and Shirkov claim that the Gell-Mann–Low treatment is wrong; I see nothing wrong with it and will use the Gell-Mann–Low approach in the following.

The subtraction momentum $\lambda$ can be chosen arbitrarily. However, the renormalized theory will have an apparently nontrivial dependence on $\lambda$. Nevertheless, the physical consequences of the theory must be independent of $\lambda$. The transformations which connect the renormalized theories with different values of $\lambda$ are the renormalization-group transformations. They are discussed in Sec. II.

The above discussion should make clear the ideas involved in generalizing the usual renormalization procedure such that subtractions are made at a momentum $\lambda$ rather than on the photon or electron mass shell.

PHYSICAL REVIEW D VOLUME 3, NUMBER 8 15 APRIL 1971

# Renormalization of Chirally Invariant Lagrangians

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(Received 1 July 1970)

We discuss the renormalization problem for chirally invariant Lagrangians constructed from the pion isovector fields alone. We find that $S$-matrix elements have ultraviolet divergences even after the damping from resummation has been effected. The counterterms necessary to remove such divergences are shown to modify the original current commutation relations; it is not possible that these divergences cancel amongst themselves. We conclude that the current commutation relations are inconsistent with the Lagrangian formulation, unless they are changed or further particles are introduced.

## I. INTRODUCTION

In this paper, we are interested in calculating higher-order corrections for scattering amplitudes arising from chirally invariant interactions. Physical results for such interactions have so far been calculated in the tree-graph approximation. Attempts have been made recently to extend this to include closed loops for nonpolynomial Lagrangians without derivative interactions. There are two main difficulties in such problems, which are that the perturbation series in the minor coupling constants have in general zero radius of convergence, and that there are ultraviolet divergences of arbitrarily high order due to the highly unrenormalizable character of the interaction. The first of these has been tackled by various resummation techniques.[1,2] The second has been satisfactorily solved by addition of a suitable number (sometimes infinite) of counterterms in the Lagrangian.[3,4] There has been a start on extending this discussion to the more interesting case of chirally invariant Lagrangians.[5] This discussion has only been given up to second order in the major coupling constant. We wish here to extend it to all orders.

In Sec. II we set up the formalism for calculating $S$-matrix elements to each order in the major coupling constant, by means of an extension of the concept of skeleton diagrams.[3] We then proceed, in Sec. III, to determine the over-all degree of ultraviolet divergence of the resulting amplitudes; we do this in coordinate space, though our results are the same as those obtained by using momentum-space methods.[2] In order to remove the divergences, we find it essential to add counterterms; it is not possible that the divergences are damped out by our resummation techniques. In Sec. IV we show that addition of chirally invariant counterterms destroys the current algebras. Since these were the original reason for setting up the Lagrangians, it would seem that an alternative to subtraction techniques is required. In Sec. V we investigate if it is possible that the ultraviolet divergences cancel among themselves. We find that this is not possible with mesons alone. We conclude with a discussion in Sec. VI of the implications of this for the general program of obtaining a Lagrangian realization of current algebras.

## II. CALCULATION OF S-MATRIX ELEMENTS

We consider the pions alone. $\boldsymbol{\pi}=(\pi^1,\pi^2,\pi^3)$ are the fields and the interaction Lagrangian is

$$\mathcal{L}_{\text{int}}=\tfrac{1}{2}f_{ab}(\boldsymbol{\pi})\partial_\mu\pi^a\partial_\mu\pi^b\,,$$

where $f_{ab}$ belongs to a certain class of (nonpolynomial) functions such that Lagrangians corresponding to different functions are related by point transformations of the fields.[6]

[1] G. V. Efimov, CERN Report No. Th. 1087, 1969 (unpublished).
[2] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969).
[3] B. W. Keck and J. G. Taylor, Proc. Phys. Soc. (London) (to be published).
[4] R. Delbourgo, K. Koller, and A. Salam, Imperial College Report No. ICTP/69/10, 1970 (unpublished).
[5] A. P. Hunt, K. Koller, and Q. Shafi, Phys. Rev. D **3**, 1327 (1971).
[6] See, e.g., K. J. Barnes and C. J. Isham, Nucl. Phys. **B15**, 333 (1970).

The $S$ matrix is calculated by expansion of $f_{ab}$ in a Taylor series, performance of the normal ordering, and resummation of each vertex. This can be expressed quantitatively using identities similar to the following. First,

$$\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)F(\phi)=F\left(\phi+\Delta\frac{\delta}{\delta\phi}\right)\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right),$$

where $\phi$ is a function of $x$,

$$\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}=\int dxdy\,\frac{\delta}{\delta\phi(x)}\Delta(x-y)\frac{\delta}{\delta\phi(y)},$$

and

$$\left(\Delta\frac{\delta}{\delta\phi}\right)(x)=\int dy\,\Delta(x-y)\frac{\delta}{\delta\phi(y)}.$$

This can be proved for

$$F(\phi)=e^{j\phi},\quad j\phi=\int dx\,j(x)\phi(x)$$

using the Baker-Hausdorff lemma

$$\exp j\left(\phi+\Delta\frac{\delta}{\delta\phi}\right)=e^{j\phi}\exp\left(j\Delta\frac{\delta}{\delta\phi}\right)\exp(\tfrac{1}{2}j\Delta j)$$

and so is true for all $F$. Second, following from this,

$$F\left(\phi+\Delta\frac{\delta}{\delta\phi}\right)G(\phi)=\left[\exp\left(\frac{1}{2}\frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi}\right)\right]F\left(\psi+\Delta\frac{\delta}{\delta\phi}\right)G(\phi)\bigg|_{\psi=\phi}.$$

The $S$ matrix (or operator) is

$$S=\sum_{n\geqslant 0}\frac{i^n}{n!}\int dx^n S(x^n),\quad x^n=(x_1,\ldots,x_n)$$

where, by Hori's formula,

$$S(x^n)=\left[\exp\left(\frac{1}{2}\frac{\delta}{\delta\pi^a}\Delta\frac{\delta}{\delta\pi^a}+\frac{\delta}{\delta\pi_\mu{}^a}\Delta_\mu\frac{\delta}{\delta\pi^a}-\frac{1}{2}\frac{\delta}{\delta\pi_\mu{}^a}\Delta_{\mu\nu}\frac{\delta}{\delta\pi_\nu{}^b}\right)\right]\times\prod_{i=1}^{n}\left[\tfrac{1}{2}f_{a_ib_i}(\pi_i)\partial_\mu\pi^{a_i}\partial^\mu\pi^{b_i}\right]_{(x_i)},$$

where

$$\Delta(x)\delta^{ab}=\langle 0|T(\pi^a(x)\pi^b(0))|0\rangle,\quad \Delta_\mu(x)=\frac{\partial}{\partial x^\mu}\Delta(x),\quad \text{etc.}$$

(a) (b)



FIG. 1. (a) Vertex. (b) Line modifying one vertex.

(a) (b)

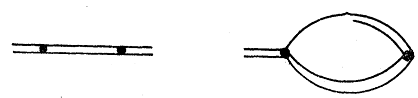

FIG. 2. (a) Example of step (i). (b) The result of applying one option of step (ii).

A simple extension of the above identities gives

$$S(x^n)=\left[\exp\left(-\frac{1}{2}\frac{\delta}{\delta\pi_\mu{}^a}\Delta_{\mu\nu}\frac{\delta}{\delta\pi_\nu{}^a}\right)\right]\times\prod_{i=1}^{n}\left(\partial\pi^{a_i}+\partial\Delta\frac{\delta}{\delta\pi^{a_i}}\right)_{(x_i)}\left(\partial\pi^{b_i}+\partial\Delta\frac{\delta}{\delta\pi^{b_i}}\right)_{(x_i)}\times\left[\exp\left(\frac{1}{2}\frac{\delta}{\delta\pi^a}\Delta\frac{\delta}{\delta\pi^a}\right)\right]\prod_{i=1}^{n}\tfrac{1}{2}f_{a_ib_i}(\pi_i).$$

The first factor is the sum of Feynman diagrams with vertices $1,\ldots,n$, where the $i$th is $\partial\pi^{a_i}\partial\pi^{b_i}$, and external legs are either $\partial\pi^a$ or $\partial\Delta(\partial/\partial\pi^a)$. The latter act on the second factor, called $S_k(x^n,a^n,b^n)$, which must be defined so as to be finite—in particular, not by expansion in Taylor series of the exp and $f_{ab}$. For $n=2$, a similar formula has been obtained by Hunt *et al.*[5] Since here

$$[\partial\Delta(\delta/\delta\pi^a)]_{(x_i)}=\sum_j\partial\Delta_{ij}(\partial/\partial\pi^{a_j}),$$

we can determine the terms of the first factor diagrammatically, as follows.

(i) Draw any diagram with $i$th vertex $\partial\pi^{a_i}\partial\pi^{b_i}$ using double lines [see Fig. 1(a)].

(ii) Extend any double line by a single line to meet a vertex [see Fig. 1(b)]. Thus, e.g., the diagram of Fig. 2(b) means that we take the diagram of Fig. 2(a) and put one of the external legs, say $\partial\pi_{(x_1)}{}^{a_1}$, equal to $\partial\Delta_{12}(\partial/\partial\pi_2{}^{a_1})$.

Our prescription for the $S$ matrix omits $g_{\mu 0}g_{\nu 0}i\delta(x)$ from $\Delta_{\mu\nu}(x)$ and uses the time-ordered exponential of $i\mathcal{L}_{\rm int}$ rather than $-i\mathcal{H}_{\rm int}$. This makes no essential difference to the argument of Sec. III, and will be discussed in Sec. V.

## III. ULTRAVIOLET DIVERGENCES

Disregarding the behavior of $S_k(x^n,a^n,b^n)$, the ultraviolet divergences that arise can be very violent. As usual for the position-space over-all power count, $\int dx^n\sim R^{4(n-1)}$, $\Delta\sim 1/R^2$, where $x_i-x_j=R\eta_{ij}$, $R\to 0$. Then, e.g., the diagram of Fig. 3 gives $R^{-2n+2}$. (This, of course, is not a vacuum-vacuum process—it involves any number of particles.)

The question for this section is whether $S_k(x^n,a^n,b^n)$ and its formal derivatives with respect to the fields fall to zero sufficiently fast as $R\to 0$ to damp these divergences, as has been found to some extent in the non-derivative-coupling case.[2,3]

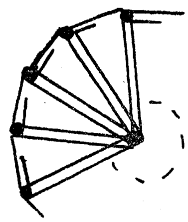

FIG. 3. Diagram with $n$ vertices whose divergence, due to the derivative interaction term, is $R^{-2n+2}$.

The ultraviolet behavior of amplitudes after resummation has been found in the past to be given by the power count $4(n-1)-nd+e$, where $4(n-1)$ is the contribution of $\int dx^n$; $d$ is the degree of the Lagrangian, counting $\partial\pi$ as $\pi^2$; and $e$ is the degree of the external lines, again counting $\partial\pi$ as $\pi^2$. (In many cases extraction of external lines does not reduce the divergence, so that this is really a lower bound.[2]) We assume that such behavior can be achieved for $S_k(x^n,a^n,b^n)$. The singular behavior of the rest can be estimated directly, not being involved in the resummation. It can be easily seen that the over-all behavior is again $R^{4(n-1)-nd+e}$ (or worse), where $d$ now is the over-all degree of $\mathcal{L}_{\text{int}}$, namely, 4, so the behavior is $R^{e-4}$. Therefore there are graphs with divergences, even with damping from $S_k(x^n,a^n,b^n)$ taken into account. This has already been remarked upon by Hunt *et al.*,[5] to second order in $f_\pi^{-1}$.

## IV. COUNTER TERMS

We proved in the Sec. III that resummation methods applied to the nonderivative factor of the chiral Lagrangian are not strong enough to damp down ultraviolet divergences arising from the derivative factor. It would thus seem essential to cancel these divergences by the addition of counterterms in a fashion extending the discussion given by us earlier for the case of nonderivative interactions.[3] Let us consider the total Lagrangian $\mathcal{L}$ obtained by adding a counterterm $\delta\mathcal{L}$ to the original chirally invariant Lagrangian $\frac{1}{2}(D_\mu\boldsymbol{\pi})$.[2,6] In order that $\mathcal{L}$ be still chirally invariant, it is necessary that $\delta\mathcal{L}$ itself be constructed from chiral invariants. To prevent the field equations from involving derivatives of the field variables higher than the second (which would then cause indefinite-metric or complex-mass states to erupt),[7] we can only construct $\delta\mathcal{L}$ from the sole chiral invariant containing the first derivative of $\boldsymbol{\pi}$, i.e. from $\frac{1}{2}(D_\mu\boldsymbol{\pi})^2$, which we denote by $X$. Thus the total Lagrangian $\mathcal{L}$ is some function $F(X)$ of $X$.

Is it possible to choose any counterterm $\delta\mathcal{L}$, or are there additional restrictions imposed by the original current-algebra structure? Indeed $\delta\mathcal{L}$ must reasonably be chosen so that this structure is not changed, for the original reason for setting up the chiral-invariant Lagrangian $X$ was that it should be a dynamical expression of the current commutation relations. It is necessary to consider the conditions imposed on $F(X)$ so that there is no change in these relations. We start by obtaining the canonical commutation relations (CCR's) for the fields. The momentum $p_a(x)$ conjugate to the field $\pi^a(x)$ is

$$p_a(x)=\frac{\partial\mathcal{L}}{\partial\dot{\pi}^a(x)}=g_{ab}\dot{\pi}^bF'(X)\,,$$

so that the CCR's are, for $x_0=y_0$,

$$[g_{ab}\dot{\pi}^b(x)F'(X),\pi^c(y)]_-=-i\delta_a{}^c\delta^3(\mathbf{x}-\mathbf{y})\,.$$

The current $J_{\mu\alpha}$ is defined as

$$J_{\mu\alpha}=\partial\delta\mathcal{L}/\partial(\partial_\mu\theta^\alpha)\,,$$

where

$$\delta\mathcal{L}=F'(X)g_{ij}\xi_\alpha{}^j\partial_\mu\pi^i\partial_\mu\theta^\alpha\,,$$

since $\delta\pi^i=\xi_\alpha{}^i\theta^\alpha$. Thus

$$J_{\mu\alpha}=F'(X)g_{ij}\xi_\alpha{}^j\partial_\mu\pi^i$$

and, in particular,

$$J_{0\alpha}=p_j\xi_\alpha{}^j\,.$$

We may easily obtain the correct commutation relations for the time component of the currents:

$$\begin{aligned}[J_{0\alpha}(x),J_{0\beta}(y)]_-&=[p_i\xi_\alpha{}^i,p_j\xi_\beta{}^j]_-\\&=[p_i,\xi_\beta{}^j]_-\xi_\alpha{}^ip_j+[\xi_\alpha{}^i,p_j]\xi_\beta{}^jp_i\\&=i[\xi_\beta{}^j{}_{,i}\xi_\alpha{}^ip_j-\xi_\alpha{}^i{}_{,j}\xi_\beta{}^jp_i]\delta^3(\mathbf{x}-\mathbf{y})\\&=-iC_{\alpha\beta}{}^\gamma J_{0\gamma}(x)\delta^3(\mathbf{x}-\mathbf{y})\,.\end{aligned}\qquad(1)$$

As in the case when $\mathcal{L}=X$, only the group law is required, along with the CCR's, to obtain (1). Let us now turn to the time-space components

$$\begin{aligned}&[J_{0\alpha}(x),J_{a\beta}(y)]_-\\&=[p_l\xi_\alpha{}^l,F'(X)g_{ij}\xi_\beta{}^j\partial_a\pi^i]\\&=p_l[\xi_\alpha{}^l,X]_-g_{ij}\xi_\beta{}^j\partial_a\pi^iF''(X)\\&\quad-i\{F'(X)(g_{ij}\xi_\beta{}^j)_{,l}\xi_\alpha{}^l\partial_a\pi^i+\xi_\alpha{}^iF'(X)g_{ij}\xi_\beta{}^j\partial_a\\&\quad+\xi_\alpha{}^lF''(X)\partial_\mu\pi^r\partial_\mu\pi^sg_{rs,l}g_{ij}\xi_\beta{}^j\partial_a\pi^i\\&\qquad+\xi_\alpha{}^lF''(X)g_{ij}\xi_\beta{}^j\partial_a\pi^ig_{ls}\partial_t\pi^s\partial_t\}\delta^3(\mathbf{x}-\mathbf{y})\,.\end{aligned}\qquad(2)$$

Using Killing's equations, as for the case when $\mathcal{L}=X$, it is possible to show that the second term on the right-hand side of (2) may be written as $-iC_{\alpha\beta}{}^\gamma J_{a\gamma}\delta^3(\mathbf{x}-\mathbf{y})$, as required. The third term is evidently the usual Schwinger term, though now multiplied by $F'(X)$. To preserve this term it is necessary to take $F'(X)\equiv1$, so that $F''(X)\equiv0$ and the remaining terms on the right-hand side of (2) vanish. It may be argued that the Schwinger term is not sacrosanct, so that $F'(X)$ may not be identically 1. In that case the extra terms on the right-hand side of (2) are of a very complicated nature, but will not vanish, nor will they be of the nature of Schwinger terms. For example, the last term involves the derivative of a $\delta$ function, though its space integral does not vanish. But in addition to these extra terms, the space-space current commutator brackets no longer

[7] J. G. Taylor, Nuovo Cimento Suppl. 1, 857 (1963), especially pp. 869, 870.

vanish when $\mathfrak{L}\neq X$. Thus, with $Y=\frac{1}{2}[D_\mu\boldsymbol{\pi}(\mathbf{y})]^2$,

$$[J_{a\alpha}(\mathbf{x}),J_{b\beta}(\mathbf{y})]_-=[F'(X)g_{ij}\xi_\alpha{}^j\partial_a\pi^i,F'(Y)g_{lm}\xi_\beta{}^l\partial_b\pi^m]_-,$$

and this will certainly not vanish. It is difficult to calculate its value purely from the canonical commutation relations; this is in general not specified. However, it appears very unlikely that it will vanish for any specific model, except when $F$ is equal to $X$. We conclude that the time-space and space-space current commutators will have extra non-Schwinger terms present in all cases except for the original Lagrangian. In other words, it is not possible to add chirally invariant counterterms to the Lagrangian without destroying the current commutation relations.

## V. SELF-CANCELLATION

If the commutation relations are regarded as unchangeable (including their Schwinger terms), the only possibility of obtaining finite results for matrix elements is by means of self-cancellation: The divergences must cancel amongst themselves. This possibility has been considered by Charap,[8] who found certain cancellations to occur if particular coordinates are used. Subsequently, however, he found[9] that this was due to the omission mentioned at the end of Sec. II and that when the calculation is done correctly the same cancellation occurs for all coordinates. We take his calculation further. In his notation

$$\mathfrak{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})^2+\tfrac{1}{2}\sum_{n=1}^{\infty}f_\pi^{-2n}[\alpha_n(\boldsymbol{\pi}^2)^n(\partial_\mu\boldsymbol{\pi})^2+\beta_n(\boldsymbol{\pi}^2)^{n-1}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2].$$

We consider the pion-pion scattering amplitude to order $f_\pi^{-4}$.

The diagrams contributing to the uncorrected amplitude (i.e., without $g_{\mu0}g_{\nu0}i\delta_x$) are shown in Figs. 4 and 5. Using the Feynman rules that the vertex contribution with isospin indices $a_1,\ldots,a_{2n}$ and momenta $p_1,\ldots,p_{2n}$ is

$$if_\pi^{-2(n-1)}2^{n-2}(n-1)!\times\left\{\delta_{a_1a_2}\delta_{a_3a_4}\cdots\delta_{a_{2n-1}a_{2n}}\left[\alpha_{n-1}\left(\sum_{i=1}^{n}p_i^2\right)+\left(\frac{\beta_{n-1}}{2(n-1)}-\alpha_{n-1}\right)(s_{12}+s_{34}+\cdots+s_{2n-1,2n})+\cdots\right]\right\},$$

where $s_{ij}=(p_i+p_j)^2$ and the other terms in the bracket $\{\ \}$ correspond to the other distinct pairings of the $2n$ lines, then the contribution from Fig. 4 is

$$A=(i/2f_\pi^4)\{\delta_{a_1a_2}\delta_{a_3a_4}N(s_{12},s_{13},s_{14})+\delta_{a_1a_3}\delta_{a_2a_4}N(s_{13},s_{12},s_{14})+\delta_{a_1a_4}\delta_{a_2a_3}N(s_{14},s_{12},s_{13})\}\,, \quad (3)$$

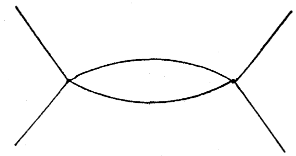

FIG. 4. Divergent diagram in two-pion scattering arising in second order in the interaction Lagrangian and to order $f_\pi^{-4}$ in the scattering amplitude.

where

$$N(s,t,u)=4(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)\chi+s(5\beta_1^2-10\alpha_1^2)T+(\beta_1-2\alpha_1)^2[s^2U(s)+\tfrac{1}{2}t^2U(t)+\tfrac{1}{2}u^2U(u)]\,, \quad (4)$$

with

$$\chi=(2\pi)^{-4}\int d^4k\,,$$

$$T=(2\pi)^{-4}\int d^4k/k^2\,,$$

$$U(s)=(2\pi)^{-4}\int d^4k/[k^2(P-k)^2]\,,$$

and $s=P^2$. In the evaluation of (4) it is necessary to treat the quadratically and quartically divergent integrals with care; no ambiguity is present if the following rules are used.

(a) $\delta^4$ functions arising at vertices may be used to choose any loop momentum as integration variables in numerators (not necessarily in denominators), where extra terms may arise due to different choices of loop momenta corresponding to shifts in integration variables. This is known to produce extra terms.[10]

(b) The integrals are linear functions of the integrands.

(c) Lorentz invariance may be used to give the general structure of the divergent integrals.

(d) Integrals of antisymmetric functions of the variables are zero.

The contribution from Fig. 5 is

$$-if_\pi^{-4}[\delta_{a_1a_2}\delta_{a_3a_4}M(s_{12})+\delta_{a_1a_3}\delta_{a_2a_4}M(s_{13})+\delta_{a_1a_4}\delta_{a_2a_3}M(s_{14})]\,,$$

where

$$M(s)=4(3\alpha_2+\beta_2)\chi+5(\beta_2-4\alpha_2)sT\,.$$

Then the total uncorrected contribution to pion-pion scattering to order $f_\pi^{-4}$ has the form (3) with $N$

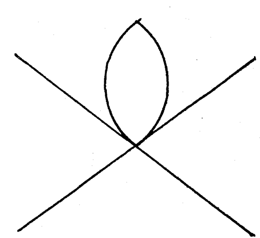

FIG. 5. Divergent diagram in two-pion scattering arising in first order in the interaction Langrangian and to order $f_\pi^{-4}$ in the scattering amplitude.

[8] J. M. Charap, Phys. Rev. D 2, 1554 (1970).

[9] J. M. Charap, this issue, Phys. Rev. D 3, 1998 (1971); see also J. Honerkamp and K. Meetz, this issue, *ibid.* 3, 1996 (1971).

[10] See, e.g., Jauch and Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), Appendix A.5.

replaced by $\bar{N}$ given by

$$\bar{N}(s,t,u)=[4(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)-8(3\alpha_2+\beta_2)]\chi \\ +[(5\beta_1^2-10\alpha_1^2)-10(\beta_2-4\alpha_2)]sT \\ +(\beta_1-2\alpha_1)^2[s^2U(s)+\tfrac{1}{2}t^2U(t)+\tfrac{1}{2}u^2U(u)]. \quad (5)$$

The correction may be calculated as follows. In Sec. II the factor

$$\exp\left(-\frac{1}{2}\frac{\delta}{\delta\pi_\mu{}^a}\Delta_{\mu\nu}\frac{\delta}{\delta\pi_\nu{}^a}\right)$$

of the Hori operator is supplemented by

$$\exp\left(-\frac{i}{2}\frac{\delta}{\delta\dot{\pi}^a}\frac{\delta}{\delta\dot{\pi}^a}\right)$$

and $i\mathcal{L}_{\text{int}}$ is replaced by $-i\mathcal{H}_{\text{int}}$, where

$$\mathcal{H}_{\text{int}}=\tfrac{1}{2}[(1+f)^{-1}-1]_{ab}\dot{\pi}^a\dot{\pi}^b+\tfrac{1}{2}f_{ab}\nabla\pi^a\cdot\nabla\pi^b.$$

Applying the extra operator to $\exp(-i\mathcal{H}_{\text{int}})$, using the identity

$$\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi}A\frac{\delta}{\delta\phi}\right)\exp(\tfrac{1}{2}\phi B\phi) \\ =\exp\left[-\tfrac{1}{2}\operatorname{tr}\ln(1-AB)+\tfrac{1}{2}\phi B\frac{1}{1-AB}\phi\right],$$

we obtain

$$\exp\left[i\mathcal{L}_{\text{int}}+\tfrac{1}{2}\chi\int dx\operatorname{tr}\ln[1+f(\boldsymbol{\pi}(x))]\right],$$

remembering that $\chi=\delta_{(0)}{}^{(4)}$. An expression for $S$ is obtained by applying the Hori operator of Sec. II to this, i.e., using the usual rules with

$$-\tfrac{1}{2}i\chi\operatorname{tr}\ln(1+f)$$

as an additional term in the Lagrangian.[11] Now

$$\operatorname{tr}\ln(1+f)=\ln\det(1+f)$$

and

$$\det(1+f)=\det(\delta_{ab}A+\pi^a\pi^bB) \\ =A^2(A+\boldsymbol{\pi}^2B),$$

where

$$A=\sum_{n\geqslant 0}f_\pi{}^{-2n}\alpha_n\boldsymbol{\pi}^{2n},\quad B=\sum_{n\geqslant 1}f_\pi{}^{-2n}\beta_n\boldsymbol{\pi}^{2(n-1)}.$$

To order $f_\pi{}^{-4}$,

$$\ln\det f=(3\alpha_1+\beta_1)f_\pi{}^{-2}\boldsymbol{\pi}^2 \\ +(3\alpha_2+\beta_2-\tfrac{3}{2}\alpha_1^2-\alpha_1\beta_1-\tfrac{1}{2}\beta_1^2)f_\pi{}^{-4}\boldsymbol{\pi}^4.$$

[11] This expression was obtained by A. Salam and J. Strathdee, [Phys. Rev. D 2, 2869 (1970)], from the "canonical" path-integral formulation, but they do not comment on its relation to the operator formalism given here and independently elsewhere; see, I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. D (to be published).

Thus the only strongly connected extra contribution to $f_\pi{}^{-4}$ is

$$\tfrac{1}{2}i\chi(3\alpha_2+\beta_2-\tfrac{3}{2}\alpha_1^2-\alpha_1\beta_1-\tfrac{1}{2}\beta_1^2)f_\pi{}^{-4}\langle 12|\boldsymbol{\pi}^4|34\rangle.$$

This cancels the first term on the right-hand side of Eq. (5), as observed by Charap.

Is it possible for the remaining terms of Eq. (5) to vanish? The answer is provided by a consequence of chiral symmetry, namely, $\beta_1-2\alpha_1=1$,[8] so that the third (logarithmically divergent) term can never vanish in a chirally invariant theory.

## VI. DISCUSSION

The results of this paper show the incompatibility of the local current-algebra commutation relations, in the form given for example by Barnes and Isham,[6] with the Lagrangian formally consistent with them. This incompatibility takes the form of ultraviolet divergences, generated by closed loops, which are not removable by any internal cancellation, nor can they be removed by addition of counterterms to the Lagrangian without considerably modifying the original current commutations relations.

How may we avoid this incompatibility? There appear to be two ways in which this can be achieved, either by changing the Lagrangian or alternatively by changing the current commutation relations. The first way can be effected by means of extra particles such as vector mesons or nucleons, which may allow the internal cancellations to take place, which pions alone were powerless to do. This appears to be rather unlikely. The other alternative corresponds to addition of new terms to the commutation relations, as was seen in Sec. IV. These terms do not enter into the commutator brackets of time components of currents with each other, but do so into the time-space and space-space brackets. In particular, the commutator bracket of the space integral of the time component with the space components acquires further terms of a very complicated nature. These may show up in soft-pion or in weak and electromagnetic interactions. We hope to return to this problem elsewhere.

There is one other way of understanding this incompatibility of the current commutation relations with the Lagrangian, namely, by accepting that there is no Lagrangian theory on which current algebras can be based. This gives justification for the program of Sugawara,[12] that the currents are all that there is with no underlying canonical fields. However, this seems a counsel of despair, and we trust that nature is sensible enough to use one of the two alternatives we put forward earlier.

## ACKNOWLEDGMENT

We would like to thank K. J. Barnes for a helpful discussion.

[12] H. Sugawara, Phys. Rev. 170, 1659 (1968).