Dispersion Sum Rules for Fixed-u πN Scattering Amplitudes*

YU-CHIEN LIU AND IAN J. MCGEE
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

(Received 22 June 1970)

We review the fixed-u finite-energy sum rules for pion-nucleon scattering amplitudes. We also derive and test the continuous-dispersion sum rules for these amplitudes. The real part of the amplitudes is restricted to appear on the right-hand cut ($N\pi \to N\pi$) only, because no information is available for the left-hand cut ($\pi\pi \to N\bar{N}$). The main result is an evaluation of the σ, f, and g coupling constants.

I. INTRODUCTION

In their prototype, dispersion relations are used to evaluate the real part of an amplitude, knowing the imaginary part from the unitarity condition. The total amplitude is then fully determined, at least in principle.[1]

The fixed-s, -t, and -u dispersion relations for πN scattering were derived by Mandelstam[2] over ten years, based on the principle of maximal analyticity of the first kind.[3] That the fixed-u (and fixed-s) dispersion relations are less well known in physics than the fixed-t ones arises from the fact that a vast unphysical region is involved in the dispersion integral. Both the numerical calculation and comparison with experiment of the fixed-u dispersion relations are very difficult, and thus have received little attention.

Nevertheless, within the scheme of pole saturation, there appeared fixed-u superconvergent dispersion sum rules[4] which, if correct, allow correlation between the MMM and the MBB coupling constants (M=meson; B=baryon). There are also fixed-u finite-energy sum rules (FESR)[5] which permit exploration of the baryon Regge trajectories[6] through analyticity.

In the present paper we have carefully examined the earlier work of Refs. 4 and 5 (Sec. II), and have redone the pertinent numerical analysis (Sec. III). In addition, we derive and test the fixed-u continuous-dispersion sum rules (CDSR) for the πN scattering amplitudes. The real part is confined to appear in the physical region of the right-hand cut ($N\pi \to N\pi$) only, in the absence of similar knowledge about the left-hand cut ($\pi\pi \to N\bar{N}$). Such a lack of symmetry under $s \rightleftarrows t$ also leads to a less convenient form of the CDSR for fixed u than for fixed t (Sec. IV).[7]

II. FESR REVISITED

Following Barger, Michael, and Phillips (BMP),[5] a suitable set of u-channel amplitudes (regularized parity-conserving helicity amplitudes) for Reggeization is

$$\tilde{F}^{\pm}(\sqrt{u},s) = \mp A(s,u) - (\sqrt{u} \pm M)B(s,u). \quad (1)$$

At fixed u and asymptotic s, the superscripts on $\tilde{F}$ label the $n=\tau P$ quantum number of the u-channel Regge exchange (N and Δ).

A convenient independent variable is chosen as

$$x = \tfrac{1}{2}(s-t) = s - M^2 - \mu^2 + \tfrac{1}{2}u = -t + M^2 + \mu^2 - \tfrac{1}{2}u, \quad (2)$$

so that the Regge amplitudes read[8]

$$\tilde{F}^{\pm}(\sqrt{u},x) + \tau \tilde{F}^{\pm}(\sqrt{u}, -x) = \gamma^{\pm,\tau}(\sqrt{u}) R(\bar{\alpha}_\tau(\pm, \sqrt{u}), x), \quad (3)$$

where

$$R(\bar{\alpha}_\tau(\sqrt{u}),x) = \frac{x^{\alpha-1/2} + \tau(-x)^{\alpha-1/2}}{\sin \pi \bar{\alpha}_\tau}, \quad (4)$$

$\alpha \equiv \alpha_\tau(\sqrt{u}) = \bar{\alpha}_\tau(\sqrt{u}) + \frac{1}{2}$, and $\tau = (-1)^{J-1/2}$.

Proceeding in the standard way,[3,5] one can derive the FESR for the crossing-odd amplitudes $\tilde{F}^{\pm}(\sqrt{u},x) - \tilde{F}^{\pm}(\sqrt{u}, -x)$, for the crossing-even ones $\tilde{F}^{\pm}(\sqrt{u},x) + \tilde{F}^{\pm}(\sqrt{u}, -x)$, and finally add them up to eliminate $\tilde{F}^{\pm}(\sqrt{u}, -x)$, obtaining

$$\int_{-N}^{N} dx\, x^{2k} [\tilde{F}^{\pm}(\sqrt{u}, x+i\epsilon) - \tilde{F}^{\pm}(\sqrt{u}, x-i\epsilon)] = 2i\gamma^{\pm,-} \frac{N^{\alpha_-(\pm)-1/2+2k+1}}{\bar{\alpha}_-(\pm)+2k+1}, \quad (5)$$

$$\int_{-N}^{N} dx\, x^{2k+1} [\tilde{F}^{\pm}(\sqrt{u}, x+i\epsilon) - \tilde{F}^{\pm}(\sqrt{u}, x-i\epsilon)] = -2i\gamma^{\pm,+} \frac{N^{\alpha_+(\pm)-1/2+2k+2}}{\bar{\alpha}_+(\pm)+2k+2}, \quad (6)$$

where k is an integer. We recall the notation used, $N_\alpha(\tau P=+, \tau=+)$, $N_\beta(-,+)$, $N_\gamma(+,-)$, $N_\delta(-,-)$, and similarly for the Δ's.

The MacDowell symmetry[9] requires

$$\tilde{F}^{-}(\sqrt{u},x) = -\tilde{F}^{+}(-\sqrt{u}, x),$$

* Work supported in part by the National Research Council of Canada.

[1] See, e.g., S. Mandelstam, Rept. Progr. Phys. **22**, 99 (1962); T. T. Chou and M. Dresden, Rev. Mod. Phys. **39**, 143 (1967).

[2] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[3] G. F. Chew, *The Dynamical S Matrix* (Benjamin, New York, 1966).

[4] D. Griffiths and W. Palmer, Phys. Rev. **161**, 1606 (1967); D. S. Beder and J. Finkelstein, *ibid.* **160**, 1363 (1967); G. G. Volkov, V. V. Ezhela, and V. S. Zamiralov, Yadern. Fiz. **8**, 1028 (1968) [Soviet J. Nucl. Phys. **8**, 593 (1969)].

[5] V. Barger, C. Michael, and R. J. N. Phillips, Phys. Rev. **185**, 1852 (1969), and references therein; B. Kayser, Phys. Rev. D **1**, 306 (1970).

[6] V. Gribov, L. Okun', and I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **45**, 1114 (1963) [Soviet Phys. JETP **18**, 769 (1964)]; R. Carlitz and M. Kislinger, Phys. Rev. Letters **24**, 186 (1970).

[7] See the formalism used in Y. C. Liu and I. J. McGee, Phys. Rev. D **1**, 3123 (1970).

[8] Writing $\tilde{F}^{\pm}(\sqrt{u},x) \pm \tilde{F}^{\pm}(\sqrt{u}, -x)$ allows the crossing relation to be valid for all values of x, not just asymptotically.

[9] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

which can be met by demanding

$$\alpha_\tau(-,\sqrt{u})=\alpha_\tau(+,\sqrt{u})$$

and

$$\gamma^{-,\tau}(\sqrt{u})=-\gamma^{+,\tau}(-\sqrt{u}).$$

Parametrizing

$$\alpha_\tau(+,\sqrt{u})=\alpha_\tau^1(u)+(\sqrt{u})\alpha_\tau^2(u),$$
$$\gamma^{+,\tau}(\sqrt{u})=\gamma_1^\tau(u)+(\sqrt{u})\gamma_2^\tau(u),$$

we obtain

$$\alpha_\tau(-,\sqrt{u})=\alpha_\tau^1(u)-(\sqrt{u})\alpha_\tau^2(u),$$
$$\gamma^{-,\tau}(\sqrt{u})=-\gamma_1^\tau(u)+(\sqrt{u})\gamma_2^\tau(u).$$

In this way we can avoid the branch point at $u=0$ in the sum rules and treat the scattering region ($u<0$) on the same footing as the spectral region ($u>0$) [see Eq. (7)].

We have conjectured that since $\sqrt{u}$ has appeared as a multiplicative factor to B in Eq. (1), it is more likely to appear in the residue function $\gamma(\sqrt{u})$ than in the trajectory function $\alpha(\sqrt{u})$ on the right-hand side of Eq. (3). Present high-energy backward scattering[5] also shows that $\alpha(\sqrt{u})$ is well described by a function of u only, so we put $\alpha_\tau(\pm,\sqrt{u})=\alpha_\tau(u)$ throughout this article. This greatly simplifies the form and the analysis of the sum rules, but is not critical.[10]

The four FESR (5) and (6) are therefore transformed into forms analytic at $u=0$ via (1) and the MacDowell symmetry $\gamma^{\pm,\tau}(\sqrt{u})=\pm\gamma_1^\tau(u)+(\sqrt{u})\gamma_2^\tau(u)$:

$$\int_{-N}^{N} dx\,x^{2k}\,\mathrm{Disc}\begin{bmatrix}B(x,u)\\ A(x,u)+MB(x,u)\end{bmatrix}=-2i\begin{bmatrix}\gamma_2^-(u)\\ \gamma_1^-(u)\end{bmatrix}\frac{N^{\alpha_-(u)-1/2+2k+1}}{\bar{\alpha}_-(u)+2k+1},$$
$$\int_{-N}^{N} dx\,x^{2k+1}\,\mathrm{Disc}\begin{bmatrix}B(x,u)\\ A(x,u)+MB(x,u)\end{bmatrix}=2i\begin{bmatrix}\gamma_2^+(u)\\ \gamma_1^+(u)\end{bmatrix}\frac{N^{\alpha_+(u)-1/2+2k+2}}{\bar{\alpha}_+(u)+2k+2},\tag{7}$$

where $\mathrm{Disc}B(x,u)=B(x+i\epsilon,u)-B(x-i\epsilon,u)$, etc. If the low-energy integral is known with confidence, one can explore the residue functions $\gamma_1^\pm(u)$ and $\gamma_2^\pm(u)$, and discover *experimentally* whether a world with parity doublets exists or not. Present information on the $\pi\pi\to N\bar{N}$ reaction[11] is too incomplete to answer such a question. Pole saturation coupled with the CDSR in the following sections may help to solve this problem.

[10] If the odd part of $\alpha(\sqrt{u})$ is retained, Eq. (7) remains analytic at $u=0$, but the right-hand side is replaced by more complicated expressions.

[11] J. Hamilton, in *High Energy Physics*, edited by E. H. S. Burhop (Academic, New York, 1967), Vol. 1.

III. NUMERICAL ANALYSIS

Two alternatives are possible for the evaluation of the discontinuities of $A(x,u)$ and $B(x,u)$ occurring in (7).

A. Pole or Narrow-Width Approximation

The effective interacting Lagrangians for exchange of particles of spin ≤ 2 are given in the Appendix. Standard Feynman rules yield[12]

$$\begin{aligned}
B_N^{(\pm)}(s,u)&=g_{NN\pi}^2/(M^2-s),\\
A_\Delta^{(\pm)}(s,u)&=\frac{1}{3}\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_{\Delta N\pi}^2}{\mu^2}\frac{A_\Delta}{M_\Delta^2-s},\\
B_\Delta^{(\pm)}(s,u)&=\frac{1}{3}\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_{\Delta N\pi}^2}{\mu^2}\frac{B_\Delta}{M_\Delta^2-s},\\
A_\sigma^{(+)}(t,u)&=M_\sigma g_{\sigma\pi\pi}g_{\sigma NN}/(M_\sigma^2-t),\\
A_\rho^{(-)}(t,u)&=-\frac{g_{\rho\pi\pi}g_t}{M_\rho^2-t}\frac{s-u}{4M},\\
B_\rho^{(-)}(t,u)&=g_{\rho\pi\pi}(g_v+g_t)/(M_\rho^2-t),\\
A_f^{(+)}(t,u)&=g_{f\pi\pi}\frac{A_f}{M_f^2-t},\\
B_f^{(+)}(t,u)&=+\frac{g_{\sigma\pi\pi}g_{fNN}^{(1)}}{M_f^2-t}\frac{s-u}{2M_fM},
\end{aligned}\tag{8}$$

and zero otherwise. In the above expressions,

$$A_\Delta=\tfrac{1}{3}(E_\Delta+M)[3(M_\Delta+M)(E_\Delta-M)\cos\theta_s+(M_\Delta-M)(E_\Delta+M)],$$
$$B_\Delta=\tfrac{1}{3}(E_\Delta+M)[3(E_\Delta-M)\cos\theta_s-(E_\Delta-M)],\tag{9}$$
$$A_f=+g_{fNN}^{(1)}\frac{M_f^2-4\mu^2}{6M_f}-g_{fNN}^{(2)}\times\frac{(s-u)^2-\frac{1}{3}(M_f^2-4M^2)(M_f^2-4\mu^2)}{8M_fM^2}.$$

To deal with unfamiliar couplings or exchange of particles with higher spin, a tractable method is to perform a partial-wave analysis and to assume the narrow-width approximation. For the s channel,[12,13] each resonance (with definite spin, J, and isospin) contributes an

[12] C. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957). The ($\pm$) in A and B refer to conventional pion isospin crossing and should not be confused with the normality quantum number $n=\tau P=\pm$ in F, or the signature factor $\tau=\pm$ in α or γ.

[13] A. Barbaro-Galtieri *et al.*, Rev. Mod. Phys. **42**, 87 (1970).

TABLE I. Superconvergent sum rules for B^{I_u} and A^{I_u}. The numbers in the parentheses are those evaluated at $u=0$, using $g_{\Delta N\pi}^2/4\pi=0.37$, $g_{\rho\pi\pi}^2/4\pi=2.4\pm0.4$, $g_{\rho\pi\pi}=g_v=g_t/3.7$, $M_\rho=0.765$ GeV, and $M_f=1.264$ GeV. A_Δ, B_Δ, and A_f are given in Eq. (9), with $A_\Delta(u=0)=0.771$ GeV^3, $B_\Delta(u=0)=-1.02$ GeV^2. The sum of N, Δ, σ, ρ, and f should add to zero, if the sum rule holds.

	$B^{I_u=1/2}$	$B^{I_u=3/2}$	$A^{I_u=1/2}$	$A^{I_u=3/2}$
N	$-g_{NN\pi}^2=4\pi(-14.6)$	$2g_{NN\pi}^2=4\pi(29.2)$	0	0
Δ	$\frac{4}{3}(g_{\Delta N\pi}/\mu)^2B_\Delta(u)$ $=4\pi(-25.8)$	$\frac{1}{3}(g_{\Delta N\pi}/\mu)^2B_\Delta(u)$ $=4\pi(-6.5)$	$\frac{4}{3}(g_{\Delta N\pi}/\mu)^2A_\Delta(u)$ $=4\pi(19.6)$	$\frac{1}{3}(g_{\Delta N\pi}/\mu)^2A_\Delta(u)$ $=4\pi(4.9)$
σ	0	0	$-g_{\sigma\pi\pi}g_{\sigma NN}M_\sigma$	$-g_{\sigma\pi\pi}g_{\sigma NN}M_\sigma$
ρ	$2g_{\rho\pi\pi}(g_v+g_t)$ $=4\pi(22.6\pm3.6)$	$-g_{\rho\pi\pi}(g_v+g_t)$ $=4\pi(-11.3\pm1.8)$	$-2g_{\rho\pi\pi}g_t(s-u)/4M$ $=4\pi(-5.8\pm0.9)$	$g_{\rho\pi\pi}g_t(s-u)/4M$ $=4\pi(2.9\pm0.5)$
f	$-g_{f\pi\pi}g_{fNN}^{(1)}(s-u)/2M_fM$ $=4\pi(-0.085g_{f\pi\pi}g_{fNN}^{(1)})$	$-g_{f\pi\pi}g_{fNN}^{(1)}(s-u)/2M_fM$ $=4\pi(-0.085g_{f\pi\pi}g_{fNN}^{(1)})$	$g_{f\pi\pi}A_f(u)$ $=4\pi(-0.200g_{f\pi\pi}g_{fNN}^{(1)}$ $+0.114g_{f\pi\pi}g_{fNN}^{(2)})$	$g_{f\pi\pi}A_f(u)$ $=4\pi(-0.200g_{f\pi\pi}g_{fNN}^{(1)}$ $+0.114g_{f\pi\pi}g_{fNN}^{(2)})$

amount given by

$$\frac{1}{4\pi}\begin{bmatrix}A\\B\end{bmatrix}=\frac{1}{q_s^3}\frac{x_J\Gamma_JM_J}{M_J^2-s}(-n)\times\left\{\begin{bmatrix}M_J-nM\\-n\end{bmatrix}(E_s+nM)P_{J+1/2}'(\cos\theta_s)+\begin{bmatrix}M_J+nM\\n\end{bmatrix}(E_s-nM)P_{J-1/2}'(\cos\theta_s)\right\}, \quad (10)$$

where $n=\tau P=\pm1$, x is the elasticity, Γ is the total width, M_J is the mass of the resonance, and

$$q_s^2=[s-(M+\mu)^2][s-(M-\mu)^2]/4s,$$
$$E_s^2=(s+M^2-\mu^2)^2/4s,\ \cos\theta_s=1+t/2q_s^2,$$

with each of them evaluated at $s=M_J^2$. For the t channel,[14,15] we have the corresponding result:

$$\frac{1}{4\pi}\begin{bmatrix}C^{(\pm)}\\B^{(\pm)}\end{bmatrix}=-\sum_{J\text{ even or odd}}^{\infty}(2J+1)\frac{1}{M_J^2-t}\times\begin{bmatrix}N_J^+P_J(\cos\theta_t)(p_tq_t)^J/p_t^2\\N_J^-P_J'(\cos\theta_t)(p_tq_t)^{J-1}/[J(J+1)]^{1/2}\end{bmatrix}, \quad (11)$$

where $C^{(\pm)}=A^{(\pm)}-(1/p_t^2)(p_tq_t\cos\theta_t)MB^{(\pm)}$, $p_t^2=\frac{1}{4}t-M^2$, $q_t^2=\frac{1}{4}t-\mu^2$, and $p_tq_t\cos\theta_t=\frac{1}{4}(s-u)$. Equations (10) and (11) are certainly convergent in the physical region of the s and t channel, respectively.

It is easy to establish the equivalence between (10), (11) and (8), (9) [and to show the correctness of the expressions in (9)]. It is however, regrettable that most of the g's and the N_J's are not known, making it difficult to perform a numerical analysis with the sum rules.

[14] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).
[15] C. Lovelace, in *Pion-Nucleon Scattering*, edited by G. L. Shaw and D. Y. Wong (Interscience, New York, 1969).

B. Phase-Shift Analysis Reconstruction

For the s channel,[12]

$$\frac{1}{4\pi}\begin{bmatrix}A^{(\pm)}\\B^{(\pm)}\end{bmatrix}=\frac{1}{E_s+M}\begin{bmatrix}\sqrt{s}+M\\1\end{bmatrix}f_1^{(\pm)}-\frac{1}{E_s-M}\begin{bmatrix}\sqrt{s}-M\\-1\end{bmatrix}f_2^{(\pm)},$$

$$f_1=\sum[f_l^+P_{l+1}'(\cos\theta_s)-f_l^-P_{l-1}'(\cos\theta_s)], \quad (12)$$
$$f_2=\sum(f_l^--f_l^+)P_l'(\cos\theta_s),$$
$$f_l=[\eta_l\exp(2i\delta_l)-1]/2iq_s.$$

For the t channel,[14] the physical region starts from $t=4M^2$. Direct experimental information in the unphysical region $4M^2<t<4\mu^2$ is not possible, although an indirect method such as extended unitarity allows one to identify the phase of the amplitude near $t=4\mu^2$ with the corresponding one from $\pi\pi$ scattering. The latter information itself is subject to great uncertainty, however.

Armed with these tools, we now evaluate the fixed-u sum rules (7). We use definite u-channel isospin amplitudes: $A^{I_u=1/2}=A^{(+)}-2A^{(-)}$, $A^{I_u=3/2}=A^{(+)}+A^{(-)}$, and similarly for B^{I_u}, corresponding to exchange of the N and the Δ, respectively.

First assume superconvergence[4] and saturation with N, Δ, σ, ρ, and f. The contributing amplitudes are shown in Table I. The numbers in the parentheses are those evaluated at $u=0$, using $g_{NN\pi}^2/4\pi=14.6$, $g_{\Delta N\pi}^2/4\pi=0.37$, $g_{\rho\pi\pi}^2/4\pi=2.4\pm0.4$, $g_{\rho\pi\pi}=g_v=g_t/3.7$, $B_\Delta(u=0)=-1.02$ GeV^2, $A_\Delta(u=0)=0.771$ GeV^3, $M_\rho=0.765$ GeV, and $M_f=1.264$ GeV.

The system N, Δ, ρ, is certainly not closed. (A notorious error has often been made by doubling the ρ contribution.) A more careful analysis, namely, inclusion of all known πN resonances, as well as inclusion of the σ and the f mesons, is again not satisfactory (e.g., subtracting $A^{1/2}$ from $A^{3/2}$), indicating that higher mesons are necessary.

Unsuccessful assumption of superconvergence leads to the use of FESR. BMP[5] considered the N_α ($\tau P=+$,

TABLE II. The FESR [Eq. (7)] for $k=0$ at $u=0$, using Eq. (14) for the left-hand cut, Eq. (12) and Ref. 16 for the right-hand cut (P. S. stands for phase shift), and adding the nucleon pole to constitute the left-hand side (LHS). The right-hand side (RHS) uses Eq. (13), i.e., the N_α and the Δ_δ trajectories only; other trajectories are set equal to zero. Evaluation of the right-hand cut (in the LHS) by means of the narrow-width approximation (NWA) is shown in parentheses, with the dominant Δ contribution extracted. Data in this Table are larger by a factor π than those of Table I. (A in units of GeV^{-1}, B in GeV^{-2}, and x in GeV^2.)

	Left-hand cut					Right-hand cut		FESR [Eq. (7)]	
	σ	ρ	f	g	N	P. S.	(NWA, Δ)	LHS	RHS
$xB^{1/2}$	0.0	29.7	3.2	−46.8	0.9	−44.3	(−59.8, −50.9)	−57.2	−38.7
$xA^{1/2}$	−1.7	−9.2	9.4	16.9	0.0	36.2	(36.8, 38.5)	51.5	1.5
$xB^{3/2}$	0.0	−14.9	3.2	23.4	−1.8	8.1	(−37.6, −12.8)	18.0	0.0
$xA^{3/2}$	−1.7	4.6	9.4	−8.4	0.0	2.0	(31.6, 9.6)	5.8	0.0
$B^{1/2}$	0.0	54.0	−4.8	25.2	−45.9	−62.4	(−91.2, −81.1)	−33.9	0.0
$A^{1/2}$	−2.4	−16.7	−14.0	−9.1	0.0	48.7	(65.0, 61.4)	6.5	0.0
$B^{3/2}$	0.0	−27.0	−4.8	−12.6	91.7	−1.0	(−15.8, −20.3)	46.4	3.0
$A^{3/2}$	−2.4	8.4	−14.0	4.6	0.0	6.1	(15.5, 15.3)	2.7	−5.1

$\tau=+$) and the Δ_δ ($\tau P=-$, $\tau=-$) trajectories at high energy, and cut off at the CERN[16] phase-shift energy (≈ 2 GeV). In our notation,

$$\bar{\alpha}_N(u)=-0.88+0.91u\,,$$
$$\bar{\alpha}_\Delta(u)=-0.29+0.84u\,,$$

$$\gamma_1^+(u)+(\sqrt{u})\gamma_2^+(u)=-16\pi[0.8(u-M^2)+18(\sqrt{u}+M)]e^{0.5u}/\Gamma(\bar{\alpha}_N(u)+1)\ \text{GeV}^{-1}, \quad (13)$$
$$-\gamma_1^-(u)+(\sqrt{u})\gamma_2^-(u)=-16\pi[0.2+0.09\sqrt{u}][1+\sqrt{u}/M_\Delta]e^{-0.7u}/\Gamma(\bar{\alpha}_\Delta(u)+1)\ \text{GeV}^{-1}.$$

The left-hand cut of (7) is saturated with the σ, ρ, f, and g, whose evaluation is made via (11), with the following parameters[5,15]:

J	M_J	N_J^+	N_J^-
σ	0.437 GeV	$N_0^+=0.642$ GeV^3	$N_0^-=0$ GeV^2
ρ	0.591 GeV	$N_1^+=0.75$ GeV	$N_1^-=4.05$ GeV^0
f	1.253 GeV	$N_2^+=4.69$ GeV^{-1}	$N_2^-=4.37$ GeV^{-2}
g	1.660 GeV	$N_3^+=2.1$ GeV^{-3}	$N_3^-=3.2$ GeV^{-4}

(14)

These parameters lead to the following conventional coupling constants:

$$g_{\sigma\pi\pi}g_{\sigma NN}/4\pi=[1/M_\sigma(M^2-\tfrac{1}{4}M_\sigma^2)]N_0^+=1.77\,,$$
$$g_{\rho\pi\pi}(g_v+g_tM_\rho^2/4M^2)/4\pi=(3/M)N_1^+=2.4\,,$$
$$g_{\rho\pi\pi}(g_v+g_t)/4\pi=(3/\sqrt{2})N_1^-=8.58\,,$$
$$g_{f\pi\pi}[g_{fNN}^{(1)}-g_{fNN}^{(2)}(1-\tfrac{1}{4}M_f^2/M^2)]/4\pi=(15/4)M_fN_2^+=22.1\,,$$
$$g_{f\pi\pi}g_{fNN}^{(1)}/4\pi=(15/2\sqrt{6})M_fMN_2^-=15.7\,.$$

The numerical results for the $k=0$ case [of Eq. (7)] are shown in Table II. With BMP we see that reasonable results are obtained for the dominant Regge amplitudes [N_α, or the first-moment FESR (S_1)], but little can be said about the smaller Regge exchange amplitudes [Δ_δ, or the zero-moment FESR (S_0)].

To test the CDSR in Sec. IV, which includes the FESR as a special case, we need a more accurate set of $N_J^\pm$. To this end we make a plausible assumption that for $u<0$ the four FESR for $B^{(\pm)}$ and $A^{(\pm)}$ are superconvergent and are saturated with the N, Δ, ρ, σ, f, and g. The N, Δ, and ρ are regarded known as before (Table I), leaving the σ, f, and g to be determined from the sum rules. The explicit form of each contribution is given in Table III, from which N_3^- can be determined from $B^{(-)}$, N_2^- from $B^{(+)}$, N_3^+ from $A^{(-)}$ and N_3^-, and N_0^+ and N_2^+ from $A^{(+)}$ and N_2^-, with N_0^+ being further separated from N_2^+ by varying u (for the contribution from the σ is u independent).

In this scheme we obtain

J	M_J	N_J^+	N_J^-
σ	0.437 GeV	1.2 GeV^3	0.0 GeV^2
ρ	0.765 GeV	1.2 GeV	5.3 GeV^0
f	1.264 GeV	9.0 GeV^{-1}	5.2 GeV^{-2}
g	1.663 GeV	4.2 GeV^{-3}	7.2 GeV^{-4}

(15)

or $g_{\sigma\pi\pi}g_{\sigma NN}/4\pi=3.31$, $g_{\rho\pi\pi}g_v/4\pi=2.4$, $g_{\rho\pi\pi}g_t/4\pi=8.9$, $g_{f\pi\pi}g_{fNN}^{(1)}/4\pi=18.9$, $g_{f\pi\pi}g_{fNN}^{(2)}/4\pi=-43.1$, in contrast to (14). (Both N_0^+ and N_2^+ are expected to be subject to great uncertainty and should not be taken too seriously.)

IV. CDSR

After the treatment of FESR, generalization to the CDSR is quite straightforward.[7] The problem is how to imbed the real part of the amplitude in the sum rules so as to obtain maximal useful information.

In the present situation reconstruction of the real part can be made through the phase-shift analysis (12). The partial-wave series is convergent in the physical region, i.e., $|\cos\theta_s|\leq 1$; therefore, useful CDSR can be derived by restricting the real part to appear in the region $x_p\leq x\leq N$, where $x_p=M^2+\mu^2-\tfrac{1}{2}u$, correspond-

[16] D. J. Herndon, A. Barbaro-Galtieri, and A. H. Rosenfeld, LRL Report No. UCRL-20030 πN, 1970 (unpublished).

ing to $\cos\theta_s=1$. For the $\pi\pi\to N\bar{N}$ cut, which starts from $x_{\pi\pi}=M^2-3\mu^2-\frac{1}{2}u$, only the narrow-width approximation is available, as discussed in Sec. III. Note that $x_p-x_{\pi\pi}=4\mu^2$ is greater than zero, so x_p always lies to the right of the $\pi\pi$ branch point.

Thus, instead of (7), we have

$$\int_{-N}^{x_p} dx(x_p-x)^\beta \,\mathrm{Disc}\begin{bmatrix}B(x,u)\\A(x,u)+MB(x,u)\end{bmatrix}+\int_{x_p}^{N} dx(x-x_p)^\beta \,\mathrm{Disc}\left\{e^{-i\pi\beta}\begin{bmatrix}B(x,u)\\A(x,u)+MB(x,u)\end{bmatrix}\right\}$$
$$=-2i\left\{\begin{bmatrix}\gamma_2^-(u)\\\gamma_1^-(u)\end{bmatrix}\frac{N^{\alpha_-(u)-1/2+\beta+1}}{\cos\frac{1}{2}\pi\bar{\alpha}_-(u)}\frac{\sin\frac{1}{2}\pi(\bar{\alpha}_-(u)+\beta+1)}{\bar{\alpha}_-(u)+\beta+1}\cos\tfrac{1}{2}\pi\beta\right.$$
$$\left.-\begin{bmatrix}\gamma_2^+(u)\\\gamma_1^+(u)\end{bmatrix}\frac{N^{\alpha_+(u)-1/2+\beta+1}}{\sin\frac{1}{2}\pi\bar{\alpha}_+(u)}\frac{\sin\frac{1}{2}\pi(\bar{\alpha}_+(u)+\beta+1)}{\bar{\alpha}_+(u)+\beta+1}\sin\tfrac{1}{2}\pi\beta+O(x_p/N)\right\},\tag{16}$$

$$\int_{-N}^{x_p} dx(x_p-x)^\beta x \,\mathrm{Disc}\begin{bmatrix}B(x,u)\\A(x,u)+MB(x,u)\end{bmatrix}+\int_{x_p}^{N} dx(x-x_p)^\beta x \,\mathrm{Disc}\left\{e^{-i\pi\beta}\begin{bmatrix}B(x,u)\\A(x,u)+MB(x,u)\end{bmatrix}\right\}$$
$$=-2i\left\{\begin{bmatrix}\gamma_2^+(u)\\\gamma_1^+(u)\end{bmatrix}\frac{N^{\alpha_+(u)-1/2+\beta+2}}{\sin\frac{1}{2}\pi\bar{\alpha}_+(u)}\frac{\sin\frac{1}{2}\pi(\bar{\alpha}_+(u)+\beta+2)}{\bar{\alpha}_+(u)+\beta+2}\cos\tfrac{1}{2}\pi\beta\right.$$
$$\left.+\begin{bmatrix}\gamma_2^-(u)\\\gamma_1^-(u)\end{bmatrix}\frac{N^{\alpha_-(u)-1/2+\beta+2}}{\cos\frac{1}{2}\pi\bar{\alpha}_-(u)}\frac{\sin\frac{1}{2}\pi(\bar{\alpha}_-(u)+\beta+2)}{\bar{\alpha}_-(u)+\beta+2}\sin\tfrac{1}{2}\pi\beta+O(x_p/N)\right\}.$$

As usual, the continuous spectral function $(x-x_p)^\beta$ becomes analytic when β is equal to an integer and (16) will reduce to the FESR .The different form of sum rule (16) (as compared to the fixed-t CDSR) is due to the use of the asymmetrical $(x-x_p)^\beta$ (under $x\to -x$) instead of a symmetrical $(x^2-x_p^2)^\beta$. This causes the presence of the factor $\cos\frac{1}{2}\pi\beta$ or $\sin\frac{1}{2}\pi\beta$ on the right-hand side of (16). Even for one-Regge-pole dominance, the diffractionlike pattern[7] (coming from the term of the form $\{\sin\frac{1}{2}\pi[\alpha(u)+\beta+\cdots]\}/[\alpha(u)+\beta+\cdots]$ for fixed u and varying β) is now "modulated" or distorted by such a factor. Thus, the "old" zeros at $\alpha(u)+\beta+\cdots=2k$ (k an integer) would still be there, but extra zeros at $\beta=$ integers also appear. Consequently, $\bar{\alpha}_\pm(0)$ would not be so easily found from the low-energy integrals as in the fixed-t situation.[7] In practice, $N\approx 4$ GeV^2 and $x_p\approx 0.90$ GeV^2 (at $u=0$), so terms up to x_p^2/N^2 have been kept on the right-hand side of (16).

The CDSR (16) are dealt with numerically by accepting (13) as the high-energy Regge contribution, adopting the "CERN experimental" set[16] as the real and the imaginary parts of the amplitude on the right-hand cut (with the narrow-width approximation as supplement), and using (15) as the left-hand cut contribution. Typical results are shown in Figs. 1 and 2. The $u=0$. $\beta=0$ case is also shown in Table IV, to be compared with Table II in which the $N_J^\pm$'s are taken from (14).

Before commenting on Figs. 1 and 2, we compare Table IV with Table II. The zero-moment sum rules (S_0) in Table IV are essentially input; the output does not, however, appear as anticipated, namely, LHS being approximately zero. This arises from the difference between the Δ dominance and the phase-shift reconstructions, used for the right-hand cut. Facing the fact that analytic continuation of the series (12) into the unphysical region ($|\cos\theta_s|>1$) may not be necessarily more accurate[17] than the narrow-width approximation, we did not try to find the $N_J^\pm$'s from Table IV.[18] Now one can envisage that Table II used the first-moment sum rules (S_1) while Table IV uses S_0 as inputs; neither produces reasonable results for the other part of the sum rules. [In (15), although $N_1^\pm$ are made larger to bring S_0 work, a larger value of M_ρ used there left S_1 worse.] In any case, were $N_1^\pm$ in (14) and (15) to assume a larger value, $N_3^\pm$ could be made smaller, so the g contribution would not exceed the ρ one.

Figures 1 and 2 show the CDSR for the amplitudes $xA^{I_u=1/2}$ and $B^{I_u=3/2}$. These amplitudes receive their high-energy contribution from the N_α and Δ_δ, respectively. The magnitude of both sides of the CDSR (16) in these figures differs, but the shape of the curves is qualitatively reasonable, although far from ideal.

Owing to imperfect knowledge on the left-hand cut, we are content in this work with testing the validity of (16) (i.e., LHS and RHS coincide), rather than attacking the whole problem associated with the fixed-u CDSR. These include, e.g., inclusion of the N_γ trajectory, investigation on the fundamental problem of the MacDowell symmetry [i.e., finding $\gamma_1^\pm(u),\gamma_2^\pm(u)$ as a function of u], etc., and, above all, testing the validity of the Mandelstam representation itself (which allows the derivation of the fixed-u dispersion sum rules); all

[17] At $u=0$ the physical threshold x_p lies in $0.28<x_p<0.92$ GeV^2. Below x_p, $\cos\theta_s$ is as big as 24.57 at $T_\pi=0.05$ GeV, or $x=0.36$ GeV^2 [$x=2M(\mu+T_\pi)+\frac{1}{2}u$].

[18] Recall that the $N_J^\pm$'s in Eq. (15) were found from Table III.

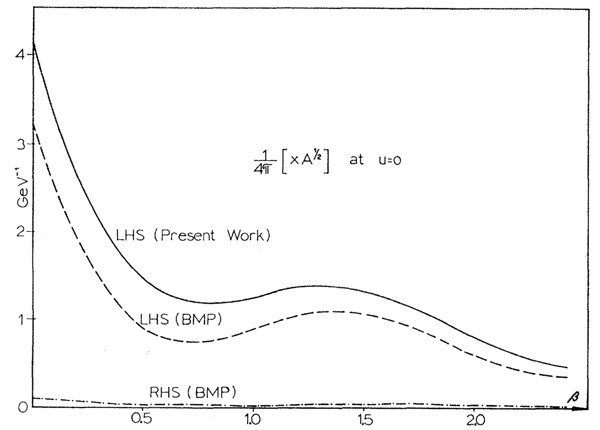

FIG. 1. Test of the fixed-u CDSR [Eq. (16)] for $xA^{I_u=1/2}$ at $u=0$ [the second equation of (16) divided by $N^{2+\beta}$]. The solid curve is the left-hand side (LHS) integral using the meson coupling constants of Eq. (15), the dashed one is the same integral using Eq. (14); see Ref. 5. The diffractionlike pattern is destroyed; this means that either the curves do not have zeros at a distance $\beta=1.0$ from each other or they have no zeros at all. (The diffractionlike pattern is characteristic of the fixed-t CDSR; see Sec. IV for details.) The dashed-dot curve shows the right-hand side (RHS) of Eq. (16), using Eq. (13). The fact that x_p is not much smaller than N prevents zeros from occurring at $\beta=$integers (i.e., both $\cos\frac{1}{2}\pi\beta$ and $\sin\frac{1}{2}\pi\beta$ are present). If one were to include the large error bars for the LHS curves, the result would look more reasonable. For this first-moment CDSR, the dashed curve is better.

of these need a more accurate calculation of the low-energy integrals. One possible way toward this aspect may be to make β negative, so that the unknown and uncertain contributions from the higher mesons are made relatively unimportant.

In conclusion, besides being intrinsically interesting in its formulation, the fixed-u CDSR have carried us one step further than the FESR[5] in investigating the amplitudes of πN backward scattering.

Note added in manuscript. Using the new parametrization of Martin and Michael[19] on the nucleon trajectory,

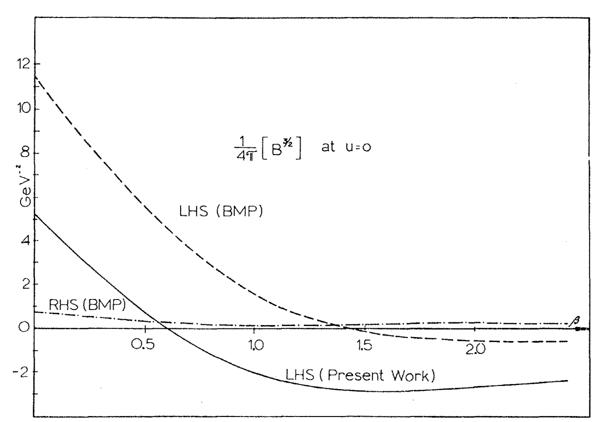

FIG. 2. Test of the fixed-u CDSR [Eq. (16)] for $B^{I_u=3/2}$ at $u=0$ [the first equation of (16) divided by $N^{1+\beta}$]. See Fig. 1 caption for details. For this zero-moment CDSR, the solid curve is better.

[19] A. D. Martin and C. Michael, Phys. Letters **32B**, 297 (1970).

TABLE III. Assumed superconvergent dispersion sum rules for $B^{(\pm)}$ and $A^{(\pm)}$ at fixed $u<0$. The data are evaluated at $u=0$. N_0^+, $N_2^\pm$, and $N_3^\pm$ are to be determined, with results given in Eq. (15). The N, Δ, and ρ have the same contributions as in Table I.

	$B^{(-)}$	$B^{(+)}$	$A^{(-)}$	$A^{(+)}$
N	$g_{NN\pi}^2/4\pi=14.6$	$g_{NN\pi}^2/4\pi=14.6$	0	0
Δ	$-\frac{1}{3}(g_{\Delta N\pi}/\mu)^2B_\Delta/4\pi=6.46$	$\frac{2}{3}(g_{\Delta N\pi}/\mu)^2B_\Delta/4\pi=-12.91$	$-\frac{1}{3}(g_{\Delta N\pi}/\mu)^2A_\Delta/4\pi=-4.88$	$\frac{2}{3}(g_{\Delta N\pi}/\mu)^2A_\Delta/4\pi=9.77$
ρ	$-g_{\rho\pi\pi}(g_v+g_t)/4\pi=-11.29$	0	$g_{\rho\pi\pi}g_t[(s-u)/4M](1/4\pi)=2.88$	0
σ	0	0	0	$(1/p_t^2)N_0^+=-1.20N_0^+$
f	0	$-(15/\sqrt{6})(p_tq_t\cos\theta_t)N_2^-$ $=-0.31N_2^-$	0	$\frac{5}{2}p_t^{-2}[3(p_tq_t\cos\theta_t)^2-p_t^2q_t^2]N_2^+$ $-(16/\sqrt{6})p_t^{-2}(p_tq_t\cos\theta_t)^2MN_2^-$ $=-0.99N_2^++0.03N_2^-$
g	$-(21/2\sqrt{12})[5(p_tq_t\cos\theta_t)^2-p_t^2q_t^2]N_3^-$ $=-1.27N_3^-$	0	$\frac{7}{2}p_t^{-2}(p_tq_t\cos\theta_t)[5(p_t^2q_t^2\cos^2\theta_t)-3p_t^2q_t^2]N_3^+$ $-(21/2\sqrt{12})[5(p_tq_t\cos\theta_t)^2-p_t^2q_t^2]MN_3^-$ $=3.01N_3^+-1.52N_3^-$	0

TABLE IV. See Table II for details, except that the left-hand cut is evaluated with Eq. (15).

	Left-hand cut σ	ρ	f	g	N	Right-hand cut P. S.	(NWA, Δ)	CDSR [Eq. (16)] for $\beta=0$ LHS	RHS
$xB^{1/2}$	0.0	22.3	3.5	−107.1	0.9	−44.3	(−59.8, −50.9)	−124.6	−38.7
$xA^{1/2}$	−3.2	−5.7	19.2	19.7	0.0	36.2	(36.8, 38.5)	66.2	1.5
$xB^{3/2}$	0.0	−11.2	3.5	53.5	−1.8	8.1	(−37.6, −12.8)	55.2	0.0
$xA^{3/2}$	−3.2	2.8	19.2	−9.9	0.0	2.0	(31.6, 9.6)	11.0	0.0
$B^{1/2}$	0.0	70.9	−5.1	57.4	−45.9	−62.4	(−91.2, −81.1)	15.0	0.0
$A^{1/2}$	−4.5	−18.1	−27.5	−10.6	0.0	48.7	(65.0, 61.4)	−11.9	0.0
$B^{3/2}$	0.0	−35.5	−5.1	−28.7	91.7	−1.0	(−15.8, −20.3)	21.6	3.0
$A^{3/2}$	−4.5	9.0	−27.5	5.3	0.0	6.1	(15.5, 15.3)	−11.6	−5.1

which in our notation reads

$$\gamma_{1N}^{+}(u)=2\pi(a_N+b_Nu)e^{c_Nu}/\Gamma(\bar{\alpha}_N(u)+1)\,,$$
$$\gamma_{2N}^{+}(u)=2\pi(a_N+b_NM^2)e^{c_Nu}/M\Gamma(\bar{\alpha}_N(u)+1)\,,$$
$$\bar{\alpha}_N(u)=-0.84+1.03u\,,$$

the right-hand side of the CDSR (16) is considerably improved. This improvement is due to their separating off the kinematic factor $\sqrt{u}$ in $\tilde{F}^{\pm}=\mp A-(\sqrt{u}\pm MB)$, in the same way as done in Eq. (7).

On the other hand, the new parametrization of Halzen *et al.*[20] on the Δ trajectory, i.e.,

$$\gamma_{1\Delta}^{-}(u)=-2\pi(a_\Delta+b_\Delta u)\bar{\alpha}_\Delta e^{c_\Delta u}/\Gamma(\bar{\alpha}_\Delta+1)\,,$$
$$\gamma_{2\Delta}^{-}(u)=2\pi(a_\Delta+b_\Delta M_\Delta{}^2)\bar{\alpha}_\Delta e^{c_\Delta u}/M_\Delta\Gamma(\bar{\alpha}_\Delta+1)\,,$$
$$\bar{\alpha}_\Delta(u)=\alpha_\Delta(u)-\tfrac{1}{2}\,,$$

has not improved the CDSR (16), although the width of the Δ is 240 MeV here as compared to the BMP value of 1 MeV. (A possible reason may be that their parameters of the nucleon trajectory yield a wrong sign for $g_{NN\pi}{}^2$.)

Note added in proof. It has been pointed out to us (by C. H. Chan) that the second CDSR of Eq. (16) can be generated from the first one by means of subtraction, namely,

$$[\text{second equation}]_\beta=x_p[\text{first equation}]_\beta-[\text{first equation}]_{\beta+1}. \quad (17)$$

Practically, one needs β as small as possible. Thus the two (actually four) CDSR in Eq. (16) are independent if β is allowed only in the same range, as in Figs. 1 and 2. Moreover, the identity (17) breaks down as $\beta\to-1$, while (16) still holds there.

[20] F. Halzen *et al.*, Phys. Letters **32B**, 111 (1970).

ACKNOWLEDGMENTS

We thank Professor R. Oehme, B. Kayser, P. Weiler, and C. Michael for useful communications.

APPENDIX

In the following effective interacting Lagrangian densities,[4] the boson fields have the dimension of mass, while the fermion fields have the dimension of (mass)$^{3/2}$, in the natural units ($\hbar=c=1$). The coupling constants g are therefore dimensionless.

$$\mathcal{L}_{\sigma\pi\pi}(x)=\tfrac{1}{2}g_{\sigma\pi\pi}M_\sigma\boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)\sigma(x)\,,$$
$$\mathcal{L}_{\sigma NN}(x)=g_{\sigma NN}\bar{\psi}(x)\psi(x)\sigma(x)\,,$$
$$\mathcal{L}_{\rho\pi\pi}(x)=-g_{\rho\pi\pi}e_{abc}\pi_a(x)\partial_\mu\pi_b(x)\rho_{\mu c}(x)\,,$$
$$\mathcal{L}_{\rho NN}(x)=ig_v\bar{\psi}(x)\gamma_\mu\tfrac{1}{2}\boldsymbol{\tau}\cdot\psi(x)\boldsymbol{\rho}_\mu(x)+(g_t/2M)\bar{\psi}(x)\sigma_{\mu\nu}\tfrac{1}{2}\boldsymbol{\tau}\cdot\psi(x)\partial_\mu\boldsymbol{\rho}_\nu(x)\,,$$
$$\mathcal{L}_{f\pi\pi}(x)=-(g_{f\pi\pi}/M_f)\partial_\mu\boldsymbol{\pi}(x)\cdot\partial_\nu\boldsymbol{\pi}(x)f_{\mu\nu}(x)\,,$$
$$\mathcal{L}_{fNN}(x)=(g_{fNN}{}^{(1)}/2M)[\partial_\nu\bar{\psi}(x)\gamma_\mu\psi(x)-\bar{\psi}(x)\gamma_\mu\partial_\nu\psi(x)]f_{\mu\nu}(x)+(g_{fNN}{}^{(2)}/M^2)\partial_\mu\bar{\psi}(x)\partial_\nu\psi(x)f_{\mu\nu}(x)\,,$$
$$\mathcal{L}_{NN\pi}(x)=ig_{NN\pi}\bar{\psi}(x)\gamma_5\boldsymbol{\tau}\cdot\psi(x)\boldsymbol{\pi}(x)\,,$$
$$\mathcal{L}_{\Delta N\pi}(x)=(g_{\Delta N\pi}/\mu)\bar{N}_\mu(x)I_{3/2}\psi(x)\partial_\mu\pi(x)+\text{H.c.}\,,$$

where $I_{3/2}$ is the isospin projection operator of the pion-nucleon system in the $I=\frac{3}{2}$ state.