression (3.8), it is easy to see that the coefficient of the isospin-one term is $6g/m^2$ so that, by identification with Eq. (3.10), we obtain the relation

$$16\pi^2 g f_\pi^2/m_\rho^2=\tfrac{1}{3}, \tag{3.11}$$

from which we can deduce the value of f_π. With $\Gamma=100$ MeV and $m_\rho=760$ MeV, we obtain $f_\pi\sim 89$ MeV in remarkable agreement with experiment. This is to be compared with the well-known KSRF relation[9] which reads, in the same normalization,

$$16\pi^2 g f_\pi^2/m_\rho^2=\tfrac{1}{2} \tag{3.12}$$

and which differs from the previous one by a factor $\frac{2}{3}$. This can be understood as follows: In one calculation, g is the residue of the ρ pole and this gives Eq. (3.11). On the other hand, Eq. (3.12), taken as such, does not agree as well with experiment ($\Gamma_\rho\sim 165$ MeV if $f_\pi=95$ MeV). Therefore, g in that formula should rather be understood as the value of the $\rho\pi\pi$ vertex when all four-momenta vanish. Note also that in usual treatments, the KSRF relation is obtained by assuming dominance of *direct* ρ exchange, whereas in our model for $\pi\pi$ scattering, *crossed* ρ poles contribute for one-third of the total amplitude at low energies.

(c) We notice that our scattering lengths could also be obtained in the linear Weinberg amplitudes,[10] using the prescription that the relative contribution of *isospin-zero and isospin-two* σ *terms is in the same ratio as the physical scattering lengths* a_0 *and* a_2.

Note added in proof. Indeed, in the linear approximation $\alpha+\beta s+\gamma(t+u)$, Eqs. (3.9) and (3.10) imply a ratio of scattering lengths $a_0/a_2=(7+5c)/(-2+2c)$, while the ratio of $I=2$ to $I=0$ t-channel amplitudes is in the limit (3.10) $\sigma_2/\sigma_0=2c/(5c+3)$; Weinberg's assumption is $c=0$, while our case corresponds to $c=-\frac{1}{3}$, hence $a_0/a_2=\sigma_0/\sigma_2$.

The previous considerations show that the present model incorporates current-algebra constraints and vector-dominance ideas which appear here to be in excellent numerical agreement. We finally recall that, as is well known,[4] since the ρ meson is universally coupled to the isospin current, the interaction is repulsive in $I=2$ $\pi\pi$ states and attractive in $I=0$ and $I=1$. This feature is very pleasant, since it forbids the

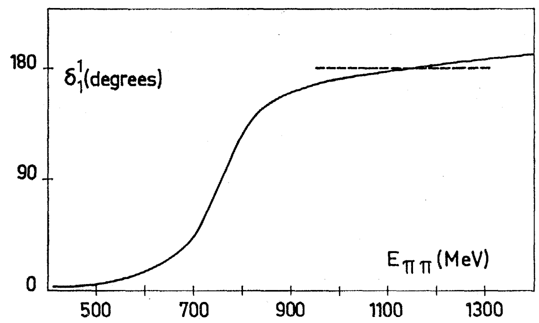

FIG. 1. $I=1$ p-wave phase shift for $g=0.17$. Notice the passage through 180°.

occurrence of exotic resonances. Furthermore, it is interesting to note that the same problem can be formulated in the more general $SU(3)$ scheme where exotic channels are repulsive and where the vector and tensor octet trajectories can be shown to be exchange degenerate up to second order of perturbation theory.[11]

IV. SECOND-ORDER TERMS AND NUMERICAL RESULTS

In second order (one-loop diagrams), the theory behaves as a renormalizable one in the sense that it requires no more subtraction constants than would be expected if it were renormalizable. Since the explicit calculation of second-order terms is rather tedious, we refer the reader to Appendix C for complete details. The important points of the calculation are the following.

(a) In the second order, the proper π^4 vertex requires one subtraction. The subtraction constant is chosen so as to impose the Adler self-consistency condition[8] on the second-order $\pi\pi\to\pi\pi$ amplitude

$$\lim_{p_{1\mu}\to 0} T_{\alpha\beta\gamma\delta}{}^2(p_1,p_2,p_3,p_4)=0,\quad p_2{}^2=p_3{}^2=p_4{}^2=m_\pi{}^2. \quad (4.1)$$

Since our Born terms also satisfy this condition, our amplitudes are consistent with the requirements of partial conservation of axial-vector current (PCAC).

(b) The subtraction constants for the $\pi\pi\rho$ vertex and for the self-mass of the ρ are chosen so that our real parameter m in Eqs. (3.1)–(3.3) is as close as possible to the physical ρ mass. Consider, for instance, the inverse ρ propagator. To second order it can be written as

$$[\Delta^\rho(s)]^{-1}=s-m^2+g\Phi(s). \quad (4.2)$$

The subtraction prescription is such that

$$\mathrm{Re}\Phi(m^2)=\mathrm{Re}\left[\frac{d}{ds}\Phi(s)\right]_{s=m^2}=0, \quad (4.3)$$

so that the final complex ρ mass M_ρ, which is given by

$$[\Delta^\rho(M_\rho{}^2)]^{-1}=0, \quad (4.4)$$

will be such that

$$\mathrm{Re}(M_\rho)\sim m. \quad (4.5)$$

(c) Once we have computed first- and second-order terms, and after projecting on to definite angular momentum and isospin channels, we obtain the perturbation series for partial-wave amplitudes,

$$t(s)=gt_1(s)+g^2t_2(s). \quad (4.6)$$

In order to sum this strong-coupling series, we use the Padé approximation technique,[5] and we build the [11,] approximant

$$t^{[1,1]}(s)=\frac{gt_1{}^2(s)}{t_1(s)-gt_2(s)}, \quad (4.7)$$

thereby obtaining partial-wave amplitudes which satisfy exact elastic unitarity in the region $4m_\pi{}^2\leqslant s\leqslant 4m_\rho{}^2$, and inelastic unitarity above.[5]

(d) In this model, the $I=J=1$ channel is special in that it contains an elementary particle, the ρ meson. In first order, the pole lies on the real axis, but, owing to the unitarity properties of our amplitudes (4.7), it is displaced onto the second Riemann sheet, as it should be, by second-order corrections. In this channel, the first- and second-order amplitudes behave in the vicinity of $s=m^2$ as

$$t_1(s)\sim\frac{\alpha}{s-m^2}, \quad (4.8)$$

$$t_2(s)\sim\frac{gA(s)}{(s-m^2)^2}+\frac{gB(s)}{s-m^2}, \quad (4.9)$$

where $A(s)$ and $B(s)$ have cuts from $s=4m_\pi{}^2$ to infinity and are subtracted according to Eq. (4.3) for $A(s)$, and Re $B(m^2)=0$. Accordingly, the Padé approximant (4.7) behaves as

$$t^{[1,1]}(s)\underset{s\sim m^2}{\sim}\frac{\alpha}{(s-m^2)[1-(g/\alpha)B(s)]-(g/\alpha)A(s)}. \quad (4.10)$$

It is clear from the preceding expression that the pole of the Padé approximant lies in the second sheet, and that its real part is close to m^2. In this sense the Padé approximation allows us to handle unstable particles in Feynman graphs as far as *direct-channel poles* are concerned.[12] Of course, there is still the problem that the perturbation series and the Padé approximants have singularities that the sum (the exact amplitude) does not have—for instance, the $\rho\rho$ cut in our case lies on the physical axis—and we cannot really trust the approximation near those singularities. However, choosing to subtract at the physical (final) ρ mass somewhat minimizes the importance of these drawbacks since the ρ width is not too large.

Since the subtraction constant of the $\pi\pi$ amplitude is fixed by the Adler self-consistency condition, our calculation contains two parameters g and m which are

[11] D. Iagolnitzer, J. Zinn-Justin, and B. Zuber (unpublished).

[12] Such a situation was already encountered in the σ model; see Ref. 7.

fixed by imposing the condition that the ρ resonance in the $I=J=1$ channel has its physical mass and width. Once this is done, all partial-wave amplitudes can be computed at once with no additional assumptions.

I=1 p Wave

We choose the value $m=5.45m_\pi$, and since there is some uncertainty in the exact value of the ρ width, we let our coupling constant g vary in a small range around $g=0.16$. The corresponding values of the (renormalized) ρ mass and width—as defined, for instance, by the position of the *second-sheet pole* of the $I=J=1$ *amplitude*, Eq. (4.10)—are

$$\begin{aligned} g&=0.16, \quad M_\rho=761 \text{ MeV}, \quad \Gamma_\rho=105 \text{ MeV}, \\ g&=0.17, \quad M_\rho=761 \text{ MeV}, \quad \Gamma_\rho=112 \text{ MeV}. \end{aligned} \tag{4.11}$$

One can also compute the pole of the renormalized ρ propagator, Eq. (4.2). It is interesting to note that the parameters of the ρ resonance obtained in these two different methods are remarkably close, although the pole structure of the four-point-function diagrams is more complicated in this order than for the two-point function. The resulting p-wave phase shift is plotted in Fig. 1, where one notices that it crosses the value 180° at $E_{\text{c.m.}}\simeq 1150$ MeV. This is because of the vanishing of the Born term

$$8\frac{(2s+m^2-4)}{s-4}Q_1\left(1+\frac{2m^2}{s-4}\right)-\frac{2}{3}\frac{s-4}{s-m^2}=0, \tag{4.12}$$

where Q_1 is the Legendre function of the second kind (see Appendix C). This feature can presumably be tested experimentally. Above this value, δ_1^1 rises again, and reaches 30° at 2 GeV, thus leaving open the possibility of an $I=J=1$ resonance in the region 1.5 GeV, as suggested by some models.[13] The p-wave scattering length is $a_1=0.37$ for $g=0.16$ and 0.39 for $g=0.17$, in good agreement with the considerations of Sec. III.[14]

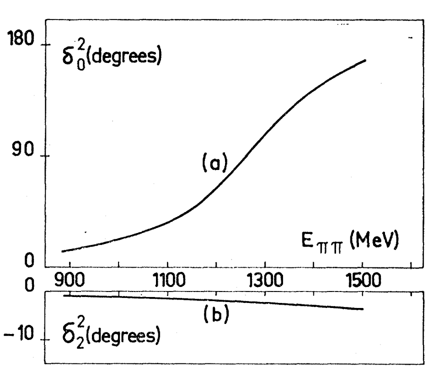

FIG. 2. d-wave phase shifts for $g=0.17$: (a) isospin zero; (b) isospin two.

[13] The Veneziano model predicts a ρ' resonance near the f_0 mass; see, for instance, M. Jacob, *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Boktryckenet, Lund, Sweden, 1969), p. 127.

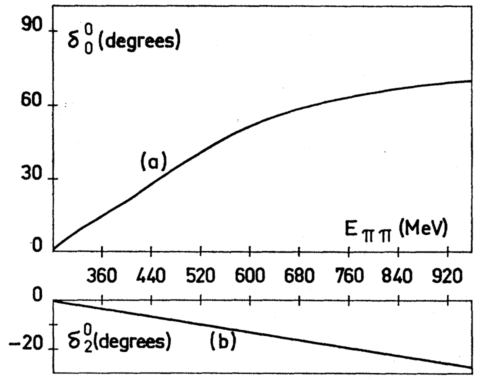

FIG. 3. s-wave phase shifts for $g=0.17$: (a) isospin zero; (b) isospin two.

I=0 and I=2 d Waves

Higher partial-wave amplitudes are nonelementary at this order in the sense that they all belong to Regge families. The Regge trajectories—which are only defined for $l>1$—can be computed according to the methods explained in Ref. 5. Their basic feature is that they do not rise indefinitely as was found in other models,[5] but they resemble potential-scattering trajectories. Since the interaction is *repulsive* in $I=2$ states, no exotic resonance occurs in this model. The $I=0$ trajectory passes through $l=2$ and gives a resonance which we identify with the f_0. In the range of parameters of interest, the mass and width obtained for the f_0 resonance are

$$\begin{aligned} g&=0.16, \quad M_{f_0}=1321 \text{ MeV}, \quad \Gamma_{f_0}=227 \text{ MeV}, \\ g&=0.17, \quad M_{f_0}=1278 \text{ MeV}, \quad \Gamma_{f_0}=260 \text{ MeV}, \end{aligned} \tag{4.13}$$

in favorable agreement with experiment[15] ($m_{f_0}=1264 \pm 10$, $\Gamma=151\pm 25$). The d-wave phase shifts are plotted in Fig. 2. Note that the $I=2$ phase shift δ_2^2 is small and negative, and it reaches $-3°$ at 1300 MeV.

s Waves

The most striking features of the s-wave amplitudes are (a) that the $I=0$ and $I=2$ phase shifts are respectively positive and negative as was expected from Born-term considerations, and (b) that the interaction generates an $I=0$ s-wave resonance that we identify with a σ resonance. Since this resonance is very broad, we compute the position of the associated second-sheet

[14] Notice that there is a numerical discrepancy of 20% between values of scattering length computed with g_A/g_V and those computed with f_π, as usual.

[15] Since previous Padé approximant calculations have yielded abnormally *small* resonance widths, we consider it significant that the f_0 width obtained here is large. Owing to imperfections of the [1,1] approximant, our present Γ_{f_0} is somewhat too large and tends to *increase* as the mass *decreases* (the Born term contains short-range forces, ρ exchange, while all other terms contain long-range forces, two-pion exchange; in turn, higher partial waves are badly approximated by the [1,1] at low energies).

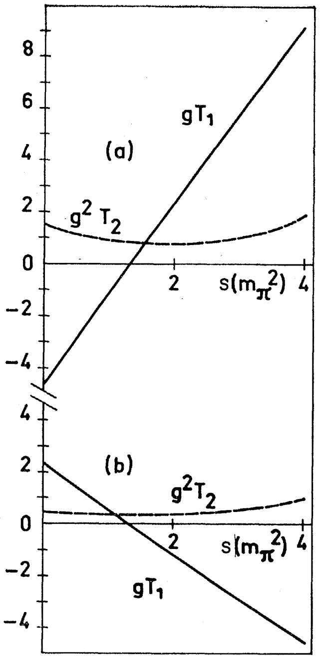

FIG. 4. First- and second-order s-wave amplitudes in the unphysical region $0 \le s \le 4m_\pi^2$: (a) isospin zero; (b) isospin two.

pole, and obtain, in the range of parameters considered,

$$g=0.16, \quad M_\sigma=431 \text{ MeV}, \quad \Gamma_\sigma=546 \text{ MeV},$$
$$g=0.17, \quad M_\sigma=424 \text{ MeV}, \quad \Gamma_\sigma=514 \text{ MeV}. \tag{4.14}$$

As a consequence of the large imaginary part of the pole, the phase shift δ_0^0 does not reach 90°, although it is large, as can be seen from Fig. 3. The $I=2$ phase shift is small and negative, in good agreement with experimental data, while δ_0^0 as it stands agrees well with the so-called "down-down" solution.[16] The s-wave scattering lengths obtained are

$$g=0.16, \quad a_0=0.17 m_\pi^{-1},$$
$$a_2=-0.057 m_\pi^{-1}, \quad a_0/a_2=-3, \tag{4.15}$$
$$g=0.17, \quad a_0=0.18 m_\pi^{-1},$$
$$a_2=-0.06\ m_\pi^{-1}, \quad a_0/a_2=-3.$$

It is interesting to note that second-order corrections have brought these values into closer agreement with Weinberg's current-algebra values (in first order, we have $a_0/a_2=-2$).

Tests of Crossing

It has already been emphasized[5,7,17] that it is important to check that the unitary Padé amplitudes are compatible with crossing symmetry at low energies. Martin's inequalities,[18] which are based on crossing symmetry and some analyticity properties, provide tests of crossing for partial-wave amplitudes in the unphysical region $0 \leqslant s \leqslant 4m_\pi^2$. In our model, these tests are positive in the sense that Roskies' relations[19] are satisfied within a few percent, and that among about forty Martin inequalities, only three are violated (by a very small amount). The basic reason for this stems from the fact that our amplitudes are based on a well-defined perturbation series which, in second order, satisfies these constraints *exactly*, as can be seen very easily. Furthermore, in the unphysical region $0 \leqslant s \leqslant 4m_\pi^2$, the second-order term g^2T_2 is much smaller than the Born term gT_1 (this can be seen in Fig. 4). As a consequence, in this region the Padé approximant (4.7) is not very different from the perturbation series.[20] In this sense, the Padé approximation is a unitarization method which is as smooth as possible, in that it maintains the basic structure of the Born term in a way compatible with crossing.

Perhaps more interesting are tests of crossing which can be made directly on *physical-region* amplitudes. These are provided, for instance, by dispersion relations. The following sum rule[21] is of particular interest since it relates s-wave scattering lengths and low-energy partial-wave cross sections:

$$L=\frac{2a_0-5a_2}{6}=\frac{1}{8\pi^2}\int_4^\infty \frac{dS}{[S(S-4)]^{1/2}} \times[\sigma^{+-}(S)-\sigma^{++}(S)] \quad (m=1), \tag{4.16}$$

where σ^{+-} and σ^{++} are the $\pi^+\pi^-$ and $\pi^+\pi^+$ total cross section. Neglecting asymptotic contributions as is usually done, we retain only s-, p-, and d-wave contributions to the integrand. The various contributions to the right-hand side of Eq. (4.16) are

$$\begin{array}{ll} I=0\ s \text{ wave:} & A_0^0=0.049 \\ I=2\ s \text{ wave:} & A_2^0=0.016 \\ I=1\ p \text{ wave } (\rho)\text{:} & A_\rho=0.036 \\ I=0\ d \text{ wave } (f_0)\text{:} & A_{f_0}=0.012 \\ \hline \text{total:} & =0.081, \end{array} \tag{4.17}$$

while we have [see Eq. (4.15)]

$$L=(2a_0-5a_2)/6=0.103. \tag{4.18}$$

The sum rule (4.16) appears therefore to be violated by ~20%. Since we believe that the value of the combination (4.18) of scattering lengths will not be changed much by higher-order corrections (this value is close to the current-algebra value), we have to conclude that

[16] J. P. Baton, G. Laurens, and J. Reignier, Phys. Letters **33B**, 525 (1970); E. Malamud and P. Schlein, in *Proceedings of Argonne Conference, 1969* (Argonne National Laboratory, Argonne, 1969), p. 106, and further references therein.

[17] J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, Nuovo Cimento **64**, 685 (1969).

[18] A. Martin, Nuovo Cimento **47A**, 265 (1967); Nuovo Cimento Letters **58A**, 303 (1968); CERN Report No. TH-1008, 1969 (unpublished). The method has been extended to the general case with isospin by G. Auberson, O. Brander, G. Mahoux, and A. Martin, Nuovo Cimento **65A**, 743 (1970).

[19] R. Roskies, Phys. Letters **30B**, 42 (1969); Nuovo Cimento **65A**, 467 (1970).

[20] Notice in Fig. 4 that we have $|g^2T_2| \ll |gT_1|$ except, of course, near a zero of gT_1. In the vicinity of such a zero, the [1,1] approximant loses its meaning, and in that case, the perturbation series itself is a much better approximation to the amplitude.

[21] M. G. Olsson, Phys. Rev. **162**, 1338 (1967).

either the contribution of the $I=0$ s wave will be increased by 40–50% in higher orders or that asymptotic contributions cannot be neglected (in fact the explicit evaluation of these contributions in Ref. 22 accounts perfectly for the present discrepancy). Note also that in this sense, our phase shift δ_0^0 does not saturate Adler's sum rule,[23] of which Eq. (4.16) is the on-shell version, with the assumption that high-energy contributions are negligible.

V. CONCLUDING REMARKS

Let us summarize what we have obtained. Our Lagrangian contains a strong isospin-one force universally coupled to the conserved isospin current, represented by ρ exchange. In second order, the subtraction constant is fixed by the Adler self-consistency condition, and we are left with two parameters which are completely determined by the mass and width of the ρ meson. We consider it significant that this interaction generates dynamically (a) the d-wave f_0 resonance in close agreement with experiment, (b) no exotic resonances, and (c) a very broad s-wave σ resonance with isospin zero. We have seen that in this framework a version of the KSRF relation obtains which is different from the usual one owing to crossed-channel ρ exchange, and which agrees very well with experiment. As a consequence, our model satisfies the consequences of PCAC and current algebra which are useful in calculating scattering lengths. Furthermore, second-order effects tend to bring our scattering lengths close to Weinberg's values. We have seen that our $I=0$ s-wave phase shift probably exhibits the "down-down" behavior.

Among all procedures to unitarize a Born term, the Padé approximation seems most satisfactory in that it preserves crossing symmetry to the maximum extent. Besides that, it has the property of exhibiting the low-energy spectrum.

In a previous calculation of the σ model,[7] it was shown that the s-wave forces, and in particular σ exchange, could generate the p-wave ρ resonance. In the present calculation, ρ exchange generates an s-wave σ resonance. In some sense, we have a reciprocal-bootstrap situation between the ρ and the σ. It is interesting to notice that the σ model[7] and the present "ρ model" seem to lead to similar features for the $\pi\pi$ amplitude. In this sense, we feel that many Lagrangians are presumably equivalent if one calculates sufficiently high orders, the main difference being the efficiency in reproducing a given feature of the dynamics at lower orders. For instance, the ρ meson itself is obviously represented more accurately in the present model than in the σ model, although in the latter it can be considered as being generated dynamically. This idea is also present in chiral Lagrangians,[24] where one can, for instance, either realize the chiral symmetry by imposing the σ particle as in the linear σ model, or by using nonlinear Lagrangians which make no reference to this particle.

The previous considerations show that in order to have the best possible representation of low-energy $\pi\pi$ scattering at low orders, the best choice would be the complete phenomenological $\pi\pi$ Lagrangian,[25] where both gauge invariance and current-algebra considerations are taken into account. Notice, however, that since the ρ, the σ, and the A_1 are already present in the Lagrangian, the predictive power of the theory (dynamically speaking) will be somewhat smaller. Finally, it is of great interest to extend the ideas developed in the present paper to the more general case of $SU(3)$ symmetry (exact or broken). Work on this subject is under way.[11]

ACKNOWLEDGMENTS

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APPENDIX A: BORN TERMS

1. Born Term of ($\pi\pi\rightarrow\pi\pi$) Amplitude

$$2g\left(\frac{s-u}{m^2-t}+\frac{s-t}{m^2-u}\right)\qquad \text{for}\quad I=0,$$

$$2g\frac{t-u}{m^2-s}+g\left(\frac{s-u}{m^2-t}-\frac{s-t}{m^2-u}\right)\qquad \text{for}\quad I=1,$$

$$-g\left(\frac{s-u}{m^2-t}+\frac{s-t}{m^2-u}\right)\qquad \text{for}\quad I=2.$$

2. Born Term of ($\pi\pi\rightarrow\rho\rho$) Amplitude

$(\pi(q),\pi(q'))\rightarrow(\rho_\mu(K),\rho_\nu(k'))$.

$$M^{\mu\nu}=g\left\{\begin{matrix}2\\1\\-1\end{matrix}\right\}\frac{(2q-k)^\mu(2q'-k')^\nu}{t-\mu^2}+g\left\{\begin{matrix}2\\-1\\-1\end{matrix}\right\}\frac{(2q'-k)^\mu(2q-k')^\nu}{u-\mu^2}$$

$$+g\left\{\begin{matrix}0\\2\\0\end{matrix}\right\}\frac{1}{s-m^2}[(q-q')^\mu(2k+k')^\nu+g^{\mu\nu}(k'-k)(q-q')-(q-q')^\nu(2k'+k)^\mu]$$

$$+g\left\{\begin{matrix}4\\0\\-2\end{matrix}\right\}g^{\mu\nu}\quad\text{for}\left\{\begin{matrix}I=0\\I=1\\I=2.\end{matrix}\right.$$

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[22] J. C. Le Guillon, A. Morel, and H. Navelet, Nuovo Cimento (to be published).

[23] S. Adler, Phys. Rev. **140**, B736 (1965).

[24] A complete review of chiral Lagrangians can be found in S. Gaziorowicz and D. A. Geffen, Rev. Mod. Phys. **41**, 531 (1969).

[25] J. Schwinger, Phys. Letters **24B**, 473 (1967); see also Ref. 24.

APPENDIX B: UNITARITY CONDITIONS IN ($\pi\pi \to \pi\pi$)

1. $\pi\pi$ Intermediate State

$$\mathrm{Im}A(s,\cos\theta)=\frac{1}{8}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\int d\Omega\, A(s,\cos\theta_1)A^*(s,\cos\theta_2).$$

2. $\rho\rho$ Intermediate State

$$\mathrm{Im}A(s,\cos\theta)=\frac{1}{8}\left(\frac{s-4m^2}{s}\right)^{1/2}\int d\Omega \times M_{\mu\nu}\left(g^{\mu\rho}-\frac{k^\mu k^\rho}{m^2}\right)\left(g^{\nu\sigma}-\frac{k'^\nu k'^b}{m^2}\right)M_{\rho\sigma}{}^*.$$

APPENDIX C: SECOND-ORDER CALCULATION

1. Preliminary Remarks

We compute the second-order ($\pi\pi\to\pi\pi$) amplitude by unitarity and crossing, as explained in Ref. 5. The absorptive part of the scattering amplitude comes from two contributions, with $\rho\rho$ and $\pi\pi$ intermediate states. We see that, owing to the preceding theoretical considerations, the most singular terms, given by the contribution of the $k^\mu k^\nu$ of the two ρ, to the absorptive part, cancel. This actually appears as a result of a Ward identity which relates $k^\mu M_{\mu\nu}$ to the $\pi\pi\rho$ vertex. The explicit calculation of the quantity $k^\mu M_{\mu\nu}$ gives

$$k^\mu M_{\mu\nu}=g\begin{Bmatrix}0\\1\\0\end{Bmatrix}\frac{1}{m^2-s}(t-u)k_\nu' \quad \text{for}\begin{cases}I=0\\I+1\\I=2.\end{cases}$$

If we compute the sum of all $k_\mu k_\nu$ terms, they give the following contribution to the $\pi\pi\to\pi\pi$ absorptive part:

$$2(t-u)\,\mathrm{Im}L(s)=g^2\begin{Bmatrix}0\\1\\0\end{Bmatrix}\frac{\pi}{6}\frac{1}{(m^2-s)^2}(s-4m^2)\times(t-u)\left(\frac{s-4m^2}{s}\right)^{1/2}\quad \text{for}\begin{cases}I=0\\I=1\\I=2.\end{cases}$$

This result justifies the claim that for the one-loop graphs, we can compute all the graphs with a $g_{\mu\nu}$ propagator for the ρ, and add a loop involving a scalar ghost field of mass m to each ρ loop.

This shows also that we need only one subtraction to reconstruct by dispersion relations the complete amplitude, starting from its absorptive parts.

In order to fix the subtraction constant, we require that the amplitude vanishes at Adler's point $s=t=u=\mu^2$ for each order. The first order obviously does. For the second order, we compute the amplitude with one pion off the mass shell and we impose Adler's condition at $s=t=u=\mu^2$.

Some contributions to the second-order amplitude have a singular absorptive part at the ρ mass. For these we restrict their real parts to be regular at the ρ mass, in order to obtain a very small change in the real part of the ρ mass in the Padé solution. Since the ρ mass is larger than 2μ, in the imaginary part of the amplitude there remains a double pole, which in the Padé solution gives the ρ width.

2. Second-Order ($\pi\pi\to\pi\pi$) Scattering Amplitude

$$(\pi\pi\to\pi\pi)=g^2\begin{cases}4[\varphi(s,t)+\varphi(s,u)]+4[\varphi(t,s)+\varphi(u,s)]+2[\varphi(t,u)+\varphi(u,t)]\\ \varphi(s,t)-\varphi(s,u)\ +\ \varphi(t,s)-\varphi(u,s)\\ \varphi(s,t)+\varphi(s,u)\ +\ \varphi(t,s)+\varphi(u,s)+2[\varphi(t,u)+\varphi(u,t)]\end{cases}\quad\text{for}\begin{cases}I=0\\I=1\\I=2,\end{cases}$$

with

$$\varphi(s,t)=B_1(s,t)+2T(t)+(s-u)[P_1(t)+2V_1(t)+L(t)]+(t-u)[P(s)+2V(s)].$$

3. Definition of Functions

We give now the definition of the different functions which corresponds as said before to graphs with a $g_{\mu\nu}$ propagator of the ρ, in terms of the scalar functions.

(a) The box function:

$$B_1(s,t)=(2s-4\mu^2+m^2)^2B(s,t)-2(2s-4\mu^2+m^2)W(s)+I(s)+4(2s-4\mu^2+m^2)W_1(t)-\frac{2s}{(t-4\mu^2)}[W_1(t)(t-2m^2)-W_1(4\mu^2)(4\mu^2-2m^2)]-\frac{4u}{(t-4\mu^2)}[I_1(t)-I_1(4\mu^2)].$$

(b) The function $T(s)$:

$$T(s)=-(\tfrac{1}{2}s+m^2-4\mu^2)W_1(s).$$

(c) The two-pion loop:

$$\mathrm{Im}P(s)=\frac{1}{3}\frac{s-4\mu^2}{(s-m^2)^2}\,\mathrm{Im}I(s).$$

The subtractions of $(s-4\mu^2)$ $I(S)$ are chosen to give a regular real part to $P(s)$ for $s=m^2$.

(d) The two-pion contribution to $\pi\pi\rho$ vertex:

$$\mathrm{Im}V(s)=\frac{1}{m^2-s}(2s-4\mu^2+m^2)\times\frac{2\pi}{[s(s-4\mu^2)]^{1/2}}Q_1\left(1+\frac{2m^2}{s-4\mu^2}\right),$$

where $Q_1(Z)$ is the Legendre function of the second kind and the subtraction is chosen to give a regular real part of $V(s)$ for $s=m^2$.

(e) The two-ρ loop and the ghost loop:

$$\mathrm{Im}P_1(s)=\frac{1}{6}\frac{1}{(m^2-s)^2}(19s+32m^2)\,\mathrm{Im}I_1(s)\,,$$

$$\mathrm{Im}L(s)=\frac{1}{12}\frac{1}{(m^2-s)^2}(s-4m^2)\,\mathrm{Im}I_1(s)\,,$$

where $P_1(s)$ and $L(s)$ are regular for $S=m^2$.

(f) The two-ρ contributions to the $\pi\pi\rho$ vertex:

$$\mathrm{Im}V_1(s)=\frac{1}{s-m^2}\frac{s+2m^2}{s-4\mu^2}\times[\tfrac{1}{2}(s+2m^2-8\mu^2)\,\mathrm{Im}W_1(s)+\mathrm{Im}I_1(s)]\,,$$

where $V_1(s)$ is subtracted to have no pole.

4. *Absorptive Part of Scalar Functions*

(a) The scalar box:

$$\mathrm{Im}_sB(s,t)=\frac{1}{4}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega}{(m^2-t_1)(m^2-t_2)}$$
$$=\frac{2\pi}{(-st)^{1/2}}\frac{1}{[\Delta(s,t)]^{1/2}}\times\ln\left\{\frac{[\Delta(s,t)]^{1/2}+[-t(s-4\mu^2)]^{1/2}}{\Delta^{1/2}-[-t(s-4\mu^2)]^{1/2}}\right\}\,,$$

$$\Delta(s,t)=4m^4-(s-4\mu^2)(t-4m^2)\,,$$

$$\mathrm{Im}_tB(s,t)=\frac{2\pi}{(-st)^{1/2}}\frac{1}{\Delta^{1/2}}\ln\left\{\frac{\Delta^{1/2}+[-s(t-4m^2)]^{1/2}}{\Delta^{1/2}-[-s(t-4m^2)]^{1/2}}\right\}.$$

(b) $W(s)$ and $I(s)$:

$$\mathrm{Im}W(s)=\frac{1}{4}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega}{m^2-t}$$
$$=\frac{\pi}{[s(s-4\mu^2)]^{1/2}}\ln\left(1+\frac{s-4\mu^2}{m^2}\right),$$

$$\mathrm{Im}I(s)=\pi\left(\frac{s-4\mu^2}{s}\right)^{1/2}.$$

(c) $W_1(s)$ and $I_1(s)$:

$$\mathrm{Im}W_1(s)=\frac{1}{4}\left(\frac{s-4m^2}{s}\right)^{1/2}\int\frac{d\Omega}{\mu^2-t}$$
$$=\frac{\pi}{[s(s-4\mu^2)]^{1/2}}\times\ln\left\{\frac{s-2m^2+[(s-4\mu^2)(s-4m^2)]^{1/2}}{s-2m^2-[(s-4\mu^2)(s-4m^2)]^{1/2}}\right\}\,,$$

$$\mathrm{Im}I_1(s)=\pi\left(\frac{s-4m^2}{s}\right)^{1/2}.$$

APPENDIX D: PARTIAL-WAVE PROJECTION

For the complex-angular-momentum-plane continuation of the partial-wave amplitude, we use the Froissart-Gribov formula:

$$a_l(s)=\frac{1}{\pi q^2}\int_{4\mu^2}^{\infty}dt'\,\mathrm{Im}_{t'}A(s,t')Q_l(Z)\,,$$

with

$$q^2=\tfrac{1}{4}(s-4\mu^2)\quad\text{and}\quad Z=1+2t/(s-4\mu^2)\,.$$

The formula for the absorptive part of $B_1(s,t)$ shows that the Froissart-Gribov formula converges for $\mathrm{Re}\,l>1$. For $l=0$ and $l=1$, we use a subtracted form of the same formula.

For the Born term, one obtains

$$g\times\begin{cases}-4\delta_{l0}+8\dfrac{(2s+m^2-4\mu^2)}{s-4\mu^2}Q_l\left(1+\dfrac{2m^2}{s-4\mu^2}\right) & \text{for } I=0\\[2ex] \tfrac{2}{3}(s-4\mu^2)\dfrac{\delta_{l1}}{m^2-s}+\dfrac{4(2s+m^2-4\mu^2)}{s-4\mu^2}\times Q_l\left(1+\dfrac{2m^2}{s-4\mu^2}\right) & \text{for } I=1\\[2ex] 2\delta_{l0}-\dfrac{4(2s+m^2-4\mu^2)}{s-4\mu^2}Q_l\left(1+\dfrac{2m^2}{s-4\mu^2}\right) & \text{for } I=2.\end{cases}$$

For the second order, the integration in the Froissart-Gribov formula is done by using the Laguerre-Gauss numerical-integration method. After a change of integration variable, we get

$$t=4M^2\cosh^2\theta\,,$$

where M^2 is μ^2 or m^2, respectively, for the two-pion or two-ρ contribution to the absorptive part.