

## Renormalization Group and Strong Interactions\*

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The renormalization-group method of Gell-Mann and Low is applied to field theories of strong interactions. It is assumed that renormalization-group equations exist for strong interactions which involve one or several momentum-dependent coupling constants. The further assumption that these coupling constants approach fixed values as the momentum goes to infinity is discussed in detail. However, an alternative is suggested, namely, that these coupling constants approach a limit cycle in the limit of large momenta. Some results of this paper are: (1) The  $e^+e^-$  annihilation experiments above 1-GeV energy may distinguish a fixed point from a limit cycle or other asymptotic behavior. (2) If electrodynamics or weak interactions become strong above some large momentum  $\Lambda$ , then the renormalization group can be used (in principle) to determine the renormalized coupling constants of strong interactions, except for  $U(3) \times U(3)$  symmetry-breaking parameters. (3) Mass terms in the Lagrangian of strong, weak, and electromagnetic interactions must break a symmetry of the combined interactions with zero mass. (4) The  $\Delta I = \frac{1}{2}$  rule in nonleptonic weak interactions can be understood assuming only that a renormalization group exists for strong interactions.

### I. INTRODUCTION

AT large momenta radiative corrections in quantum electrodynamics grow logarithmically. At an energy of about  $10^{40}$  eV the radiative corrections are of order 1 instead of order  $\alpha$  ( $\alpha$  is the fine-structure constant), and at infinite energy the radiative corrections are infinite. As a result, the Born approximation to quantum electrodynamics is unreliable at energies of  $10^{40}$  eV or higher. This raises a challenge: Can one find an approximation to electrodynamics which is valid for these energies?

The academic nature of this challenge is evident. Quantum electrodynamics neglects the interactions of photons with hadrons and the weak interactions of electrons, not to mention interactions not yet discovered. Any of these interactions could appreciably alter the electron-photon interaction at high energies. In any case,  $10^{40}$  eV is an energy hopelessly beyond the range of any conceivable accelerator. Nonetheless, some notable authors have tried to meet this challenge.<sup>1,2</sup> This paper is concerned with the work of Gell-Mann and Low,<sup>1</sup> who studied in particular the behavior of the photon propagator in the limit of large  $k^2$ ,  $k$  being the photon momentum. To study the photon propagator, Gell-Mann and Low used a method which has since become known as the renormalization-group approach. The renormalization group was invented by Stueckelberg and Petermann<sup>3</sup>; its role in the Gell-Mann-Low analysis is discussed in the book of Bogoliubov and Shirkov.<sup>4</sup>

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<sup>1</sup> M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

<sup>2</sup> L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk. SSSR **95**, 773 (1954); **95**, 1177 (1954); **96**, 261 (1954).

<sup>3</sup> E. C. G. Stueckelberg and A. Petermann, Helv. Phys. Acta **26**, 499 (1953).

<sup>4</sup> N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), Chap. VIII. An important recent reference on the renormalization

Gell-Mann and Low suggest that their analysis may apply to theories of strong interactions as well as electrodynamics, and that in strong interactions their results might apply at energies more accessible than  $10^{40}$  eV. In practice, the storage-ring experiments to measure the total cross section for  $e^+e^-$  annihilation into hadrons above 1-GeV momentum transfer (and perhaps the SLAC deep-inelastic scattering experiments) explore a range of momenta relevant to the Gell-Mann-Low theory. Clearly the time has come to explore in detail the consequences of the Gell-Mann-Low theory for strong interactions.

The basic formula in the Gell-Mann-Low theory for electrodynamics is a differential equation for a quantity  $e_\lambda$ . Let the renormalized photon propagator be written  $k^{-2}d_e(k^2/m^2, e^2)$ , where  $k$  is the photon four-momentum,  $e$  is the renormalized electron charge, and  $m$  is the renormalized electron mass. Then  $e_\lambda$  is defined by

$$e_\lambda^2 = e^2 d_e(-\lambda^2/m^2, e^2). \quad (1.1)$$

Gell-Mann and Low set up a generalization of the usual renormalization procedure in which  $e_\lambda$  is defined to be the renormalized coupling constant, for some arbitrary chosen value of  $\lambda$ , in place of  $e$ . They also argue that  $e_\lambda$ , considered as a function of  $\lambda$ , interpolates between the physical charge  $e$  and the bare charge, namely,  $e_\lambda$  for  $\lambda \rightarrow 0$  is  $e$  and  $e_\lambda$  for  $\lambda \rightarrow \infty$  is the bare charge. The bare charge will be denoted  $e_\infty$  in this paper. The Gell-Mann-Low formula is of the form

$$\frac{de_\lambda^2}{d(\ln \lambda^2)} = \psi(m^2/\lambda^2, e_\lambda^2). \quad (1.2)$$

Gell-Mann and Low suggest that  $\psi(m^2/\lambda^2, e_\lambda^2)$  has a nonzero limit as  $m \rightarrow 0$ , i.e.,  $\psi(0, e_\lambda^2)$  exists and is not identically zero. If this is true, then for  $\lambda \gg m$ ,  $e_\lambda$  satisfies

group is M. Astaud and B. Jovet, Nuovo Cimento **63A**, 5 (1969); **66A**, 11 (1970).

approximately

$$\frac{de_\lambda^2}{d(\ln\lambda^2)} = \psi(0, e_\lambda^2). \quad (1.3)$$

Gell-Mann and Low assume that this equation holds for any value of  $e_\lambda^2$ , although it can be justified, at best, only for small  $e_\lambda^2$ ; they then discuss qualitative features of the large momentum behavior of  $e_\lambda$  based on qualitative features of  $\psi$ . In particular, they show that the limit  $e_\infty$  of  $e_\lambda$  must either be infinite or, if finite, must be a root of the equation  $\psi(0, e_\infty^2) = 0$ . In either case,  $e_\infty$  is independent of the value of the physical charge  $e$ .

The purpose of this paper is to propose that the Gell-Mann-Low theory when applied to theories of strong interactions can produce some startling consequences. For example, it will be shown that if strong interactions are described by one of the popular renormalizable models (e.g., the gluon model<sup>5</sup> or the  $\sigma$  model<sup>6</sup>) then one can use the renormalization-group equations [analogous to Eq. (1.2)] of the model to determine one or more of the *renormalized* coupling constants of the model. In order to derive this result, a physical assumption is made, namely, that strong interactions will cease to be isolated from weak or electromagnetic interactions at some cutoff momentum  $\Lambda$  much larger than 1 GeV/c, and that any model theory of strong interactions is valid only below the cutoff. There will be small cutoff-dependent errors in the prediction of the renormalized coupling constants.

This is a consequence of the Gell-Mann-Low theory of theoretical interest. An experimental consequence which is a possible but far from certain prediction of the Gell-Mann-Low theory is that the cross section  $\sigma_{\text{tot}}(q^2)$  for  $e^+e^-$  annihilation into hadrons at large momentum transfers  $q$  will oscillate as a function of  $q^2$ . To be precise the oscillations would have the form

$$\sigma_{\text{tot}}(q^2) = (q^2)^{-1} f(\ln q^2), \quad (1.4)$$

where  $f(t)$  is a periodic function of  $t$  with period  $t_0$ . The period  $t_0$  and the amplitude of the oscillations cannot be predicted. This behavior is only one of the alternatives made possible by the Gell-Mann-Low theory, and it is possible only for theories with at least two renormalized coupling constants. For the gluon model, which has only one renormalized coupling constant, the prediction for  $\sigma_{\text{tot}}(q^2)$  at large  $q^2$  is that  $q^2\sigma_{\text{tot}}(q^2)$  be a constant. If  $q^2\sigma_{\text{tot}}(q^2)$  is constant at large  $q^2$  it is likely (according to the Gell-Mann-Low theory) that strong interactions are scale invariant at short distances. The hypothesis of broken scale invariance has been extensively discussed elsewhere.<sup>7</sup>

<sup>5</sup> The gluon model is a quark model with quarks coupled to a neutral vector meson.

<sup>6</sup> M. Lévy, *Nuovo Cimento* 52A, 23 (1967); M. Gell-Mann and M. Lévy, *ibid.* 16, 705 (1960).

<sup>7</sup> G. Mack, *Nucl. Phys.* B5, 499 (1968); K. Wilson, *Phys. Rev.* 179, 1499 (1969); S. Ciciarelli, R. Gatto, G. Sartori, and M. Tonin, *Phys. Letters* 30B, 546, (1969); K. Wilson, SLAC Report No. SLAC-PUB-737, 1970 (unpublished); G. Mack, *Phys. Rev.*

The renormalization-group differential equations such as Eq. (1.3) are best understood by setting up an analogy with equations of motion in classical mechanics or electric circuit theory. Let  $t = \ln\lambda^2$  and  $x = e_\lambda^2$ ; then Eq. (1.3) becomes

$$dx/dt = \psi(0, x). \quad (1.5)$$

This is a simple equation of motion with  $\psi$  being the analog of a time-independent force (except the equation involves  $dx/dt$ , not  $d^2x/dt^2$ ). If the field theory has two renormalized coupling constants, say,  $x$  and  $y$ , the corresponding renormalization-group equations have the form [neglecting masses, as in Eq. (1.3)]

$$dx/dt = \psi_1(x, y), \quad (1.6)$$

$$dy/dt = \psi_2(x, y), \quad (1.7)$$

i.e., they are two coupled equations with time-independent forces.

The crucial feature of the renormalization-group equations is that they must be solved over a large or infinite range of  $t$ , namely, from  $\lambda$  of order  $m$  to  $\lambda = \Lambda$  or  $\infty$ , leading to the range  $\ln m^2 < t < (\ln \Lambda^2 \text{ or } \infty)$  for  $t$ . Furthermore, the equations are nonlinear [in perturbation theory  $\psi(0, x)$ , or  $\psi_1(x, y)$  and  $\psi_2(x, y)$ , have power-series expansions in  $x$  and  $y$  and are not linear in  $x$  and  $y$ ]. The essential question in solving the renormalization-group equations is to determine the behavior of the solution in the limit of large  $t$ . This is analogous to the central problem of nonlinear mechanics.<sup>8</sup> Nonlinear mechanics is concerned with finding equilibrium points or other asymptotic solutions of equations like Eqs. (1.5) or (1.6) and (1.7) and studying the stability of these solutions. The predictions for strong interactions cited earlier arise from applying the theory of asymptotic solutions and their stability to the renormalization-group equations.

The most serious drawback of the Gell-Mann-Low theory is that it is highly speculative; it requires extrapolation of functions like  $\psi(0, x)$  from perturbation theory (small  $x$ ) to strongly interactions ( $x \sim 1$ ) and there is no way to check the validity, even qualitatively, of this extrapolation. Furthermore, as one studies the consequences of the theory for strong interactions it becomes clear that there is a glaring omission in the Gell-Mann-Low equations. The omission is the omission of coupling constants associated with nonrenormalizable interactions. In perturbation theory there are well-known reasons for distinguishing renormalizable interactions from nonrenormalizable interactions, but in strong coupling this distinction becomes blurred, as will be shown.

It is beyond the scope of this paper to discuss how to

*Letters* 25, 400 (1970); M. Gell-Mann, Caltech Report, 1970 (unpublished); and references contained in all the above.

<sup>8</sup> An example of a book on nonlinear mechanics is N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, Princeton, N. J., 1962). For limit cycles, see Chap. 3. The theorem cited in Sec. III H of the present paper occurs on p. 84 of Minorsky's book.

incorporate nonrenormalizable interactions into the renormalization group, but one can see that however this is done the resulting differential equations will be very complicated. This is because there are an infinite number of nonrenormalizable interactions. Hence it is important to understand the practical importance of the renormalization group for strong interactions, and to understand the reasons why one must add nonrenormalizable interactions to the renormalization group. For these reasons it is worth discussing the renormalization group without nonrenormalizable interactions even if nonrenormalizable interactions are necessary for a correct treatment. The emphasis in this paper will be on qualitative features of solutions of the renormalization group equations resulting from the existence of equilibrium or other asymptotic solutions; this analysis is generalizable to any number of coupling constants.

Anyone who has studied the Gell-Mann-Low theory, either in the original paper of Gell-Mann and Low<sup>1</sup> or in the review of Bogoliubov and Shirkov,<sup>4</sup> has found it extraordinarily difficult to understand. Accordingly, Sec. II and the Appendix of this paper give a thorough review of the Gell-Mann-Low theory for electrodynamics. However, there will be no attempt here to show that the limit of  $\psi(m^2/\lambda^2, e_\lambda^2)$  for  $m \rightarrow 0$  exists beyond fourth order in  $e_\lambda$ ; there will only be a brief review of work done on this question.

In Sec. III solutions of the renormalization-group equations will be discussed in detail, including applications both to strong and electromagnetic interactions. It will be shown that if the asymptotic solution is a fixed point, then the corresponding field theory exhibits broken scale invariance. In Sec. IV the possible mechanisms for breaking scale invariance will be discussed, and the renormalization group will be extended to include a  $\lambda$ -dependent mass  $m_\lambda$ . In Sec. V the renormalization group for strong interactions will be discussed based on the work of Secs. III and IV. It will be argued that mass terms in strong, electromagnetic, and weak interactions must break a symmetry common to all three interactions. In Sec. VI it will be argued that the  $\Delta I = \frac{1}{2}$  rule in weak interactions can be understood if strong interactions have a renormalization group, independently of the type of asymptotic solution of the renormalization-group equations. Section VII contains final remarks.

## II. DERIVATION OF RENORMALIZATION-GROUP EQUATIONS

However one defines a renormalization program for quantum electrodynamics to all orders in perturbation theory, there is one stage in the program where one makes infinite or cutoff-dependent subtractions in the  $n$ th-order vacuum polarization, electron self-energy, and vertex function. In addition to the cutoff-dependent subtractions, one makes finite subtractions which can be chosen arbitrarily. This arbitrariness is customarily

removed by specifying *ad hoc* normalizations for the renormalized fields and by specifying that the renormalized mass and charge parameters are the physical mass and charge of the electron. Gell-Mann and Low define alternative conditions for removing the arbitrariness: Gell-Mann and Low specify unconventional normalization conditions for the fields and define a charge parameter  $e_\lambda$  which is not the physical electron charge. However, they use the conventional definition for the renormalized mass. The Gell-Mann-Low conditions involve a "renormalization parameter"  $\lambda$  which one can choose arbitrarily. The renormalized fields of Gell-Mann and Low for any given value of  $\lambda$  are related to the conventional fields through a finite renormalization. The reason for considering the unconventional renormalization conditions of Gell-Mann and Low is that they apparently define fields which are finite off the mass shell when the electron mass is zero.

The renormalization-group equation results from comparing renormalized theories for two different values of the renormalization momentum  $\lambda$ ; if these theories both exist for zero electron mass, then Eq. (1.2) derived from these theories also exists for zero electron mass.

In this section a modified form of the Gell-Mann-Low renormalization conditions will be defined; then the Gell-Mann-Low renormalized theory will be expressed in terms of the conventionally renormalized theory, and finally the renormalization-group differential equation, Eq. (1.2), will be derived. In the Appendix the motivation for setting up the Gell-Mann-Low renormalization conditions will be reviewed in detail with illustrations from low orders of perturbation theory.

The renormalization conditions of Gell-Mann and Low can be applied to any method of renormalization. This means that one does not have to use or understand Ward's renormalization program<sup>9</sup> used in Gell-Mann and Low's paper. One can have the Bogoliubov-Parasiuk-Hepp<sup>10</sup> or any other method in mind in reading this section.

The Gell-Mann-Low conditions consist of one restriction on vacuum polarization, two on the electron self-energy, and one on the vertex function. There is considerable arbitrariness in how these conditions are formulated but it seems to be inevitable that they look awkward due to spin complications. The conditions proposed here will not be the ones given by Gell-Mann and Low; Bogoliubov and Shirkov evade the problem of stating precise conditions for the electron self-energy. For comparison purposes the restrictions used in conventional renormalization theory will be stated also.

It is convenient to use the following notation. Let

<sup>9</sup> See Ref. 1 for this program.

<sup>10</sup> N. N. Bogoliubov and D. V. Shirkov, *Fortschr. Physik* **4**, 438 (1956); N. N. Bogoliubov and O. Parasiuk, *Acta Math.* **97**, 227 (1957); K. Hepp, *Commun. Math. Phys.* **2**, 301 (1966). See also Ref. 4.

$\Pi_{c\mu\nu}(k)$ ,  $\Sigma_c(p)$ , and  $\Gamma_{c\mu}(p, q, k)$  be the vacuum polarization, electron self-energy, and vertex function for conventionally renormalized electrodynamics. In the vertex function  $p$  and  $q$  are electron momenta and  $k = q - p$  is the photon momentum. Let  $\Pi_{\lambda\mu\nu}(k)$ ,  $\Sigma_\lambda(p)$ , and  $\Gamma_{\lambda\mu}(p, q, k)$  be the corresponding renormalized functions satisfying the Gell-Mann-Low restrictions. It is convenient to define invariant functions as follows:

$$\Pi_{c\mu\nu}(k) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi_c(k^2), \quad (2.1)$$

$$\Pi_{\lambda\mu\nu}(k) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi_\lambda(k^2), \quad (2.2)$$

$$\Sigma_c(p) = \not{p}A_c(p^2) + B_c(p^2), \quad (2.3)$$

$$\Sigma_\lambda(p) = \not{p}A_\lambda(p^2) + B_\lambda(p^2), \quad (2.4)$$

$$\Gamma_{c\mu}(p, p, 0) = \gamma_\mu \Gamma_{c1}(p^2) + \not{p}_\mu \Gamma_{c2}(p^2) + \gamma_\mu \not{p} \Gamma_{c3}(p^2) + \not{p}_\mu \not{p} \Gamma_{c4}(p^2), \quad (2.5)$$

$$\Gamma_{\lambda\mu}(p, p, 0) = \gamma_\mu \Gamma_{\lambda1}(p^2) + \not{p}_\mu \Gamma_{\lambda2}(p^2) + \gamma_\mu \not{p} \Gamma_{\lambda3}(p^2) + \not{p}_\mu \not{p} \Gamma_{\lambda4}(p^2), \quad (2.6)$$

where  $\not{p}$  is  $\gamma^\mu p_\mu$ . One looks at the vertex function for  $k=0$  because it is connected to the electron self-energy by Ward's identity:

$$\begin{aligned} \Gamma_{c\mu}(p, p, 0) &= \gamma_\mu - \frac{\partial \Sigma_c(p)}{\partial p^\mu} \\ &= \gamma_\mu - \gamma_\mu A_c(p^2) \\ &\quad - 2\not{p} \not{p}_\mu \frac{\partial A_c(p^2)}{\partial p^2} - 2\not{p}_\mu \frac{\partial B_c(p^2)}{\partial p^2}, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \Gamma_{\lambda\mu}(p, p, 0) &= \gamma_\mu - \gamma_\mu A_\lambda(p^2) \\ &\quad - 2\not{p} \not{p}_\mu \frac{\partial A_\lambda(p^2)}{\partial p^2} - \not{p}_\mu \frac{\partial B_\lambda(p^2)}{\partial p^2}. \end{aligned} \quad (2.8)$$

These identities give

$$\Gamma_{c1}(p^2) = 1 - A_c(p^2), \quad (2.9)$$

$$\Gamma_{c2}(p^2) = -2 \frac{\partial B_c(p^2)}{\partial p^2}, \quad (2.10)$$

$$\Gamma_{c3}(p^2) = 0, \quad (2.11)$$

$$\Gamma_{c4}(p^2) = -2 \frac{\partial A_c(p^2)}{\partial p^2}, \quad (2.12)$$

and analogously

$$\Gamma_{\lambda1}(p^2) = 1 - A_\lambda(p^2), \quad \text{etc.} \quad (2.13)$$

There are four subtraction constants to be fixed, namely, one constant independent of momentum in each of the following functions:  $\Pi_c(k^2)$ ,  $A_c(p^2)$ ,  $B_c(p^2)$ , and  $\Gamma_{c1}(p^2)$ . These subtraction constants are con-

ventionally determined by the following conditions:

$$\Pi_c(0) = 0, \quad (2.14)$$

$$\Sigma_c(p) |_{\not{p}=m} = mA_c(m^2) + B_c(m^2) = 0, \quad (2.15)$$

$$\left[ \frac{\partial \Sigma_c(p)}{\partial \not{p}} \right]_{\not{p}=m} = A_c(m^2) + \left[ 2m^2 \frac{\partial A_c(p^2)}{\partial p^2} + 2m \frac{\partial B_c(p^2)}{\partial p^2} \right]_{p^2=m^2} = 0, \quad (2.16)$$

and

$$\Gamma_{c1}(m^2) + m\Gamma_{c2}(m^2) + m^2\Gamma_{c4}(m^2) = 1. \quad (2.17)$$

The conditions on the electron propagator are chosen so that neither the mass nor the residue of the pole of the electron propagator at  $\not{p}=m$  is changed by interaction. The subtraction constant in  $A_c(p^2)$  is determined by Eq. (2.16) [note that the derivatives  $\partial A_c(p^2)/\partial p^2$  and  $\partial B_c(p^2)/\partial p^2$  do not involve the subtraction constant]; the subtraction constant in  $B_c(m^2)$  is then fixed by Eq. (2.15). The peculiar equation for  $\Gamma_{c1}(m^2)$  is dictated by the requirement that  $\Gamma_{c1}(m^2)$  satisfy the Ward identity. Due to Eqs. (2.9)–(2.12), the condition for  $\Gamma_{c1}(m^2)$  is a consequence of Eq. (2.16). (As part of any renormalization program one must prove that all Ward identities are satisfied if the subtraction constant in the vertex function is chosen so that the vertex function satisfies one Ward identity at one value of the momentum.)

The alternative conditions of Gell-Mann and Low (somewhat modified by the author) are as follows:

$$\Pi_\lambda(-\lambda^2) = 0, \quad (2.18)$$

$$\Sigma_\lambda(p) |_{\not{p}=m} = mA_\lambda(m^2) + B_\lambda(m^2) = 0, \quad (2.19)$$

$$A_\lambda(-\lambda^2) = 0, \quad (2.20)$$

$$\Gamma_{\lambda1}(-\lambda^2) = 1. \quad (2.21)$$

The first restriction on  $\Sigma_\lambda(p)$  ensures that the mass  $m$  in the free-electron propagator is the physical electron mass. This restriction determines the subtraction constant in  $B_\lambda(p^2)$ . The third equation [Eq. (2.20)] (which was one of many possible choices) fixes the subtraction constant in  $A_\lambda(p^2)$ . The last equation fixes the subtraction constant in  $\Gamma_{\lambda1}(p^2)$  and is consistent with the Ward identity [Eq. (2.13)] for  $\Gamma_{\lambda1}(p^2)$ .

These conditions are chosen so as not to introduce singularities in the renormalized functions when the renormalized electron mass is zero. This is explained in detail in the Appendix for low orders of perturbation theory. The Gell-Mann-Low renormalized amplitudes are also free of infrared divergences when the photon mass is zero and the electron mass is nonzero. This is also explained in the Appendix.

Will the Gell-Mann-Low subtraction conditions combined with a renormalization program to all orders give renormalized amplitudes which are finite for

$m \rightarrow 0$  in all orders? This is a hard question and cannot be pursued here. Baker and Johnson<sup>11</sup> have an indirect proof that the photon propagator as renormalized by Gell-Mann and Low is finite for  $m \rightarrow 0$ ; the author has not checked their proof. Kinoshita<sup>12</sup> did an extensive study of mass singularities in field theory but did not study the effects of renormalizing according to the Gell-Mann–Low specifications. A full proof of the existence of the zero-mass limit does not exist to the author's knowledge. If the zero-mass limit does not exist in higher orders, the conclusions of this paper may be incorrect.

Given that electrodynamics renormalized according to the Gell-Mann–Low specifications differs only by a renormalization from conventionally renormalized electrodynamics,<sup>13</sup> it is possible to express all the amplitudes of the Gell-Mann–Low theory in terms of the conventional theory. The formulas connecting the two theories will now be obtained. Let the conventionally renormalized electromagnetic potential be  $A_{e\mu}(x)$  and the conventionally renormalized electron field be  $\psi_e(x)$ . Let the corresponding fields of the Gell-Mann–Low theory be  $A_{\lambda\mu}(x)$  and  $\psi_\lambda(x)$ . Then one requires that

$$A_{\lambda\mu}(x) = (z_{3\lambda})^{1/2} A_{e\mu}(x), \quad (2.22)$$

$$\psi_\lambda(x) = (z_{2\lambda})^{1/2} \psi_e(x). \quad (2.23)$$

The Gell-Mann–Low theory is parametrized differently from the conventional theory; the coupling constant in the Gell-Mann–Low theory is denoted  $e_\lambda$ . The mass parametrization is in terms of the physical electron mass  $m$  in both theories. The restrictions (2.18)–(2.21) imposed on the Gell-Mann–Low theory are sufficient to determine  $z_{2\lambda}$ ,  $z_{3\lambda}$ , and  $e_\lambda$  in terms of conventionally renormalized amplitudes; when these are known all amplitudes of the Gell-Mann–Low theory are determined through Eqs. (2.22) and (2.23).

It is convenient to rewrite two of the Gell-Mann–Low restrictions in terms of the complete propagators. Let  $S_\lambda(p)$  be the electron propagator. One has (in the Feynman gauge for the exact propagator<sup>14</sup>)

$$D_{\lambda^{-1}\mu\nu}(k) = -g_{\mu\nu}k^2 - g_{\mu\nu}k^2 \Pi_\lambda(k^2), \quad (2.24)$$

$$S_\lambda^{-1}(p) = \not{p} - m - \not{p} A_\lambda(p^2) - B_\lambda(p^2). \quad (2.25)$$

It is convenient to introduce invariant functions related to  $D_{\lambda\mu\nu}(k)$  and  $S_\lambda(p)$ , and indicate explicitly their dependence on  $m$ ,  $\lambda$ , and  $e_\lambda$  as well as momentum:

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = [1 + \Pi_\lambda(k^2)]^{-1}, \quad (2.26)$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = [1 - A_\lambda(p^2)]^{-1}. \quad (2.27)$$

Then

$$D_{\lambda\mu\nu}(k) = -g_{\mu\nu}(k^2)^{-1} d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2). \quad (2.28)$$

There is a second invariant function for the electron propagator which will be defined later (Sec. IV). In the zero-mass limit one has  $\gamma_5$  invariance, which means  $B_\lambda(p^2)$  is zero. In this limit, one has

$$S_\lambda(p) = (\not{p})^{-1} s(p^2/\lambda^2, 0, e_\lambda^2). \quad (2.29)$$

The functions  $d$  and  $s$  are dimensionless,<sup>15</sup> which is why they are functions only of ratios of the variables  $p^2$ ,  $m^2$ , and  $\lambda^2$ . The Gell-Mann–Low restrictions (2.18) and (2.20) become

$$d(-1, m^2/\lambda^2, e_\lambda^2) = 1, \quad (2.30)$$

$$s(-1, m^2/\lambda^2, e_\lambda^2) = 1. \quad (2.31)$$

There are corresponding functions  $d_e(k^2/m^2, e^2)$  and  $s_e(p^2/m^2, e^2)$  for the conventionally renormalized theory;  $d_e$  satisfies

$$d_e(0, e^2) = 1. \quad (2.32)$$

The conditions on  $\Sigma_e(p)$  cannot be expressed in terms of  $s_e$  alone.

From relations (2.22) and (2.23), one must have

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D_{e\mu\nu}(k), \quad (2.33)$$

$$S_\lambda(p) = z_{2\lambda} S_e(p), \quad (2.34)$$

which in turn means that

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{3\lambda} d_e(k^2/m^2, e^2), \quad (2.35)$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{2\lambda} s_e(p^2/m^2, e^2). \quad (2.36)$$

Putting  $k^2$  and  $p^2$  equal to  $-\lambda^2$  and using the conditions (2.30) and (2.31), one gets

$$(z_{3\lambda})^{-1} = d_e(-\lambda^2/m^2, e^2), \quad (2.37)$$

$$(z_{2\lambda})^{-1} = s_e(-\lambda^2/m^2, e^2). \quad (2.38)$$

In order to determine  $e_\lambda$  in terms of  $e$ , one must know how  $\Gamma_{\lambda\mu}(p, q, k)$  and  $\Gamma_{e\mu}(p, q, k)$  are related to vacuum expectation values of the fields. The formulas are as follows:

$$\Gamma_{\lambda\mu}(p, q, k) = (e_\lambda)^{-1} S_\lambda^{-1}(p) D_{\lambda^{-1}\mu\nu}(k) F_{\lambda\nu}(p, q, k) S_\lambda^{-1}(q), \quad (2.39)$$

$$F_{\lambda\nu}(p, q, k)$$

$$= - \int \int e^{ip \cdot x} e^{-iq \cdot y} \langle \Omega | T \psi_\lambda(x) \bar{\psi}_\lambda(y) A_{\lambda\nu}(0) | \Omega \rangle \times d^4x d^4y. \quad (2.40)$$

The factor  $e_\lambda^{-1}$  occurs in the definition of  $\Gamma_{\lambda\mu}(p, q, k)$  along with the inverse propagators to ensure that  $\Gamma_{\lambda\mu}(p, q, k)$  in lowest order is  $\gamma_\mu$  (not  $e_\lambda \gamma_\mu$ ); using the inverse of the exact propagators in Eq. (2.39) is necessary to remove self-energy insertions from external lines of vertex graphs.

<sup>15</sup> For a discussion of dimensional analysis in perturbation theory, see Sec. III C.

<sup>11</sup> M. Baker and K. Johnson, Phys. Rev. **183**, 1292 (1969).

<sup>12</sup> T. Kinoshita, J. Math. Phys. **3**, 650 (1962).

<sup>13</sup> This should be part of any proof of renormalizability of electrodynamics; see, e.g., Ref. 4, Sec. 31.2.

<sup>14</sup> The choice of gauge in the Gell-Mann–Low theory is explained in the Appendix.

Analogous formulas with  $e$  replacing  $e_\lambda$  and  $S_e^{-1}$ ,  $D_e^{-1}$ , etc., replacing  $S_\lambda^{-1}$ ,  $D_\lambda^{-1}$ , etc., give the vertex  $\Gamma_{c\mu}(p, q, k)$ . It follows from these formulas and Eqs. (2.22), (2.23), (2.33), and (2.34), that

$$e_\lambda \Gamma_{\lambda\mu}(p, q, k) = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} e \Gamma_{c\mu}(p, q, k). \quad (2.41)$$

A particular consequence of this equation is that

$$e_\lambda \Gamma_{\lambda 1}(p^2) = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} e \Gamma_{c1}(p^2). \quad (2.42)$$

Using the Ward identities (2.9) and (2.13) and the definitions of  $s$  and  $s_e$ , this becomes

$$e_\lambda [s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)]^{-1} = (z_{2\lambda})^{-1} (z_{3\lambda})^{-1/2} \times e [s_e(p^2/m^2, e^2)]^{-1}. \quad (2.43)$$

Comparing with Eq. (2.36), one must have

$$e_\lambda = (z_{3\lambda})^{-1/2} e. \quad (2.44)$$

Using Eq. (2.37), this is

$$e_\lambda^2 = e^2 d_e(-\lambda^2/m^2, e^2). \quad (2.45)$$

[This is Eq. (1.1) of the Introduction.]

One can now completely reconstruct the Gell-Mann-Low form of renormalized quantum electrodynamics if the conventional renormalized form is known. To construct  $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$ , for example, one uses Eqs. (2.35) and (2.37) to give

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = d_e(k^2/m^2, e^2)/d_e(-\lambda^2/m^2, e^2). \quad (2.46)$$

To reparametrize the ratio  $d_e(k^2/m^2, e^2)/d_e(-\lambda^2/m^2, e^2)$  in terms of  $e_\lambda^2$ , one must first solve Eq. (2.45) to give  $e^2$  as a function  $e_\lambda^2$ . In perturbation theory this is a tedious but straightforward process and gives  $e^2$  as a power series in  $e_\lambda^2$ .

To derive the renormalization-group equations, one looks at the formulas connecting the Gell-Mann-Low renormalized fields for two different values of  $\lambda$ , say,  $\lambda$  and  $\lambda'$ . The two sets of fields are connected by renormalization constants:

$$A_{\lambda\mu}(x) = z_{3\lambda\lambda'} A_{\lambda'\mu}(x), \quad (2.47)$$

$$\psi_\lambda(x) = z_{2\lambda\lambda'} \psi_{\lambda'}(x), \quad (2.48)$$

where

$$z_{3\lambda\lambda'} = z_{3\lambda}/z_{3\lambda'}, \quad (2.49)$$

$$z_{2\lambda\lambda'} = z_{2\lambda}/z_{2\lambda'}. \quad (2.50)$$

In consequence, one has

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{3\lambda\lambda'} d(k^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2), \quad (2.51)$$

$$s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = z_{2\lambda\lambda'} s(p^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2). \quad (2.52)$$

Also, from Eq. (2.44) one has

$$e_\lambda^2 = (z_{3\lambda\lambda'})^{-1} e_{\lambda'}^2. \quad (2.53)$$

It follows from putting  $p^2 = k^2 = -\lambda^2$  that

$$(z_{3\lambda\lambda'})^{-1} = d(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2), \quad (2.54)$$

$$(z_{2\lambda\lambda'})^{-1} = s(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2), \quad (2.55)$$

$$e_\lambda^2 = e_{\lambda'}^2 d(-\lambda^2/\lambda'^2, m^2/\lambda'^2, e_{\lambda'}^2). \quad (2.56)$$

The advantage of these equations is that they remain finite when  $m \rightarrow 0$ . It should also be noted that the function  $s(p^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$  does not have infrared divergences for finite mass  $m$  while the function  $s_e(p^2/m^2, e^2)$  does. This means  $z_{2\lambda}$  is infrared divergent but  $z_{2\lambda\lambda'}$  is not.

In the limit  $m \rightarrow 0$ , Eq. (2.56) becomes (with  $\lambda$  and  $\lambda'$  interchanged, for no good reason)

$$e_{\lambda'}^2 = e_\lambda^2 d(-\lambda'^2/\lambda^2, 0, e_\lambda^2). \quad (2.57)$$

This is an equation which is true for any  $\lambda'$  and  $\lambda$  provided both are nonzero. As a result it defines a rather simple transformation group on a single variable  $x$ . Think of  $x = e_\lambda^2$  as a point on the positive real line; let  $T_s$  be the transformation which takes  $x$  into the point  $x' = e_{\lambda'}^2$ , i.e.,

$$x' = x d(-s, 0, x), \quad (2.58)$$

with  $s = \lambda'^2/\lambda^2$ . So  $T_s$  is a transformation on a space of coupling constants. The transformations  $T_s$  have the group property: If

$$s_2 = s_1 s, \quad (2.59)$$

then

$$T_{s_2} = T_{s_1} T_s. \quad (2.60)$$

To prove this one first defines

$$(\lambda'')^2 = s_2(\lambda^2), \quad \text{i.e., } s_2 = \lambda''^2/\lambda^2. \quad (2.61)$$

Then

$$s_1 = s_2/s = (\lambda''/\lambda)^2. \quad (2.62)$$

Consider any point  $x > 0$ . Choose  $e_\lambda^2 = x$ . Since Eq. (2.57) holds for any  $\lambda'$ , one has in addition

$$e_{\lambda'}^2 = e_\lambda^2 d(-\lambda'^2/\lambda^2, 0, e_\lambda^2). \quad (2.63)$$

Since  $\lambda$  is arbitrary one also has

$$e_{\lambda''}^2 = e_{\lambda'}^2 d(-\lambda''^2/\lambda'^2, 0, e_{\lambda'}^2). \quad (2.64)$$

Let  $x'$  be  $e_{\lambda'}^2$  and  $x''$  be  $e_{\lambda''}^2$ . Then  $T_s$  takes  $x$  to  $x'$  and  $T_{s_1}$  takes  $x'$  to  $x''$ ; but  $T_{s_2}$  [by Eq. (2.63)] takes  $x$  directly to  $x''$ . This proves the group property. In other words, if one is given the coupling constant  $e_\lambda^2$  for subtraction momentum  $\lambda$ , the corresponding coupling constant  $e_{\lambda'}^2$  for subtraction momentum  $\lambda'$  is given by Eq. (2.57); the coupling constant  $e_{\lambda''}^2$  for subtraction momentum  $\lambda''$  is given by Eq. (2.63). By "corresponding" is meant that  $e_\lambda^2$ ,  $e_{\lambda'}^2$ , and  $e_{\lambda''}^2$  all lead to the same physics but in different parametrizations. Now since  $e_{\lambda'}^2$  and  $e_{\lambda''}^2$  lead to the same physics they too must be related, namely, by Eq. (2.64); this is what gives the group property. The group has an identity transformation, namely, the transformation  $T_1$ ; this is the identity due to the normalization condition (2.30). The transformation inverse to  $T_s$  is  $T_{s^{-1}}$ .

The group of transformations  $T_s$  will be called here the renormalization group. This is a less ambitious definition than is given by Bogoliubov and Shirkov<sup>4</sup> or Petermann and Stueckelberg<sup>3</sup>; however, it is easier to work with the transformations  $T_s$  than to consider the

more complex transformations which make up the Petermann-Stueckelberg renormalization group.

The transformations  $T_s$  are nonlinear, because  $xd(-s, 0, x)$  is nonlinear in  $x$ . So the renormalization group is different from the symmetry groups of quantum mechanics, which are groups of linear transformations. Instead the renormalization group is analogous to the group of time translations in classical mechanics. In classical mechanics with time-independent potentials the equations of motion define infinitesimal transformations on phase space; the corresponding finite transformations are also nonlinear and also form a one-parameter group. The analogy to classical mechanics will become important in Sec. III.

In practice one works mostly with the infinitesimal transformation of the renormalization group. The infinitesimal transformation determines the derivative  $de_{\lambda^2}/d\lambda^2$ . It is convenient to derive an equation for  $de_{\lambda^2}/d\lambda^2$  which is valid for finite mass. This is obtained by differentiating Eq. (2.56) with respect to  $\lambda$  and then setting  $\lambda' = \lambda$ . The result is of the form

$$\frac{d(e_{\lambda^2})}{d(\lambda^2)} = \lambda^{-2} \psi\left(\frac{m^2}{\lambda^2}, e_{\lambda^2}\right), \quad (2.65)$$

where

$$\psi\left(\frac{m^2}{\lambda^2}, e_{\lambda^2}\right) = e_{\lambda^2} \left[ \frac{\partial d(-s, m^2/\lambda^2, e_{\lambda^2})}{\partial s} \right]_{s=1}. \quad (2.66)$$

The function  $\psi$  has a limit for  $m \rightarrow 0$  because  $d$  does, and the function  $\psi(0, x)$  gives the infinitesimal transformation of the renormalization group. In practice one can replace  $\psi(m^2/\lambda^2, e_{\lambda^2})$  by  $\psi(0, e_{\lambda^2})$  when  $\lambda^2 \gg m^2$ ; hence solutions of the equation

$$d(e_{\lambda^2})/d(\lambda^2) = \lambda^{-2} \psi(0, e_{\lambda^2}) \quad (2.67)$$

give the asymptotic behavior of  $e_{\lambda}$  for  $\lambda \gg m$ . The function  $\psi(0, x)$  is explicitly known to order  $x^3$  when  $x$  is small<sup>4</sup>:

$$\psi(0, x) = (12\pi^2)^{-1} [x^2 + (3/16\pi^2)x^3 + \dots]. \quad (2.68)$$

One can also derive differential equations for  $z_{2\lambda}$  and  $z_{3\lambda}$ . However,  $z_{3\lambda}$  is related to  $e_{\lambda^2}$  by Eq. (2.44), so we only give the equation for  $z_{2\lambda}$ . First differentiate Eq. (2.55) with respect to  $\lambda^2$  and then put  $\lambda'^2 = \lambda^2$  (and note that  $z_{2\lambda\lambda} = 1$ ):

$$\left. \frac{\partial z_{2\lambda\lambda'}}{\partial \lambda^2} \right|_{\lambda'=\lambda} = \lambda^{-2} \sigma\left(\frac{m^2}{\lambda^2}, e_{\lambda^2}\right), \quad (2.69)$$

where

$$\sigma\left(\frac{m^2}{\lambda^2}, e_{\lambda^2}\right) = \left[ \frac{\partial s(y, m^2/\lambda^2, e_{\lambda^2})}{\partial y} \right]_{y=-1}. \quad (2.70)$$

Since  $z_{2\lambda} = (z_{2\lambda\lambda'}) (z_{2\lambda'})$ , one has

$$\frac{dz_{2\lambda}}{d(\lambda^2)} = \left( \frac{\partial z_{2\lambda\lambda'}}{\partial \lambda^2} \right) z_{2\lambda'}. \quad (2.71)$$

Putting  $\lambda'^2 = \lambda^2$  gives

$$\frac{dz_{2\lambda}}{d\lambda^2} = (\lambda^2)^{-1} z_{2\lambda} \sigma\left(\frac{m^2}{\lambda^2}, e_{\lambda^2}\right). \quad (2.72)$$

The function  $\sigma$  has a zero-mass limit because the function  $s$  does, so when  $\lambda \gg m$  one has approximately

$$dz_{2\lambda}/d\lambda^2 = (\lambda^2)^{-1} z_{2\lambda} \sigma(0, e_{\lambda^2}). \quad (2.73)$$

Bogoliubov and Shirkov<sup>4</sup> also write down differential equations for the whole vertex function; they will not be needed here.

### III. SOLUTIONS OF RENORMALIZATION-GROUP EQUATIONS

#### A. Introduction

In Sec. II the basic equation of the Gell-Mann-Low theory for electrodynamics was derived:

$$d(e_{\lambda^2})/d(\ln \lambda^2) = \psi(0, e_{\lambda^2}), \quad (3.1)$$

valid for  $\lambda \gg m$ .

Imagine now that equations similar to this hold for other field theories besides electrodynamics, in particular, for possible field theories of strong interactions (e.g., the gluon model or the  $\sigma$  model).<sup>16</sup> Imagine also that the renormalization-group equation is valid for large values of the coupling constant (e.g., large values of  $e_{\lambda^2}$  in the case of electrodynamics).

To do a general analysis of all possible renormalization-group equations would be a hopeless task. One has no information on the behavior of  $\psi(0, e_{\lambda^2})$  when  $e_{\lambda^2}$  is of order one or larger, and in some theories of strong interactions the renormalization-group equations involve coupled differential equations for several  $\lambda$ -dependent coupling constants [e.g., Eqs. (1.6) and (1.7)]. However, as pointed out in the Introduction, the renormalization-group equations are analogous to equations of motion in nonlinear mechanics; in this analogy  $\ln \lambda^2$  is analogous to the time  $t$  and  $e_{\lambda^2}$  is analogous to a coordinate  $x$ .

In classical nonlinear mechanics there are standard types of asymptotic behavior for large  $t$  which occur in many different kinds of systems. The simplest asymptotic behavior is a fixed (equilibrium) point. For Eq. (3.1), a fixed point is a point  $x$  for which  $\psi(0, x) = 0$ . For Eqs. (1.6) and (1.7) a fixed point is a pair of values  $x, y$  for which  $\psi_1(x, y)$  and  $\psi_2(x, y)$  both vanish. It is quite common, especially in electric-circuit problems, to have a system which starts off with a transient (time-dependent) behavior but which settles into a time-independent state (fixed point) as  $t \rightarrow \infty$ ; this can occur no matter how many dependent variables ( $x$ , or  $x, y$ , or etc.) are needed to describe the system. A limit cycle is another type of asymptotic behavior

<sup>16</sup> The renormalization group-equations for pseudoscalar meson theory are formulated in Bogoliubov and Shirkov (Ref. 4).

which also occurs in many systems. A limit cycle is a periodic solution of a nonlinear set of equations of motion; because of the nonlinearity, the amplitude of the solution is fixed as well as the period.<sup>17</sup> Other types of asymptotic behavior are also possible but are not so easily characterized.

The purpose of this section is to discuss the possibility that the asymptotic solution of the renormalization-group equations is either a fixed point or limit cycle. These are not the only possibilities but other possibilities are more difficult to analyze. Studying the consequences of a fixed point or a limit cycle makes clear the importance of the renormalization group for field theory.

The asymptotic behavior of the solution of the renormalization-group equations is important for several reasons. For example, there are two experimental quantities which should reflect directly the qualitative behavior of the coupling constants for large  $\lambda$ . One is the total cross section for  $e^+e^-$  annihilation into hadrons discussed in the Introduction; the other is the Callan-Gross integral over deep-inelastic electron-scattering cross sections.<sup>18</sup> The latter involves a more complicated analysis and will not be discussed further here. The consequence of a fixed-point or limit-cycle behavior for  $e^+e^-$  annihilation will be discussed later in this section. Contrasting fixed-point behavior with limit-cycle behavior suggests experimental tests which distinguish fixed points from limit cycles and probably from other types of behavior as well. [For example, if the cross section  $\sigma_{\text{tot}}(q^2)$  for annihilation of  $e^+e^-$  to hadrons behaves as  $1/q^2$  for large  $q^2$ , then the asymptotic solution of the renormalization group for strong interactions is probably a fixed point. See Sec. III H.] Hence one may learn from experiment whether one must study asymptotic behaviors other than the fixed point.

In order to discuss the possibility of a fixed point, one must study functions  $\psi(0,x)$  which have fixed points. It is convenient to discuss a function  $\psi(0,x)$  which has at least three positive roots  $x$ ; this allows one to consider various types of behavior connected with fixed points. Only the case of one coupling constant, as in Eq. (3.1), will be discussed explicitly; the discussion is easily generalized to the case of two or more coupling constants.

To discuss a limit cycle, one must have at least two coupling constants, so the limit cycle will be discussed using Eqs. (1.6) and (1.7).

In applying the renormalization group to strong

interactions, one must distinguish two situations. The first alternative is that strong interactions remain distinguishable from weak or electromagnetic or other interactions for arbitrarily large momenta, and that there is a theory of strong interactions valid for all momenta which neglects these other interactions. Given this alternative, the discussion of the renormalization group for strong interactions, assuming fixed-point asymptotic behavior, is similar to Gell-Mann and Low's discussion of electrodynamics.<sup>1</sup> In particular, one can predict the values of the bare coupling constants of strong interactions but the physical coupling constants are undetermined theoretically and can only be found experimentally. The second alternative is that at some cutoff  $\Lambda$  large compared to 1 GeV the electromagnetic or weak or other corrections to strong interactions become too large to be treated as a perturbation. In this case a theory of strong interactions in isolation is only valid for momenta small compared to the cutoff and one must allow for large corrections to the  $\lambda$ -dependent coupling constants when  $\lambda$  is of order  $\Lambda$ . The consequence of this (as will be shown below) is that the physical coupling constants are predictable theoretically if one knows the precise form of the renormalization group equations (and if the solution of these equations is a fixed point for  $\lambda \gg 1$  GeV but  $\lambda \ll \Lambda$ ). Both alternatives will be explained in detail when the fixed-point hypothesis is applied to strong interactions.

The renormalization-group equations can be derived in detail only after a field theory has been solved since the function  $\psi(0,x)$  is defined in terms of a propagator of the interacting theory (in theories other than electrodynamics,  $\psi$  depends on vertex functions as well), and the propagator is not known unless the theory has been solved. Any property of the field theory which one deduces by solving the renormalization-group equations ought in principle to be discernible directly from the solution of the theory, making a discussion of the renormalization-group equation unnecessary. However, one does not have solutions to field theories except in perturbation theory; the analysis of the renormalization-group equations described here is one way of groping towards the nature of strongly interacting field theories.

## B. Integration of Differential Equation

Now consider solutions of the renormalization-group equations with fixed-point asymptotic behavior. Equation (3.1) for electrodynamics will be discussed explicitly but the analysis applies to other field theories as well. Suppose that the function  $\psi(0,x)$  has at least three positive roots  $x_1, x_2,$  and  $x_3$  with  $0 < x_1 < x_2 < x_3$ . Let these roots all be simple roots. From the perturbation-theory formula (2.68)  $\psi(0,x)$  has a double root at  $x=0$  and is positive when  $x$  is positive and small. Suppose that  $\psi(0,x)$  is bounded, continuous, and differentiable for all  $x$ . One consequence of these assump-

<sup>17</sup> For a detailed discussion of limit cycles, see Ref. 8. The idea that there might be limit-cycle behavior in field theory has been suggested by H. Mitter, *Nuovo Cimento* **32**, 1789 (1964). (Mitter discusses an invariance to discrete scale transformations; this is equivalent to limit-cycle behavior.)

<sup>18</sup> C. Callan and D. Gross, *Phys. Rev. Letters* **21**, 311 (1968); J. D. Bjorken, *Phys. Rev.* **179**, 1547 (1969); C. Callan and D. Gross, *Phys. Rev. Letters* **22**, 156 (1969); S. Ciciarello *et al.*, (Ref. 7); G. Mack, *Phys. Rev. Letters* **25**, 400 (1970).



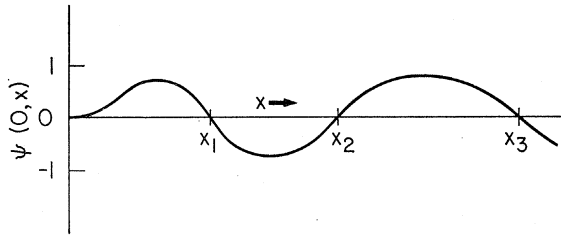


FIG. 1. Example of a function  $\psi(0,x)$  which has a double root at  $x=0$  and single roots at  $x=x_1, x_2$ , and  $x_3$ .

tions is that  $\psi(0,x)$  is positive for  $0 < x < x_1$ , negative for  $x_1 < x < x_2$ , and positive again for  $x_2 < x < x_3$ . The range  $x > x_3$  will not be discussed here. A function  $\psi(0,x)$  with these properties is shown in Fig. 1. These assumptions are made so that the solutions of the renormalization-group equation will illustrate several forms of fixed-point asymptotic behavior; there is no way of knowing whether these assumptions are true for quantum electrodynamics or any other given field theory.

For  $\psi(0,x)$  to be negative contradicts the Källén-Lehmann representation<sup>19</sup> for the photon propagator. The Källén-Lehmann representation gives

$$d_c\left(\frac{k^2}{m^2}, e^2\right) = 1 + k^2 \int_0^\infty \rho\left(\frac{k'^2}{m^2}, e^2\right) \frac{1}{k^2 - k'^2 - i\epsilon} dk'^2, \quad (3.2)$$

where  $\rho$  is a positive spectral function. This equation holds provided no subtractions are needed to make the integral converge. It follows from this formula and Eq. (2.45) that

$$\frac{de_\lambda^2}{d\lambda^2} = e^2 \int_0^\infty \rho\left(\frac{k'^2}{m^2}, e^2\right) \frac{k'^2}{(\lambda^2 + k'^2)^2} dk'^2, \quad (3.3)$$

which is positive but might go to zero when  $\lambda \rightarrow \infty$ .

The function  $\psi(0,x)$  will be permitted to be negative anyway. The reason is that Eq. (3.1) is regarded here only as a prototype for the renormalization-group equations for arbitrary field theories, in most of which  $de_\lambda^2/d\lambda^2$  involves vertex functions as well as propagators and can be negative. Also, it is known from the example of the Lee model<sup>20</sup> that a renormalized theory does not necessarily satisfy the requirement of a positive definite metric which is assumed in the proof that  $\rho(k'^2/m^2, e^2)$  is positive. So it may be that  $\psi(0,x)$  does go negative even in electrodynamics.

Now consider the renormalization-group differential equation (3.1). This is the zero-mass equation; the effects of a finite mass will be discussed later. The zero-mass equation can be integrated to give

$$\ln(\lambda^2) - \ln(\lambda'^2) = F(e_\lambda^2) - F(e_{\lambda'}^2), \quad (3.4)$$

<sup>19</sup> G. Källén, *Helv. Phys. Acta* **25**, 417 (1952); H. Lehmann, *Nuovo Cimento* **11**, 342 (1954).

<sup>20</sup> T. D. Lee, *Phys. Rev.* **95**, 1329 (1954).

and  $\lambda$  and  $\lambda'$  are arbitrary, and

$$F(x) = \int_c^x [\psi(0,x')]^{-1} dx' \quad (3.5)$$

and  $c$  is an arbitrary constant which cancels out in the difference  $F(e_\lambda^2) - F(e_{\lambda'}^2)$ . It is convenient to distinguish three ranges of the variable  $x$ :  $0 < x < x_1$ ,  $x_1 < x < x_2$ , and  $x_2 < x < x_3$ . The region beyond  $x_3$  will not be discussed here. First suppose that  $x$  is in the range  $0 < x < x_1$ . Let  $c$  also lie between 0 and  $x_1$ , so the integration region in Eq. (3.5) does not cross any zeros of  $\psi$ .  $F(x)$  has the following properties in the range  $0 < x < x_1$ : It is increasing [since  $\psi(0,x')$  is positive for  $0 < x' < x_1$ ]; it goes to  $+\infty$  logarithmically as  $x \rightarrow x_1$  [due to the simple zero in  $\psi(0,x')$  for  $x' \rightarrow x_1$ ]; and it goes to  $-\infty$  proportional to  $x^{-1}$  when  $x \rightarrow 0$  due to the double zero in  $\psi(0,x')$  as  $x' \rightarrow 0$ . The function  $F(x)$  is plotted in Fig. 2.

Now look at Eq. (3.4). Let  $\lambda'$  be fixed, and let  $e_{\lambda'}^2$  be chosen arbitrarily but in the range  $0 < e_{\lambda'}^2 < x_1$ . As  $\lambda$  increases,  $\ln \lambda^2$  increases; therefore,  $F(e_\lambda^2)$  must increase and this means  $e_\lambda^2$  must increase. For  $\lambda^2 \rightarrow \infty$ ,  $F(e_\lambda^2)$  must go to  $\infty$  also, which means  $e_\lambda^2 \rightarrow x_1$ . (This argument should be checked using Fig. 2.) In the other limit  $\lambda^2 \rightarrow 0$ ,  $\ln \lambda^2$  goes to  $-\infty$ , and this forces  $e_\lambda^2$  to go to 0. This is true for any value of  $e_{\lambda'}^2$  in the range  $0 < e_{\lambda'}^2 < x_1$ . In fact, if one lets  $f(y)$  be the function inverse to  $F(x)$  [i.e.,  $x = f(y)$  if  $y = F(x)$ ], then the solution to Eq. (3.4) is

$$e_\lambda^2 = f[\ln(\lambda^2) - \ln(\lambda'^2) + F(e_{\lambda'}^2)], \quad (3.6)$$

which means a change in  $e_{\lambda'}^2$  is equivalent to translating the solution for  $e_\lambda^2$  by a fixed distance in the "time" variable  $\ln \lambda^2$ . A set of solution curves for  $e_\lambda^2$  as a function of  $\ln \lambda^2$  is shown in Fig. 3. All the curves which lie

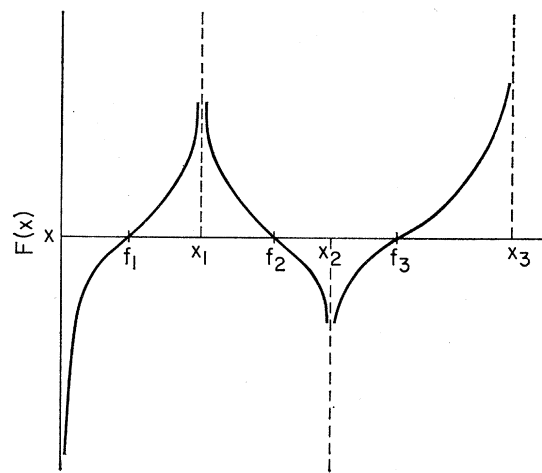


FIG. 2. Plot of the function  $F(x)$  assuming  $\psi(0,x)$  is the function shown in Fig. 1. The constant of integration  $c$  is also shown;  $c = f_1$  if  $0 < x < x_1$ ;  $c = f_2$  if  $x_1 < x < x_2$ ;  $c = f_3$  if  $x_2 < x < x_3$ . The constants  $f_1, f_2$ , and  $f_3$  are chosen arbitrarily.

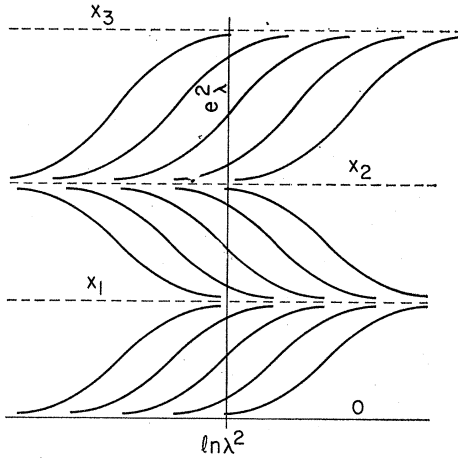


FIG. 3. Solutions of the zero-mass renormalization-group equations for  $e_\lambda^2$  plotted vs  $\ln \lambda^2$ .

in the range  $0 < e_\lambda^2 < x_1$  go to  $x_1$  as  $\lambda^2 \rightarrow \infty$ ; they all go to 0 as  $\lambda^2 \rightarrow 0$ .

Similar analyses can be performed for solution curves lying in the intervals  $x_1 < e_\lambda^2 < x_2$  and  $x_2 < e_\lambda^2 < x_3$ . The curves lying in the range  $x_1 < e_\lambda^2 < x_2$  go to  $x_1$  for  $\lambda^2 \rightarrow \infty$  and go to  $x_2$  for  $\lambda^2 \rightarrow 0$ ; curves lying in the range  $x_2 < e_\lambda^2 < x_3$  go to  $x_3$  for  $\lambda^2 \rightarrow \infty$  and go to  $x_2$  for  $\lambda^2 \rightarrow 0$ .

There are also special solutions of the differential equation, namely, the fixed points:  $e_\lambda^2 \equiv 0$ ,  $e_\lambda^2 \equiv x_1$ ,  $e_\lambda^2 \equiv x_2$ , or  $e_\lambda^2 \equiv x_3$ .

What one sees from this analysis is that in the limits  $\lambda^2 \rightarrow \infty$  or  $\lambda^2 \rightarrow 0$  any solution of the zero-mass differential equation (3.1) must approach one of the fixed points 0,  $x_1$ ,  $x_2$ , or  $x_3$ . This is despite the fact that for a fixed and finite  $\lambda$ ,  $e_\lambda^2$  is continuously variable. Underlying this result is an amplification and deamplification effect in the solution of the differential equation. Suppose one considers two initial conditions  $x_A$  and  $x_B$  for  $x$  at  $t=0$  and then integrates Eq. (1.5); one gets two solutions  $x_A(t)$  and  $x_B(t)$ . Suppose that  $x_B \simeq x_A$ . As  $t$  increases, the difference  $x_B(t) - x_A(t)$  may increase, in which case the difference can become large when  $t$  is large: This is amplification. It occurs, for example, if  $x_B$  and  $x_A$  are both near  $x_2$ ; it is clear from Fig. 3 that the solutions  $x_B(t)$  and  $x_A(t)$  will separate as  $t$  increases. [If  $x_A(t)$  and  $x_B(t)$  both approach the limit  $x_1$  or both approach  $x_3$  when  $t \rightarrow \infty$ , the separation will eventually reach a maximum and then decrease again.] It is also possible for the curves  $x_A(t)$  and  $x_B(t)$  to approach each other as  $t$  increases, becoming equal as  $t \rightarrow \infty$ ; this is deamplification. Amplification and deamplification can be discussed quantitatively by defining solutions  $x(t, x_A)$  of Eq. (1.5) which depend on the initial value,

$$x(0, x_A) = x_A, \quad (3.7)$$

and then looking at  $\partial x(t, x_A) / \partial x_A$ . If there is strong amplification between 0 and  $t$  then  $\partial x(t, x_A) / \partial x_A$  will be much larger than 1. Strong deamplification means that

$\partial x(t, x_A) / \partial x_A \ll 1$ . Since the function  $x(t, x_A)$  satisfies

$$\frac{\partial x}{\partial t}(t, x_A) = \psi[0, x(t, x_A)], \quad (3.8)$$

one gets by differentiation

$$\frac{\partial}{\partial t} \left[ \frac{\partial x}{\partial x_A}(t, x_A) \right] = \frac{\partial x}{\partial x_A}(t, x_A) \frac{\partial \psi}{\partial x}[0, x(t, x_A)]. \quad (3.9)$$

If one thinks of  $\partial \psi[0, x(t, x_A)] / \partial x$  as being a known function  $u(t)$ , this equation can be integrated. One needs a boundary condition: From the boundary condition (3.7) one has

$$\frac{\partial x}{\partial x_A}(0, x_A) = 1. \quad (3.10)$$

The result is

$$\frac{\partial x}{\partial x_A}(t, x_A) = \exp \left( \int_0^t u(t') dt' \right). \quad (3.11)$$

If  $\partial \psi / \partial x$  is positive,  $\partial x(t, x_A) / \partial x_A$  increases with  $t$ ; if  $\partial \psi / \partial x$  is negative,  $\partial x / \partial x_A$  decreases. To make  $\partial x / \partial x_A$  be near zero or infinity requires that the integral in the exponent be large. With the form of  $\psi(0, x)$  being postulated here,  $\partial \psi / \partial x$  is not itself large and a large integral can come only from a large value of  $t$  (near  $+\infty$  or  $-\infty$ ).

Now the case of finite mass will be discussed briefly. The main qualitative effect of having finite mass is that large amplification and deamplification effects are confined to the region  $\lambda \gg m$ . What happens for  $\lambda \ll m$  is that  $\partial \psi / \partial x$  is proportional to  $\lambda^2 / m^2$  and is too small to cause much amplification and deamplification. The appearance of the factor  $\lambda^2$  is best seen from Eq. (2.45): Differentiating it, one gets

$$\psi \left( \frac{m^2}{\lambda^2}, e_\lambda^2 \right) = \lambda^2 \frac{d(e_\lambda^2)}{d\lambda^2} = \frac{\lambda^2}{m^2} \left[ \frac{\partial d_c(y, e^2)}{\partial y} \right]_{y=\lambda^2/m^2}. \quad (3.12)$$

The function  $d_c$  is well behaved for  $\lambda \rightarrow 0$  when  $m$  is finite; hence  $\partial d_c(-\lambda^2/m^2, e^2) / \partial \lambda^2$  is finite for  $\lambda \rightarrow 0$  (or perhaps mildly singular due to the three-photon contribution to the spectral function, which has a threshold at  $k^2=0$ ). Expressing  $e^2$  in terms of  $e_\lambda^2$  causes no trouble, at least in perturbation theory, nor does differentiation with respect to  $e_\lambda^2$ . So  $\partial \psi(m^2/\lambda^2, x) / \partial x$  has a factor  $\lambda^2/m^2$  when  $\lambda \ll m$ , which is  $m^{-2}e^{2t}$  in terms of  $t$ , which makes the region  $t \ll \ln m$  negligible. Thus large amplification and deamplification effects which require  $\partial \psi / \partial x$  to be finite over a large range of  $t$  occur only for  $\lambda \gg m$ . For qualitative purposes one can avoid integrating the finite-mass equation, instead integrating the zero-mass equation but stopping at  $\lambda = m$  instead of going to  $\lambda = 0$ . The value of  $e_m^2$  obtained this way should qualitatively be similar to  $e^2$  obtained from integrating the exact

equation; in particular,  $\partial(e_m^2)/\partial e^2$  should be neither very small nor very large.

### C. Fixed-Point Solutions and Scale Invariance

What is the significance of the solutions described above of the renormalization-group differential equation? First the special fixed-point solutions will be discussed. Consider, for example, the solution  $e_\lambda^2 = x_1$  for all  $\lambda$ , with  $m$  being zero. It will now be shown that this solution defines a scale-invariant field theory.<sup>21</sup> Consider a dimensional analysis of the fields  $A_{\lambda\mu}(z)$  and  $\psi_\lambda(z)$ , with  $e_\lambda^2$  being a constant independent of  $\lambda$  and  $m$  being zero. The only variables left are  $z$  and  $\lambda$ . In the free-field limit, the dimensions of  $A_{\lambda\mu}(z)$  and  $\psi_\lambda(z)$  are fixed by the canonical commutation rules, giving  $A_{\lambda\mu}(z)$  the dimensions of mass and  $\psi_\lambda(z)$  the dimensions (mass)<sup>3/2</sup>.

When one does perturbation theory with a finite cutoff, the dimensions of the unrenormalized fields are still fixed by the canonical commutation rules. The renormalized fields differ from the unrenormalized fields by cutoff-dependent factors which in principle could carry dimensions<sup>22</sup>; in practice the renormalization conditions (2.30) and (2.31) imposed on the renormalized fields ensure that they also have canonical dimensions<sup>23</sup> and that the renormalization constants are dimensionless. To be specific, setting  $d(-1, m^2/\lambda^2, e_\lambda^2)$  and  $s(-1, m^2/\lambda^2, e_\lambda^2)$  equal to 1 for all  $e_\lambda^2$  means that  $d$  and  $s$  are dimensionless for all  $e_\lambda$ , not just  $e_\lambda = 0$ , and this means the dimensions of  $A_{\lambda\mu}$  and  $\psi_\lambda$  are also independent of  $e_\lambda$ . One can apply dimensional analysis to an arbitrary vacuum expectation value of the fields, e.g.,

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = \langle \Omega | T A_{\lambda\mu}(z_1) A_{\lambda\nu}(z_2) \psi_\lambda(z_3) \bar{\psi}_\lambda(z_4) | \Omega \rangle. \quad (3.13)$$

<sup>21</sup> The argument that follows shows that scale invariance is exact, i.e., the vacuum is invariant to scale transformations. P. Carruthers [Phys. Rev. D 2, 2265 (1970)] has proposed models of scale-invariant theories with a noninvariant vacuum. The author does not know what kind of solution of the renormalization-group equations corresponds to a theory with scale invariance but a noninvariant vacuum.

<sup>22</sup> Consider, e.g., the constant  $Z_2$  relating the unrenormalized electron field to the renormalized electron field. The only absolute requirement on  $Z_2$  is that it contain all the cut-off dependence of the unrenormalized field. In second order  $Z_2$  will have the form  $1 + \eta e^2 [\ln \Lambda^2 + \eta']$ , where  $\eta$  is a calculable constant and  $\eta'$  is arbitrary. It is not necessary for  $\eta'$  to be a logarithm of a mass so that  $[\ln \Lambda^2 + \eta'] = [\ln(\Lambda^2/M^2)]$ , making the argument of the logarithm dimensionless. Any  $\eta'$  will ensure that  $Z_2$  includes all cut-off dependence of the unrenormalized field. One can put  $\eta' = 0$ , for example. In this case the renormalized field does not have a well-defined dimension. If  $Z_2$  to all orders has the special form  $\exp(\eta e^2 \ln \Lambda^2)$  (when parameters like  $\eta'$  are set equal to zero) then  $Z_2 = (\Lambda^2)^{\eta e^2}$  and the renormalized field does have dimensions, but not the same as the unrenormalized field.

<sup>23</sup> One is talking here about "mass dimensions," namely, the dimensions that fields have in dimensional analysis. Later in this section "scale dimensions" will be defined which give the dimensions of fields under operator scale transformations. The two dimensions can be distinct because in a scale transformation of dimensional analysis,  $\lambda$  is scaled, while an operator scale transformation does not change  $\lambda$ .

The product  $\lambda^{-5} T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$  is dimensionless and can depend only on the products  $z_1\lambda$ ,  $z_2\lambda$ ,  $z_3\lambda$ , and  $z_4\lambda$ . Similar results apply to arbitrary vacuum expectation values. These results can be represented schematically by the equation

$$\lambda^{-1} A_{\lambda\mu}(z) = A_{1\mu}(\lambda z), \quad (3.14)$$

$$\lambda^{-3/2} \psi_\lambda(z) = \psi_1(\lambda z). \quad (3.15)$$

[ $A_{1\mu}(z)$  means  $A_{\lambda'\mu}(z)$  with  $\lambda' = 1$ ;  $\lambda'$  is put equal to 1 whenever an arbitrary fixed value of  $\lambda'$  is needed.] These equations are to mean that vacuum expectation values such as  $T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$  satisfy the identities that result from making the substitutions (3.14) and (3.15) in the vacuum expectation value. The equations cannot be understood as operator relations since one might have instead

$$\lambda^{-1} A_{\lambda\mu}(z) = U_\lambda^\dagger A_{1\mu}(\lambda z) U_\lambda, \quad (3.16)$$

$$\lambda^{-1} \psi_\lambda(z) = U_\lambda^\dagger \psi_1(\lambda z) U_\lambda, \quad (3.17)$$

where  $U_\lambda$  is a unitary transformation; these equations will also reproduce the result of dimensional analysis on  $T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda)$ .

In addition to the dimensional relations, there are the renormalization relations

$$A_{\lambda\mu}(z) = (z_{3\lambda 1})^{1/2} A_{1\mu}(z), \quad (3.18)$$

$$\psi_\lambda(z) = (z_{2\lambda 1})^{1/2} \psi_1(z). \quad (3.19)$$

For  $T_{\mu\nu}$  these relations give

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = z_{3\lambda 1} z_{2\lambda 1} T_{\mu\nu}(z_1, z_2, z_3, z_4, 1). \quad (3.20)$$

Using both Eqs. (3.14) and (3.15) and (3.18) and (3.19) one can eliminate the fields  $A_{\lambda\mu}(z)$  and  $\psi_\lambda(z)$ , leaving formulas relating  $A_{1\mu}(z)$  and  $\psi_1(z)$  to  $A_{1\mu}(\lambda z)$  and  $\psi_1(\lambda z)$ , i.e., scaling relations. First we need to determine  $z_{3\lambda 1}$  and  $z_{2\lambda 1}$ . The constant  $z_{3\lambda 1}$  has the form [from Eq. (2.53)]

$$z_{3\lambda 1} = e_1^2 / e_\lambda^2. \quad (3.21)$$

Since  $e_\lambda^2$  is constant,  $e_\lambda^2 = e_1^2$  and  $z_{3\lambda 1}$  is 1. To obtain  $z_{2\lambda 1}$  one solves Eq. (2.72) for  $z_{2\lambda}$ ; since  $z_{2\lambda 1}$  is  $z_{2\lambda} / z_{21}$ , one obtains

$$z_{2\lambda 1} = \lambda^{2\sigma}, \quad (3.22)$$

with

$$\sigma = \sigma(0, x_1). \quad (3.23)$$

Thus one obtains the relation

$$\lambda^5 T_{\mu\nu}(\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, 1) = T_{\mu\nu}(z_1, z_2, z_3, z_4, \lambda) = \lambda^{2\sigma} T_{\mu\nu}(z_1, z_2, z_3, z_4, 1). \quad (3.24)$$

This equation gives a scaling law for  $T_{\mu\nu}(z_1, z_2, z_3, z_4, 1)$ , namely, it is equal to  $\lambda^{5-2\sigma} T_{\mu\nu}(\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, 1)$ . The scaling law for arbitrary vacuum expectation values is represented schematically by the equations

$$A_{1\mu}(z) = \lambda A_{1\mu}(\lambda z), \quad (3.25)$$

$$\psi_1(z) = \lambda^{3/2-\sigma} \psi_1(\lambda z). \quad (3.26)$$

If Eqs. (3.16) and (3.17) hold, then the actual operator relations are

$$U_\lambda A_{1\mu}(z) U_\lambda^\dagger = \lambda A_{1\mu}(\lambda z), \quad (3.27)$$

$$U_\lambda \psi_1(z) U_\lambda^\dagger = \lambda^{3/2-\sigma} \psi_1(\lambda z). \quad (3.28)$$

As a result of these equations the unitary transformations  $U_\lambda$  generate scale transformations of the fields  $A_{1\mu}$  and  $\psi_1$ , so the theory is scale invariant. I cannot justify Eqs. (3.27) and (3.28), but in any case all vacuum expectation values of  $A_{1\mu}(z)$  and  $\psi_1(z)$  scale as if the theory were scale invariant, which presumably means the theory is scale invariant.

The quantity  $\frac{3}{2}-\sigma$  is the "scale dimension" of the field  $\psi_1(z)$ . Unless  $\sigma$  is 0, the scale dimension of  $\psi_1(z)$  is different from its dimension in mass units. The field  $\lambda^{-\sigma} \psi_\lambda(z)$  is [from Eqs. (3.19) and (3.22)] equal to  $\psi_1(z)$  and therefore independent of  $\lambda$ ; this field has both scale and mass dimensions  $\frac{3}{2}-\sigma$ . The fact that interacting fields can have different scale dimensions from free fields has been noted in many circumstances.<sup>24-26</sup> The scale dimension of  $A_{1\mu}(z)$  is 1, from Eq. (3.27), and is not changed by interaction. This is presumably because of the field equation which makes  $\square A_{\lambda\mu}(z)$  proportional to the electromagnetic current  $j_{\lambda\mu}(z)$ . The scale dimension of the current has to be 3 in order that the equal-time commutator,

$$[j_{10}(\mathbf{z}, z_0), \psi_1(\mathbf{y}, z_0)] = \delta^3(\mathbf{z}-\mathbf{y}) \psi_1(\mathbf{y}, z_0), \quad (3.29)$$

be scale invariant. Hence  $\square A_{1\mu}(z)$  must have scale dimension 3, and  $A_{1\mu}(z)$  itself must have scale dimension 1.

#### D. Renormalization Group, Fixed Points, and Strong Interactions in Isolation

Suppose now that Eq. (3.1) is the renormalization-group equation for strong interactions. This would mean that there is only one renormalized coupling constant in strong interactions, as in the gluon model. The coupling constant  $e$  is interpreted in this section to be the renormalized coupling constant of strong interactions, and  $m$  will be assumed to be a typical strong-interaction mass (about 1 GeV). Assume that strong interactions can be isolated from other interactions approximately for all momenta (the opposite assumption will be discussed later). In this case the renormalized coupling constant  $e$  is an arbitrary parameter in the theory and the finite-mass renormalization-group equation should be solved with the boundary condition  $e_\lambda \rightarrow e$  as  $\lambda \rightarrow 0$ . In practice it is more convenient to discuss the zero-mass renormalization-group

equation; the solution  $e_\lambda$  of the finite-mass equation satisfies the zero-mass equation approximately when  $\lambda$  is much larger than  $m$ . One can now discuss a solution  $e_\lambda'$  of the zero-mass equation such that  $e_\lambda' \rightarrow e_\lambda$  as  $\lambda \rightarrow \infty$ , the difference being some power of  $m$ .

The quantity  $e_m'$  should depend in a reasonable way on  $e$ ; that is, one does not expect large amplification or deamplification effects in the dependence of  $e_m'$  on  $e$ . So in the following,  $e_m'$  will be taken to be the input parameter instead of  $e$ , and we shall discuss only the zero-mass renormalization-group equation. In the following, we shall write  $e_\lambda$  for  $e_\lambda'$ . The function  $\psi$  will be assumed to have the form discussed in Sec. III B. Suppose for convenience that  $e_m^2$  lies between 0 and  $x_2$ . If this is so,  $e_\lambda^2$  will approach  $x_1$  as  $\lambda \rightarrow \infty$ . It is not necessary for  $\lambda$  to be enormous before  $e_\lambda^2$  is close to  $x_1$ : Since  $\psi(0, x)$  is of order 1 when  $x$  is of order 1 (except when  $x$  is near  $x_1, x_2$ , etc.), the solution curves change reasonably rapidly with  $\lambda$  and soon approach  $x_1$ . The only exception is if  $e_m^2$  is close to  $x_2$  or 0; but if  $e_m^2$  is an arbitrary parameter fixed by experiment, it would seem unlikely that it would be close to any preassigned number such as  $x_2$ . (It is also assumed that  $e_m$  is not small.)

One can determine the rate of approach of  $e_\lambda^2$  to  $x_1$  as  $\lambda \rightarrow \infty$ . Let  $x(t)$  be near  $x_1$ . Then one can expand  $\psi(0, x)$  about  $x = x_1$ :

$$\psi(0, x) \simeq \psi(0, x_1) + [x(t) - x_1] \partial \psi(0, x_1) / \partial x. \quad (3.30)$$

It is convenient to introduce the constants

$$a_n = \partial \psi(0, x_n) / \partial x \quad (3.31)$$

for each fixed point. Then, approximately,

$$dx/dt \simeq a_1 [x(t) - x_1]. \quad (3.32)$$

The solution of this equation is

$$x(t) = x_1 + c_1 e^{a_1 t}, \quad (3.33)$$

where  $c_1$  is an arbitrary constant. It is evident from Fig. 1 that  $a_1$  is negative, so  $x(t) \rightarrow x_1$  as  $t \rightarrow \infty$  exponentially. Translated in terms of  $e_\lambda^2$ , this gives

$$e_\lambda^2 \simeq x_1 + c_1 \lambda^{2a_1}. \quad (3.34)$$

The constant  $c_1$  will depend on the initial condition  $e_m^2$ . If  $e_m^2 - x_1$  is small also, then  $c_1$  will be linear in  $e_m^2 - x_1$ , but if  $e_m^2 - x_1$  is large, the linearized equation Eq. (3.32) will be incorrect for  $t$  near  $\ln m^2$  and the relation of  $c_1$  to  $e_m^2 - x_1$  will be nonlinear. In any case,  $e_\lambda^2$  approaches its limiting value  $x_1$  as an inverse power of  $\lambda$ .

When  $e_\lambda$  was introduced in Sec. II it arose as an alternative definition of the renormalized coupling constant; in the Gell-Mann-Low renormalization program all Feynman amplitudes are considered as functions of  $e_\lambda$  not  $e$ . This raises a problem. If one has a function such as  $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$  which depends on  $e_\lambda^2$ , one would suppose that a small change in  $e_\lambda^2$  implies a small change in  $d$ . But if  $\lambda$  is very large, a small change

<sup>24</sup> K. Wilson, Phys. Rev. 179, 1499 (1969).

<sup>25</sup> K. Wilson, SLAC Report No. SLAC-PUB-737, 1970 (unpublished).

<sup>26</sup> K. Wilson, Phys. Rev. D 2, 1473 (1970); 2, 1478 (1970); C. Callan, *ibid.* 2, 1541 (1970); S. Coleman and R. Jackiw, MIT report (unpublished); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); J. Lowenstein and B. Schroer, Pittsburgh report (unpublished); H. Georgi, Phys. Rev. D 2, 2908 (1970).

in  $e_\lambda^2$  away from  $x_1$  is amplified to give a large and non-linear change in  $e_m^2$ ; so  $e_m^2$  and any quantity which depends directly on  $e_m^2$  will not change just a little when  $e_\lambda^2$  changes a little. This means it is not very appropriate to use  $e_\lambda^2$  for a large value of  $\lambda$  as one's renormalized coupling constant for amplitudes which are best parametrized in terms of  $e_m^2$ . Another way to say this is that physically the qualitative nature of a given amplitude should be determined by a qualitative knowledge of the physical couplings which determine that amplitude. If one has to specify a coupling constant to 1% accuracy in order to determine the amplitude to 50% accuracy, there is something wrong. The response to this problem is, I think, to consider the coupling constant  $e_\lambda^2$  for different ranges of the momentum  $\lambda$  to be physically distinct coupling constants governing distinct physical processes. The natural thing to expect is that  $e_\lambda$  with  $\lambda \sim m$  governs amplitudes with momenta of order  $m$  or less;  $e_\lambda$  with  $\lambda \sim 10m$  covers amplitudes whose momenta are of order  $10m$ , and so forth. Within a particular order of magnitude range of momenta one can specify arbitrarily the exact value of  $\lambda$  for which  $e_\lambda$  is the coupling constant.

If  $\lambda$  is much larger than  $m$ , the amplitudes for which  $e_\lambda^2$  is the relevant coupling constant are off-mass-shell amplitudes with virtual masses of order  $\lambda$ , such as the photon propagator with  $k^2 \sim \lambda^2$ . A scattering amplitude with large energies and momentum transfers, but for particles on the mass shell, cannot be said to involve only  $e_\lambda^2$  for  $\lambda$  large; the mass-shell condition suggests that the amplitude will also be strongly affected by the value of  $e_m^2$ . The behavior of scattering amplitudes at large energies and momentum transfers is outside the scope of this paper.

The way this picture shows up in perturbation theory is that a function such as  $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$  has a perturbation expansion in  $e_\lambda^2$  whose coefficients are of order 1 (apart from factors of  $2\pi$  which are considered to be of order 1 in this paper) when  $k^2$  is of order  $\lambda^2$ ; but if  $k^2/\lambda^2$  or  $\lambda^2/k^2$  is large, the coefficient of  $(e_\lambda^2)^n$  contains the large factor<sup>27</sup>  $[\ln(k^2/\lambda^2)]^n$  which makes  $d$  much more sensitive to changes in  $e_\lambda^2$ . So perturbation theory confirms the hypothesis that  $e_\lambda^2$  is the appropriate coupling constant for momenta of order  $\lambda$  and not so appropriate for momenta much larger or smaller than  $\lambda$ .

In summary, one defines a sequence of coupling constants, say,  $e_m, e_{10m}, e_{100m}$ , etc. All of these have to be specified in order to determine qualitatively the physics at arbitrary momenta, owing to the large amplification or deamplification effects that can occur in the relation between different coupling constants in the sequence.

For large momentum  $\lambda$ ,  $e_\lambda^2$  is close to  $x_1$  and  $m/\lambda$  is small; this means propagators and other amplitudes at large momenta will be close to the scale-invariant amplitudes for  $e_\lambda^2 = x_1$  and  $m = 0$ . So scale invariance is

a broken symmetry of the theory. Furthermore, as seen earlier in the case of the field  $\psi$ , one must expect scale dimensions of local fields to differ from free-field dimensions as was assumed in Ref. 24. One consequence of broken scale invariance is that  $\sigma_{\text{tot}}(q^2)$ , the  $e^+e^-$  annihilation cross section into hadrons, behaves as  $(q^2)^{-1}$  when  $q^2$  is large.<sup>25</sup>

### E. Strong Interactions with Cutoff

Suppose now that there is a cutoff momentum  $\Lambda$  beyond which one cannot distinguish strong interactions from other interactions.<sup>28</sup> It will be assumed that  $\Lambda \gg m$ , i.e.,  $\Lambda$  is much larger than 1 GeV/c. The effect of the cutoff is to shift the boundary condition on the renormalization-group equation from  $\lambda = m$  to  $\lambda = \Lambda$ . The reason is this. With a cutoff, the strong-interaction theory that one develops for laboratory energies is no more than an infrared approximation to a more complicated theory combining strong and other interactions. The more complicated theory covers energies of order  $\Lambda$  or less; strong-interaction theory is valid only for momenta much less than  $\Lambda$ . The coupling constant  $e_\lambda$  for  $\lambda$  of order  $\Lambda$  will now be determined by properties of the more complicated theory rather than by strong interactions alone. Under these circumstances, the chances are negligible that  $e_\lambda$  with  $\lambda \sim \Lambda$  will be close to the fixed point  $x_1$ , since the location of  $x_1$  is a property of the isolated strong-interaction theory and not of the combined theory. Another way of saying this is that if weak interactions, say, become strong at momenta of order  $\Lambda$ , then there will be large corrections to  $e_\lambda$  when  $\lambda$  is of order  $\Lambda$  due to weak interactions, making it unlikely that  $e_\lambda^2$  will be near  $x_1$ . Suppose, therefore, that  $e_\lambda^2$  has a random value, but for convenience assume that it lies between  $x_1$  and  $x_3$ . (The case that  $e_\lambda^2$  is less than  $x_1$  will be discussed later: See Sec. III G.) Now if one solves the renormalization-group equation, one sees from Fig. 3 that  $e_\lambda^2$  rapidly approaches  $x_2$  as  $\lambda$  decreases. In fact, one sees that for  $\lambda \ll \Lambda$ ,  $e_\lambda^2$  has the form

$$e_\lambda^2 \simeq x_2 + c_2(\lambda/\Lambda)^{2a_2}, \quad (3.35)$$

where  $a_2$  is given by Eq. (3.31) and  $c_2$  is a constant of order 1 since  $e_\lambda^2$  differs from  $x_2$  by of order 1. In particular,  $e_m^2$  has the form

$$e_m^2 \simeq x_2 + c_2(m/\Lambda)^{2a_2}. \quad (3.36)$$

This means  $e_m^2$  is very little affected by the exact value of  $e_\lambda^2$ ; the dependence of  $e_m^2$  on  $e_\lambda^2$  is through the constant  $c_2$  and this constant is multiplied by the small coefficient  $(m/\Lambda)^{2a_2}$ . This in turn means that the ordinary renormalized coupling constant  $e^2$  is also only slightly dependent on the value of  $e_\lambda^2$ . This assumes that  $e_\lambda^2$  is restricted to the range  $x_1 < e_\lambda^2 < x_3$ ; if  $e_\lambda^2$  lies outside this range,  $e_m^2$  will be very different.

<sup>27</sup> If  $k^2 \ll m^2$  the factor is  $[\ln(m^2/\lambda^2)]^n$  (and similarly if  $\lambda^2 \ll m^2$ ).

<sup>28</sup> In this section, as in Sec. III D,  $e_\lambda$  refers to a coupling constant for strong interactions.

The consequence of introducing the cutoff  $\Lambda$  and requiring  $e_\Lambda^2$  to be arbitrary is that  $e_m^2$  and therefore  $e^2$  is fixed precisely except for corrections which are small when  $m/\Lambda$  is small. In other words, there is a bootstrap condition for the renormalized coupling constant, apart from small cutoff-dependent effects. The bootstrap condition is *not* a consistency condition: The field theory of strong interactions in isolation exists and is (by assumption) well behaved for any value of  $e$ ; it is the influence of other interactions at the cutoff  $\Lambda$  that forces  $e$  to be the solution of the bootstrap.

### F. Precise Formulation of Bootstrap Condition

The bootstrap condition will now be discussed in more detail.<sup>23</sup> Consider the exact equation for  $e_\lambda^2$ , Eq. (2.65), with finite mass. The equation is to be solved with the value of  $e_\Lambda^2$  being the boundary condition. It will be assumed that  $e_\Lambda^2$  lies somewhere between  $x_1$  and  $x_3$ ; what happens when  $e_\Lambda^2$  is outside this range is discussed later in this section. It is also assumed that all cutoff-dependent effects are absorbed into the constant  $e_\Lambda^2$  and do not, for example, change the function  $\psi(m^2/\lambda^2, e_\lambda^2)$ . This assumption is surely an oversimplification, but the essential features of the bootstrap are not changed by making this simplification. It will also be assumed that  $\Lambda/m$  is sufficiently large so that over a sizable range of  $\lambda$ ,  $m/\lambda$  and  $\lambda/\Lambda$  are both small. In the range  $\lambda < \Lambda$  but  $\lambda \gg m$ , the exact function  $\psi(m^2/\lambda^2, e_\lambda^2)$  is approximately  $\psi(0, e_\lambda^2)$ ; therefore, the solution  $e_\lambda^2$  behaves as shown in Fig. 3 and is close to  $x_2$  for  $\lambda \ll \Lambda$ . This suggests that one define a solution of the finite-mass equation which is exactly  $x_2$  when  $\lambda \gg m$ , and compare this solution with the exact solution. So one has two functions, say,  $e_\lambda$  and  $e_{\lambda 1}$ , where

$$de_\lambda^2/d(\ln \lambda^2) = \psi(m^2/\lambda^2, e_\lambda^2), \quad (3.37)$$

$$de_{\lambda 1}^2/d(\ln \lambda^2) = \psi(m^2/\lambda^2, e_{\lambda 1}^2), \quad (3.38)$$

and  $e_{\lambda 1}^2$  satisfies the boundary condition

$$\lim_{\lambda \rightarrow \infty} e_{\lambda 1}^2 = x_2, \quad (3.39)$$

while the boundary condition for  $e_\lambda^2$  is the value of  $e_\Lambda^2$ . The functions  $e_\lambda^2$  and  $e_{\lambda 1}^2$  are illustrated in Fig. 4.

The functions  $e_\lambda^2$  and  $e_{\lambda 1}^2$  are both close to  $x_2$  in the range  $\lambda \gg m$  but  $\lambda \ll \Lambda$ . This means that  $e_\lambda^2$  is close to  $e_{\lambda 1}^2$  for  $\lambda$  of order  $m$  also because the differential equation deamplifies the difference between  $e_\lambda^2$  and  $e_{\lambda 1}^2$  as  $\lambda$  decreases (see below). The advantage of defining  $e_{\lambda 1}$  is that it has no cutoff dependence. It is also uniquely defined since  $e_{\lambda 1}$  must satisfy the zero-mass renormalization-group equation for large  $\lambda$ , and only one solution of the zero-mass equation goes to  $x_2$  when  $\lambda \rightarrow \infty$ . This is a situation reminiscent of the one-dimensional Schrödinger equation in the bound-state region; the Schrödinger equation has one solution which goes to zero at infinity, while it has many solutions which blow

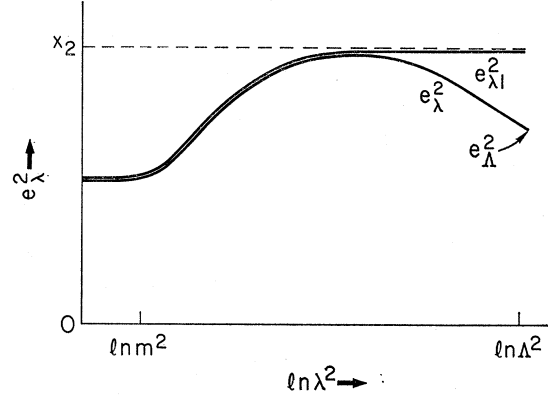


FIG. 4. Solutions  $e_\lambda^2$  and  $e_{\lambda 1}^2$  of the finite-mass renormalization-group equation, with boundary conditions  $e_\Lambda^2$  for  $e_\lambda^2$  and  $e_{\lambda 1}^2 \rightarrow x_2$  for  $\lambda \rightarrow \infty$ .

up at infinity. Since  $e_{\lambda 1}$  can depend only on  $\lambda$  and  $m$  and is dimensionless, it must have the form

$$e_{\lambda 1}^2 = \xi(\lambda^2/m^2) \quad (3.40)$$

for some function  $\xi$ . In particular, the value of  $e_{\lambda 1}^2$  for  $\lambda=0$  is  $\xi(0)$ , which is a fixed number independent of the mass  $m$ .

As long as  $e_\lambda^2$  is near  $e_{\lambda 1}^2$  one can write a linearized equation for the difference  $e_\lambda^2 - e_{\lambda 1}^2$ :

$$\frac{d(e_\lambda^2 - e_{\lambda 1}^2)}{d(\ln \lambda^2)} = \frac{\partial \psi}{\partial x} \left( \frac{m^2}{\lambda^2}, e_{\lambda 1}^2 \right) (e_\lambda^2 - e_{\lambda 1}^2). \quad (3.41)$$

When  $\lambda \gg m$  but  $\lambda \ll \Lambda$ , one has  $e_{\lambda 1}^2 \simeq x_2$  and  $m^2/\lambda^2 \ll 1$ , giving  $\partial \psi / \partial x \simeq a_2$ . Since  $a_2$  is positive, the difference  $e_\lambda^2 - e_{\lambda 1}^2$  decreases as  $\lambda$  decreases, except that for  $\lambda \sim m$ ,  $\partial \psi / \partial x$  might change sign due to finite-mass effects. However, large amplification cannot occur for  $\lambda \lesssim m$  (see Sec. III B) so even if  $\partial \psi / \partial x$  does change sign for  $\lambda \sim m$ , the difference  $e_\lambda^2 - e_{\lambda 1}^2$  cannot become large.

More precisely, for  $m \ll \lambda \ll \Lambda$  one has

$$e_\lambda^2 - e_{\lambda 1}^2 \simeq c_2 (\lambda/\Lambda)^{2a_2} \quad (\lambda \gg m, \lambda \ll \Lambda) \quad (3.42)$$

for some constant  $c_2$ . Using this as a boundary condition, the solution of Eq. (3.41) is

$$e_\lambda^2 = e_{\lambda 1}^2 + c_2 \exp \left[ - \int_{\lambda^2}^{\Lambda^2} u \left( \frac{\lambda'^2}{m^2} \right) (\lambda'^2)^{-1} d(\lambda'^2) \right], \quad (3.43)$$

with

$$u \left( \frac{\lambda^2}{m^2} \right) = \frac{\partial \psi}{\partial x} \left[ \frac{m^2}{\lambda^2}, \xi \left( \frac{\lambda^2}{m^2} \right) \right]. \quad (3.44)$$

When  $\lambda^2 \gg m^2$ ,  $\xi(\lambda^2/m^2)$  is approximately  $x_2$ ,  $u(\lambda^2/m^2)$  is approximately  $a_2$ , and Eq. (3.43) reduces to Eq. (3.42). When  $\lambda^2$  is 0, one has

$$e^2 = e_{01}^2 + c_2 \exp \left[ - \int_0^{\Lambda^2} (\lambda'^2)^{-1} u \left( \frac{\lambda'^2}{m^2} \right) d\lambda'^2 \right]. \quad (3.45)$$

The large part of the integral in the exponent comes from the region of integration  $m \ll \lambda' \ll \Lambda$ . The result is

$$e^2 \simeq e_{01}^2 + c_2 (m^2/\Lambda^2)^{a_2} w, \quad (3.46)$$

where  $w$  is a constant, namely,

$$w = \exp \left\{ - \int_0^{m^2} (\lambda')^{-1} u \left( \frac{\lambda'^2}{m^2} \right) d\lambda'^2 - \int_{m^2}^{\Lambda^2} \lambda'^2 \left[ u \left( \frac{\lambda'^2}{m^2} \right) - a_2 \right] d\lambda'^2 \right\}. \quad (3.47)$$

The function  $u(\lambda'^2/m^2) - a_2$  goes to zero as  $\lambda' \rightarrow \infty$ ; it is assumed here that it goes to zero fast enough so that the region  $\lambda'^2 > \Lambda^2$  is negligible. The quantity  $e_{01}^2$  is  $\xi(0)$  and is the bootstrap value of the renormalized coupling constant; the actual value for  $e^2$  differs from the bootstrap value by an amount of order  $(m^2/\Lambda^2)^{a_2}$ .

Note that the bootstrap value of the renormalized coupling constant is computed by a complex procedure. First one must locate the fixed point  $x_2$  of the zero-mass renormalization-group equation. This means solving the equation  $\psi(0,x)=0$ , which is the bootstrap condition. Having found the solution  $x_2$ , one must then solve the differential equation for  $e_{\lambda 1}^2$  with the boundary condition  $e_{\lambda 1}^2 \rightarrow x_2$  as  $\lambda \rightarrow \infty$ ; the value of  $e_{\lambda 1}^2$  for  $\lambda=0$  is the bootstrap value of the renormalized coupling constant. The bootstrap condition is a condition on a zero-mass theory, a theory which very likely has no  $S$  matrix because of infrared divergences.<sup>25</sup> The bootstrap condition is determined by the function  $\psi$ , which is in turn defined in terms of a propagator well off the mass shell; this is a very different bootstrap from those that have been proposed in the context of  $S$ -matrix theory.

For momenta between  $m$  and  $\Lambda$  the theory is approximately scale invariant, since  $e_{\lambda}^2$  is close to the fixed point  $x_2$  and the mass can be neglected. However, the scale-invariant theory differs from the scale-invariant theory associated with the fixed point  $x_1$ ; for example, the scale dimension of the field  $\psi$  is different in the two cases since  $\sigma(0,x_2)$  need not be the same as  $\sigma(0,x_1)$ .

Whether or not there is a cutoff  $\Lambda$ , the short-distance behavior of strong interactions is determined by a fixed point of the renormalization group. However, a different fixed point is involved if there is a cutoff. What is the difference? The fixed point  $x_2$  has the property that solutions of the zero-mass renormalization-group equation in the vicinity of  $x_2$  approach  $x_2$  as  $\lambda \rightarrow 0$ . In contrast, solutions in the vicinity of  $x_1$  or  $x_3$  for finite  $\lambda$  go away from  $x_1$  or  $x_3$  as  $\lambda \rightarrow 0$ . One can say that  $x_2$  is "infrared-stable," while  $x_1$  and  $x_3$  are "infrared unstable." With a cutoff, one has to allow  $e_{\Lambda}^2$  to be arbitrary which means it is unlikely to be equal to  $x_1$  or  $x_3$ , or to any infrared-unstable fixed point. Hence the solution of the renormalization-group equation will approach one of the infrared-stable fixed points of

$\psi(0,x)$ . The infrared-stable fixed points are  $x=0$ ,  $x=x_2$ , plus possibly roots beyond  $x_3$ . If  $e_{\Lambda}^2$  lies between  $x_1$  and  $x_3$ , then  $x_2$  is the relevant fixed point; if  $e_{\Lambda}^2$  is less than  $x_1$ , then  $x=0$  is the relevant fixed point. The significance of  $x=0$  as a fixed point will be discussed in Sec. III G. What happens if  $e_{\Lambda}^2 > x_3$  will not be discussed here. Without a cutoff it is the renormalized coupling constant  $e^2$  that is arbitrary; one then follows solutions of the differential equation out to  $\lambda = \infty$ , and it is the fixed points which are stable in this limit that become the possible asymptotic solutions of  $e_{\lambda}^2$  for large  $\lambda$ . The possible fixed points are  $x_1$  and  $x_3$ ; these might be called "ultraviolet-stable" fixed points.

The bootstrap condition resulting from the presence of a cutoff is, precisely, that  $\psi(0,x)$  be zero and that  $x$  be an infrared-stable root. Any infrared-stable root is acceptable, so if there is more than one, the correct one has to be determined experimentally.

### G. Electrodynamics with Cutoff

To conclude the discussion of fixed points, the case of electrodynamics will be considered. Now  $e$ ,  $e_{\lambda}$ , and  $\psi$  refer to electrodynamics rather than to strong interactions. The same form is assumed for  $\psi$  as in Sec. III B. While the renormalized coupling constant  $e^2$  of electrodynamics is small, one sees from Fig. 3 that  $e_{\lambda}^2 \rightarrow x_1$  for  $\lambda \rightarrow \infty$ . The constant  $x_1$  is fixed independent of  $e^2$  and so cannot be arbitrarily small. This suggests that all particles will couple strongly to photons at sufficiently high momenta; but this would mean that electrodynamics and strong interactions would mix strongly, suggesting that pure electrodynamics is valid only below a cutoff momentum  $\Lambda$ . Suppose this is the case and that  $e_{\Lambda}^2$  is therefore arbitrary as discussed earlier, but happens to be smaller than  $x_1$ . Then for  $\lambda \ll \Lambda$ ,  $e_{\lambda}^2$  is close to the fixed point zero. One can as before define a bootstrap value for the renormalized coupling constant by solving the renormalization-group equation for a function  $e_{\lambda 1}^2$  which goes to zero as  $\lambda \rightarrow \infty$ . The solution is  $e_{\lambda 1}^2 \equiv 0$ . The departure of  $e^2$  from 0 is therefore a cutoff-dependent effect, as discussed earlier. Because of the special nature of the fixed point  $x=0$ ,  $e^2$  does not vary as a power of the cutoff. The function  $\psi(0,x)$  has no term linear in  $x$  for  $x$  near zero, so one cannot find the dependence of  $e^2$  on  $\Lambda$  from a linearized equation. Instead one must keep the quadratic term in  $\psi$ , giving the approximate equation for  $e_{\lambda}^2$  small:

$$d e_{\lambda}^2 / d (\ln \lambda^2) = (12\pi^2)^{-1} e_{\lambda}^4. \quad (3.48)$$

[This equation neglects all contributions to vacuum polarization except for electrons; other particles will increase the factor  $(12\pi^2)^{-1}$  by a presently unknown factor.] The boundary condition is that  $e_{\Lambda}^2$  be arbitrary which presumably means of order 1. The solution is

$$(e_{\lambda}^2)^{-1} - (e_{\Lambda}^2)^{-1} = (12\pi^2)^{-1} \ln(\Lambda^2/\lambda^2) \quad (3.49)$$

or

$$e_{\lambda}^2 = e_{\Lambda}^2 [1 + (12\pi^2)^{-1} e_{\Lambda}^2 \ln(\Lambda^2/\lambda^2)]^{-1}. \quad (3.50)$$

For  $\lambda \ll \Lambda$ , for which  $e^2 \Lambda \ln(\Lambda^2/\lambda^2)$  is large, one gets the approximate form

$$e_{\lambda^2} \approx (12\pi^2) [\ln(\Lambda^2/\lambda^2)]^{-1}. \quad (3.51)$$

So  $e_{\lambda^2}$  decreases as an inverse logarithm of  $\lambda^2$  for  $\lambda \ll \Lambda$ . One has to have an astronomical value of  $\Lambda/m$  to make the renormalized coupling constant  $e^2$  be as small as  $1/137$ .

In the case of electrodynamics, the small cutoff dependence of the coupling constant is very noticeable since it is only the cutoff-dependent term that makes  $e^2$  nonzero. For strong interactions, where the bootstrap value of the renormalized coupling constant should be of order 1, a small cutoff dependence would be much harder to detect experimentally.

### H. Limit Cycles

If there are at least two renormalized coupling constants in strong interactions, as in pseudoscalar meson theory, there is an intriguing alternative to a fixed point, namely, a limit cycle.<sup>8,17</sup> To illustrate the hypothesis of a limit cycle, suppose there are two renormalized coupling constants; then the zero-mass renormalization-group equations have the form

$$dx/dt = \psi_1(x, y), \quad (3.52)$$

$$dy/dt = \psi_2(x, y), \quad (3.53)$$

where the functions  $\psi_1$  and  $\psi_2$  are analogous to the function  $\psi(0, x)$ ,  $t = \ln \lambda^2$ , and  $x$  and  $y$  are the momentum-dependent coupling constants, say,  $x = g_{\lambda}$ ,  $y = h_{\lambda}$ . The functions  $\psi_1$  and  $\psi_2$  will be mass dependent when  $\lambda$  is of order the masses in the theory, but this dependence will not be exhibited explicitly. It is assumed that the mass dependence can be neglected for large enough  $\lambda$ .

The solutions of Eqs. (3.52) and (3.53) will define trajectories in a two-dimensional space with coordinates  $x$  and  $y$ .

A limit cycle is a special trajectory which is a closed orbit, namely, a solution  $\{x(t), y(t)\}$  which satisfies

$$x(t+\tau) = x(t) \quad (3.54)$$

and

$$y(t+\tau) = y(t), \quad (3.55)$$

where  $\tau$  is a constant giving the period of the limit cycle. Generally the trajectories in the neighborhood of a limit cycle are not closed. Instead they either approach the cycle as  $t \rightarrow \infty$ , or they move away from the cycle as  $t$  increases, in neither case closing on themselves. There are general conditions on the functions  $\psi_1$  and  $\psi_2$  which ensure the existence of a limit cycle without determining  $\tau$  or the exact form of the cycle.<sup>8</sup> For instance, if one can find an annulus with the property that trajectories can go into the annulus but not out of it (that is, not cross out of the outer ring nor cross inside the inner ring of the annulus) then there is a limit cycle contained in the annulus. The condition that

trajectories only go into the annulus is easily checked since this means that the velocity vector  $\{\psi_1(x, y), \psi_2(x, y)\}$  must point into the annulus for all points  $\{x, y\}$  on the inner and outer rings of the annulus. For this theorem to hold, there must not be any fixed points inside the annulus.

A detailed discussion of limit cycles will not be given here. There is one important observation to make. If the coupling constants  $g_{\lambda}$  and  $h_{\lambda}$  approach a limit cycle as  $\lambda \rightarrow \infty$  instead of a fixed point, there is a chance that this will be experimentally observable. Consider the total cross section  $\sigma_{\text{tot}}(q^2)$  for  $e^+e^-$  annihilation into hadrons:  $q$  is the momentum transfer. Assuming electrodynamics is treated to lowest order,  $\sigma_{\text{tot}}(q^2)$  is the absorptive part of the propagator for the electromagnetic current of hadrons, times known factors. By dimensional analysis,  $\sigma_{\text{tot}}(q^2)$  has the form

$$\sigma_{\text{tot}}(q^2) = \alpha(q^2)^{-1} f(q^2/\lambda^2, m/\lambda, g_{\lambda}, h_{\lambda}), \quad (3.56)$$

where  $m$  stands for all possible mass parameters in the theory and  $\alpha$  is the fine-structure constant. The mass dependence should be negligible for  $q^2$  and  $\lambda^2$  large. The formula is valid for any value of  $\lambda$ . In particular, one can set  $\lambda^2 = q^2$ ; if  $q^2$  is large,  $m^2/q^2$  is negligible and

$$\sigma_{\text{tot}}(q^2) = \alpha(q^2)^{-1} f(1, 0, g_q, h_q). \quad (3.57)$$

It is unlikely that  $f$  is independent of  $g_{\lambda}$  and  $h_{\lambda}$ , since vacuum polarization in electrodynamics is also the current-current propagator in electrodynamics and it is coupling-constant dependent. So as  $g_q$  and  $h_q$  change with  $q^2$ , so will  $q^2 \sigma_{\text{tot}}(q^2)$  and one will see perpetual oscillations in the  $e^+e^-$  total hadronic cross section in the limit of large  $q^2$ .

There are other forms of asymptotic behavior besides fixed points and limit cycles. There will be no discussion of further alternatives here. The example of the limit cycle suggests that any asymptotic behavior other than a fixed point will mean that  $q^2 \sigma_{\text{tot}}(q^2)$  will not approach a constant in the limit of large  $q^2$ .

### I. Conclusions and Remarks

There are three basic results of this section. First, if the asymptotic solution of the renormalization-group equations for strong interactions is a fixed point, then strong interactions will have broken scale invariance as a symmetry. Second, if in addition there is a large but finite cutoff  $\Lambda$  above which strong interactions cannot be isolated from other interactions, the fixed point must be infrared stable and there is a bootstrap condition which determines renormalized coupling constants of strong interactions. Third, if the asymptotic solution of the renormalization-group equations is not a fixed point, then  $q^2 \sigma_{\text{tot}}(q^2)$  for  $e^+e^-$  annihilation will not be constant for large  $q^2$ ; if there is a limit cycle in particular, then  $q^2 \sigma_{\text{tot}}(q^2)$  will oscillate perpetually for large  $q^2$ , with a fixed period if plotted vs  $\ln q^2$ .



All these results are crucially dependent on the assumption that functions such as  $\psi(m^2/\lambda^2, x)$  have a nontrivial limit for  $m \rightarrow 0$ . If instead  $\psi(m^2/\lambda^2, x)$  were to go to zero as  $m \rightarrow 0$  for all  $x$ , the analysis given above would collapse and the asymptotic limit of  $e_\lambda^2$  for  $\lambda \rightarrow \infty$  would be a straightforward function of  $e^2$ . If in perturbation theory  $\psi(m^2/\lambda^2, x)$  were to contain logarithms of  $m^2/\lambda^2$  it would be easy for the sum of the perturbation series to approach zero as  $m \rightarrow 0$ , for example, if the sum of the series gave something like

$$\psi(m^2/\lambda^2, e_\lambda^2) = e_\lambda^4 \exp[-e_\lambda^2 \ln(\lambda^2/m^2)] \quad (3.58)$$

(a more complicated example is needed if one is to fit the known term of order  $e_\lambda^6$ ). This does not mean that proving the existence of the zero-mass limit for  $\psi(m^2/\lambda^2, x)$  to all orders in  $x$  is the crucial problem. The crucial problem is to determine whether the physics of a strongly coupled field theory is such that the results obtained above are reasonable.

#### IV. BREAKING OF SCALE INVARIANCE

In Sec. III it was found that the fixed-point solutions of the renormalization-group equations for zero mass define scale-invariant field theories. For the specific type of function  $\psi(0, x)$  discussed in Sec. III, there are at least three distinct scale-invariant theories defined by the fixed points  $x_1$ ,  $x_2$ , and  $x_3$ . The purpose of this section is to study the nature of scale-invariance breaking. One can break scale invariance in two ways. One way is to let the mass be finite instead of zero. The other way is to choose a solution  $e_\lambda^2$  of the zero-mass renormalization-group equations which is not equal to a fixed point. Then the variation of  $e_\lambda^2$  with  $\lambda$  breaks scale invariance, as will be seen below. Both these forms of scale-invariance breaking will be investigated in this section. In particular, we shall study amplitudes at high momenta for which the scale-invariance breaking is small and develop a perturbation method for computing the terms which break scale invariance.

A theory of scale-invariance breaking has been proposed elsewhere.<sup>24</sup> In this theory the nature of scale-invariance-breaking corrections at large momenta is determined by the scale dimensions of the terms in the Lagrangian density which break scale invariance. The interaction Lagrangian density associated with the renormalized charge  $e_\lambda$  should be a renormalized form of the local product  $\bar{\psi}_\lambda(x)\gamma_\mu\psi_\lambda(x)A_\lambda^\mu(x)$ ; the interaction Lagrangian density associated with the renormalized mass should be a renormalized form of the product  $\bar{\psi}_\lambda(x)\psi_\lambda(x)$ . The problem of renormalizing composite fields such as  $\bar{\psi}_\lambda(x)\gamma_\mu\psi_\lambda(x)A_\lambda^\mu(x)$  and showing that these fields are connected with the renormalized coupling constant and mass will not be discussed here. It will simply be assumed that the part of the Lagrangian density which determines the charge  $e_\lambda$  and the mass  $m$  involves two local interactions denoted  $\mathcal{L}_{e\lambda}(x)$  and  $\mathcal{L}_{m\lambda}(x)$ , respectively.

Explicit calculations (based on the renormalization-group equations) of scale-invariance-breaking corrections will be compared with the general theory of Ref. 24. One result will be that the scale dimension of  $\mathcal{L}_{e\lambda}(x)$  is  $4+2a_n$  in the scale-invariant theory defined by the fixed point  $x_n$ . The constant  $a_1$  is negative and the constant  $a_2$  is positive, so the dimension of  $\mathcal{L}_{e\lambda}(x)$  can be either less than 4 or greater than 4 depending on which fixed point one uses. It was pointed out in Ref. 24 that interactions with dimension  $>4$  act like nonrenormalizable interactions while interactions with dimension  $<4$  act like super-renormalizable interactions or mass terms. (Interactions with dimension  $<4$  were called generalized mass terms in Ref. 24.) So the interaction  $\mathcal{L}_{e\lambda}(x)$  changes its character considerably when one switches from the fixed point  $x_1$  to the fixed point  $x_2$ .

To discuss the dimension of  $\mathcal{L}_{m\lambda}(x)$  will require an extension of the renormalization-group equations to include a  $\lambda$ -dependent mass parameter.<sup>29</sup> The result will be to conclude that  $\mathcal{L}_{m\lambda}(x)$  could be either a generalized mass term or a nonrenormalizable interaction at either  $x_1$  or  $x_2$ , the choice being determined by detailed dynamics which are unknown at present.

#### A. Scale-Invariance Breaking through Nonconstant $e_\lambda$

To start with, scale-invariance breaking due to nonconstant  $e_\lambda$  will be considered. The mass  $m$  is taken to be zero. Suppose that over some range of  $\lambda$ ,  $e_\lambda^2$  is close to the fixed point  $x_n$ . As shown in Sec. III D, the approximate form for  $e_\lambda^2$  as long as  $e_\lambda^2$  is close to  $x_n$  is

$$e_\lambda^2 = x_n + c_n \lambda^{2a_n}, \quad (4.1)$$

where  $c_n$  is an arbitrary small constant and  $a_n$  is given by Eq. (3.31). Equation (4.1) neglects terms quadratic in the difference  $e_\lambda^2 - x_n$ , i.e., terms of order  $c_n^2$ . Expansions will now be sought for  $z_{3\lambda\lambda'}$ ,  $z_{2\lambda\lambda'}$ , and vacuum expectation values, also valid to first order in  $c_n$ . Given these expansions, one can construct scaling laws for first-order scale-invariance-breaking terms in the vacuum expectation values.

Since  $z_{3\lambda\lambda'} = e_\lambda^2/e_{\lambda'}^2$  [Eq. (2.53)], one has

$$z_{3\lambda\lambda'} \simeq [x_n + c_n (\lambda')^{2a_n}] [x_n + c_n \lambda^{2a_n}]^{-1} \simeq 1 + (x_n)^{-1} c_n [(\lambda')^{2a_n} - \lambda^{2a_n}]. \quad (4.2)$$

The equation for  $z_{2\lambda\lambda'}$  is [cf. Eqs. (2.73) and (2.50)]

$$\frac{\partial z_{2\lambda\lambda'}}{\partial(\ln \lambda^2)} = z_{2\lambda\lambda'} \sigma(0, e_\lambda^2), \quad (4.3)$$

with the boundary condition

$$z_{2\lambda'\lambda'} = 1. \quad (4.4)$$

<sup>29</sup> This extension was suggested by K. E. Eriksson, Nuovo Cimento **30**, 1423 (1963); see also M. Astaud and B. Jovet (Ref. 4).

The solution of this equation is

$$z_{2\lambda\lambda'} = \exp \left[ \int_{\lambda'}^{\lambda} \sigma(0, e_{\lambda''}) (\lambda'')^{-1} 2d\lambda'' \right]. \quad (4.5)$$

If

$$\sigma_n = \sigma(0, x_n), \quad (4.6)$$

$$\tau_n = \frac{\partial \sigma(0, x_n)}{\partial x}, \quad (4.7)$$

then to first order in  $c_n$

$$z_{2\lambda\lambda'} \simeq \exp \{ \sigma_n \ln(\lambda^2/\lambda'^2) + (\tau_n c_n/a_n) [\lambda^{2a_n} - (\lambda')^{2a_n}] \} \\ \simeq (\lambda/\lambda')^{2\sigma_n} \{ 1 + (\tau_n c_n/a_n) [\lambda^{2a_n} - (\lambda')^{2a_n}] \}. \quad (4.8)$$

Consider now a typical vacuum expectation value, e.g.,

$$\tau_{\mu\nu}(z_1\lambda, z_2\lambda, z_3\lambda, z_4\lambda, m/\lambda, e_{\lambda^2}) \\ = \lambda^{-5} \langle \Omega | T A_{\lambda\mu}(z_1) A_{\lambda\nu}(z_2) \psi_{\lambda}(z_3) \bar{\psi}_{\lambda}(z_4) | \Omega \rangle. \quad (4.9)$$

( $T_{\mu\nu}$  is dimensionless so it depends only on dimensionless variables, as indicated.) Putting  $m=0$  and using the renormalization relations, one has<sup>30</sup>

$$T_{\mu\nu}(z_1\lambda, z_2\lambda, z_3\lambda, z_4\lambda, 0, e_{\lambda^2}) \\ = \lambda^{-5} z_{3\lambda 1} z_{2\lambda 1} T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, e_1^2). \quad (4.10)$$

Write

$$T_{\mu\nu}(z_1, z_2, z_3, z_4) = T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, x_n), \quad (4.11)$$

$$U_{\mu\nu}(z_1, z_2, z_3, z_4) = \left. \frac{\partial T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, x)}{\partial x} \right|_{x=x_n}. \quad (4.12)$$

Then to first order in  $c_n$

$$T_{\mu\nu}(z_1, z_2, z_3, z_4, 0, e_1^2) \simeq T_{\mu\nu}(z_1, z_2, z_3, z_4) \\ + c_n U_{\mu\nu}(z_1, z_2, z_3, z_4) \quad (4.13)$$

and similarly for  $T_{\mu\nu}(z_1\lambda, \dots, z_4\lambda, 0, e_{\lambda^2})$ . Equation (4.10), to first order in  $c_n$ , is

$$T_{\mu\nu}(z_1\lambda, \dots, z_4\lambda) + c_n \lambda^{2a_n} U_{\mu\nu}(z_1\lambda, \dots, z_4\lambda) \\ = \lambda^{2\sigma_n - 5} [1 + c_n (\tau_n a_n^{-1} - x_n^{-1}) (\lambda^{2a_n} - 1)] \\ \times T_{\mu\nu}(z_1, \dots, z_4) + \lambda^{2\sigma_n - 5} c_n U_{\mu\nu}(z_1, \dots, z_4). \quad (4.14)$$

For  $c_n=0$  this gives the scaling law found in Sec. III C:

$$T_{\mu\nu}(\lambda z_1, \dots, \lambda z_4) = \lambda^{2\sigma_n - 5} T_{\mu\nu}(z_1, \dots, z_4). \quad (4.15)$$

The terms proportional to  $c_n$  in Eq. (4.14) give the following result. Let

$$U_{\mu\nu}(z_1, \dots, z_4) = U_{\mu\nu}'(z_1, \dots, z_4) \\ + (\tau_n a_n^{-1} - x_n^{-1}) T_{\mu\nu}(z_1, \dots, z_4). \quad (4.16)$$

Then

$$U_{\mu\nu}'(\lambda z_1, \dots, \lambda z_4) = \lambda^{2\sigma_n - 2a_n - 5} U_{\mu\nu}'(z_1, \dots, z_4). \quad (4.17)$$

One can now derive to order  $c_n$  the scaling law for  $T_{\mu\nu}(z_1, \dots, z_4, 0, e_{\lambda^2})$  with  $\lambda$  held fixed at  $\lambda=1$ , i.e.,  $e_{\lambda^2}$

fixed at  $e_1^2$ :

$$T_{\mu\nu}(s z_1, \dots, s z_4, 0, e_1^2) \\ = s^{2\sigma_n - 5} [1 + c_n (\tau_n a_n^{-1} - x_n^{-1})] T_{\mu\nu}(z_1, \dots, z_4) \\ + c_n s^{2\sigma_n - 2a_n - 5} U_{\mu\nu}'(z_1, \dots, z_4). \quad (4.18)$$

Equation (4.18) shows that the first-order change in  $T_{\mu\nu}(z_1, \dots, z_4, 0, e_1^2)$  when  $e_1^2 \neq x_n$  is to renormalize the scale-invariant term  $T_{\mu\nu}(z_1, \dots, z_4)$  by the factor  $[1 + c_n (\tau_n a_n^{-1} - x_n^{-1})]$  and to add the term

$$c_n U_{\mu\nu}'(z_1, \dots, z_4)$$

which obeys a separate scaling law. It is easily seen that when the above analysis is generalized to an arbitrary vacuum expectation value, there is a renormalization which is equivalent to a renormalization of the fields:

$$A_{1\mu}(z) \rightarrow (1 - \frac{1}{2} c_n x_n^{-1}) A_{1\mu}(z), \quad (4.19)$$

$$\psi_1(z) \rightarrow (1 + \frac{1}{2} c_n \tau_n a_n^{-1}) \psi_1(z), \quad (4.20)$$

and there is an extra term which always scales by an extra factor  $s^{-2a_n}$  relative to the scale-invariant term.

Now consider the Lagrangian description of scale-invariance breaking.<sup>31</sup> For convenience the subtraction momentum  $\lambda$  will be set equal to 1. Assume that changing the coupling constant from  $x_n$  to  $e_1^2$  is equivalent to adding a term  $(e_1^2 - x_n) \mathcal{L}_{e_1}(z)$  to the Lagrangian density, where  $\mathcal{L}_{e_1}(z)$  is a finite local field. Then the term of first order in  $c_n$  in  $T_{\mu\nu}$  can be obtained from lowest-order perturbation theory:

$$T_{\mu\nu 1}(z_1, \dots, z_4, 0, e_1^2) \simeq T_{\mu\nu}(z_1, \dots, z_4) \\ + c_n \int \langle \Omega | T A_{1\mu}(z_1) A_{1\nu}(z_2) \\ \times \psi_1(z_3) \bar{\psi}_1(z_4) \mathcal{L}_{e_1}(y) | \Omega \rangle d^4 y, \quad (4.21)$$

where [from Eq. (4.1)]  $e_1^2 - x_n$  has been replaced by  $c_n$  and the vacuum expectation value multiplying  $c_n$  is computed in the unperturbed (scale-invariant) theory. Ultraviolet divergences could arise in the integral owing to singularities when  $y = z_1, z_2, z_3$ , or  $z_4$ ; an infrared divergence could occur for  $y \rightarrow \infty$ . It will be assumed here that these divergences are absent or unimportant. If so, the scaling properties of the integral are determined by scale invariance. Write

$$W_{\mu\nu}(z_1, \dots, z_4, y) \\ = \langle \Omega | T A_{1\mu}(z_1) A_{1\nu}(z_2) \psi_1(z_3) \bar{\psi}_1(z_4) \mathcal{L}_{e_1}(y) | \Omega \rangle. \quad (4.22)$$

The scale dimension of  $A_{1\mu}$  is 1; the scale dimension of  $\psi_1$  is  $\frac{3}{2} - \sigma_n$  (cf. Sec. III C); let the scale dimension of  $\mathcal{L}_{e_1}$  be  $d_e$ . Then from scale invariance

$$W_{\mu\nu}(s z_1, \dots, s y) = s^{2\sigma_n - 5 - d_e} W_{\mu\nu}(z_1, \dots, y). \quad (4.23)$$

<sup>30</sup> For convenience it is assumed that  $\lambda=1$  is included in the range of  $\lambda$  for which  $e_{\lambda^2} \sim x_n$ .

<sup>31</sup> What follows is a nonrigorous argument deriving the scale-invariance breaking theory of Ref. 24 to first order.

Hence the scaling behavior of the integral in Eq. (4.21) is

$$\begin{aligned} & \int d^4y W_{\mu\nu}(sz_1, \dots, sz_4, y) \\ &= s^4 \int d^4y' W_{\mu\nu}(sz_1, \dots, sz_4, sy') \\ &= s^{2\sigma_n - 5} s^{4-d_e} \int d^4y' W_{\mu\nu}(z_1, \dots, z_4, y'). \end{aligned} \quad (4.24)$$

So the prediction of the Lagrangian theory is that the term of order  $c_n$  in  $T_{\mu\nu}$  scales with an extra factor  $s^{4-d_e}$  relative to the scale-invariant term. There is no term of order  $c_n$  in Eq. (4.21) which renormalizes the scale-invariant term. However, we are using the unsubtracted form of perturbation theory which means one has no freedom to specify a normalization for the perturbed fields. In the renormalization-group calculation a normalization is specified for the perturbed fields, and to achieve this normalization one must expect to add a renormalization term to Eq. (4.21). Hence it is fair to interpret the explicit  $c_n$  term in Eq. (4.21) as corresponding to  $c_n U_{\mu\nu}'(z_1, \dots, z_4)$ . Comparing the scaling law for the  $c_n$  term of Eq. (4.21) with the scaling law for  $U_{\mu\nu}'$ , one must have

$$4-d_e = -2a_n, \quad (4.25)$$

i.e.,

$$d_e = 4 + 2a_n. \quad (4.26)$$

In Ref. 24, the theory of scale-invariance breaking was stated in the form of a simple rule. Applied to this first-order calculation, the rule is that if the coefficient  $c_n$  of  $\mathcal{L}_{e_1}$  is assigned the dimensions (mass) $^{4-d_e}$ , then the term of first order in  $c_n$  will have the same dimensions as the scale-invariant term. The calculation performed here confirms this rule. In Ref. 24 this rule was hedged in that the terms of first order (or higher) in  $c_n$  might involve logarithms that would spoil a strict scaling law. Logarithms might occur if the integral in Eq. (4.21) required subtractions. The renormalization-group calculation indicates that no logarithms occur in this case.

For the fixed point  $x_1$ ,  $a_1$  is negative and  $e_{\lambda^2}$  goes away from  $x_1$  as  $\lambda$  decreases. In this case the scale-invariance breaking should become important for low momenta, i.e., large distances. It is clear from the scaling law (4.18) that the scale-invariance breaking term does increase relative to the scale-invariant term as the scale length  $s$  increases. The dimension of  $\mathcal{L}_{e_1}(z)$  in this case is less than 4, which means that it is a generalized mass term in the language of Ref. 24. In contrast, for the fixed point  $x_2$ ,  $a_2$  is positive and the departure from scale invariance increases as one goes to large momenta or short distances. This is what one expects of a nonrenormalizable interaction. In this case  $\mathcal{L}_{e_1}(z)$  has dimension greater than 4, which means that

it is a nonrenormalizable interaction in the notation of Ref. 24.

If  $\mathcal{L}_{e_1}(z)$  is the interaction associated with the constant  $e_1$  in the Lagrangian, a term proportional to  $\mathcal{L}_{e_1}(z)$  should be present in the Lagrangian even for  $e_1 = x_n$ . But if  $\mathcal{L}_{e_1}(z)$  is a scale-invariance-breaking interaction, it obviously cannot be present in the Lagrangian of a scale-invariant theory. Evidently, the part of the Lagrangian which describes the scale-invariant theory must be distinguished from the term proportional to  $\mathcal{L}_{e_1}(z)$  which describes departures of  $e_1^2$  from  $x_n$ . This distinction must somehow arise in the process of defining the renormalized field  $\mathcal{L}_{e_1}(z)$ . It is difficult to study this problem in the context of this paper; it will not be discussed further.

In all the discussion of this paper it is assumed that none of the  $a_n$  are zero. For  $a_n$  to be zero means  $\psi(0, x)$  has a double root (at least) at  $x = x_n$  and the discussion of scale-invariance breaking is more complicated. Since the root of  $\psi(0, x)$  at  $x = 0$  is a double root, one has some experience with the scale-invariance breaking accompanying a double root, from ordinary perturbation theory. For general field theories a double root for  $x_n \neq 0$  seems unlikely and will not be discussed further.

## B. Breaking of Scale Invariance through Finite Mass

The next problem to be discussed is scale-invariance breaking due to nonzero mass. To analyze this problem it seems to be necessary to extend the renormalization group by defining a  $\lambda$ -dependent mass  $m_\lambda$  and obtaining a differential equation for  $m_\lambda$ . A method for doing this has been suggested by Ericksson.<sup>28</sup> The idea is to replace the subtraction condition on the mass shell for  $\Sigma_\lambda(p)$  [Eq. (2.19)] by a second condition at momentum equal to  $\lambda$ . This will mean essentially that the propagator  $S_\lambda(p)$  reduces to  $(p - m_\lambda)^{-1}$  when  $p^2 = \lambda^2$  so the mass parameter  $m_\lambda$  is defined in terms of the behavior of  $S_\lambda(p)$  for  $p \sim \lambda$ . The parameter  $m_\lambda$  is then used instead of  $m$  to parametrize the mass dependence of amplitudes renormalized at momentum  $\lambda$ . The precise form of the new subtraction condition for  $\Sigma_\lambda(p)$  has been chosen arbitrarily from many possibilities. The new subtraction condition is

$$B_\lambda(-\lambda^2) = 0 \quad (4.27)$$

[ $B_\lambda(p^2)$  was defined in Eq. (2.4)]. This condition replaces Eq. (2.19).

Because of the new condition for  $\Sigma_\lambda$ , the fields  $A_{\lambda\mu}$  and  $\psi_\lambda$  discussed below are different from the fields defined in Sec. II; also they and the functions  $d$ ,  $s$ , etc., are functions of  $m_\lambda^2/\lambda^2$  instead of  $m^2/\lambda^2$ . In addition to the function  $s(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_{\lambda^2})$ , it is convenient to define a second function

$$\begin{aligned} s_M(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_{\lambda^2}) = & [1 - A_\lambda(p^2)]^{-1} \\ & \times [1 + (m_\lambda)^{-1} B_\lambda(p^2)]. \end{aligned} \quad (4.28)$$

The relation of  $S_\lambda(p)$  to  $\Sigma_\lambda$  is

$$S_\lambda^{-1}(p) = \not{p} - m_\lambda - \Sigma_\lambda(p). \quad (4.29)$$

From this and the definitions of  $s$  and  $s_M$ , one gets

$$S_\lambda(p) = s(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2) \times [\not{p} - m_\lambda s_M(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2)]^{-1}. \quad (4.30)$$

From the renormalization conditions (2.20) and (4.27) one gets

$$s_M(-1, m_\lambda^2/\lambda^2, e_\lambda^2) = 1. \quad (4.31)$$

[The normalization conditions for  $s(-1, m_\lambda^2/\lambda^2, e_\lambda^2)$  and  $d(-1, m_\lambda^2/\lambda^2, e_\lambda^2)$  are that both equal 1, as before.]

Given the values of  $e_\lambda$  and  $m_\lambda$  and the functional form of  $s_M$  one can compute the physical mass  $m$  from the condition that  $S_\lambda(p)$  have a pole for  $\not{p} = m$ , namely, one must have

$$m = m_\lambda s_M(m^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2). \quad (4.32)$$

This is in general an implicit equation for  $m$ ; in perturbation theory it has a unique solution for  $m$  in the form

$$m = m_\lambda + (\text{power series in } e_\lambda^2), \quad (4.33)$$

where the power series begins with a term of order  $e_\lambda^2$ .

The renormalization-group differential equations can now be derived as in Sec. II. The equations for  $e_\lambda^2$ ,  $z_{3\lambda}$ , and  $z_{2\lambda}$  are unchanged in form. However,  $\psi$  and  $\sigma$  will be different functions than the functions in Sec. II because of the different renormalization conditions; also they depend on  $m_\lambda^2/\lambda^2$  instead of  $m^2/\lambda^2$ . In addition to these equations there is an equation for  $m_\lambda$ . Since  $S_\lambda^{-1}(p)$  must be  $(z_{2\lambda\lambda'})^{-1}$  times  $S_\lambda^{-1}(p)$ , one has in particular

$$[s(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2)]^{-1} m_\lambda s_M(p^2/\lambda^2, m_\lambda^2/\lambda^2, e_\lambda^2) = (z_{2\lambda\lambda'})^{-1} [s(p^2/\lambda'^2, m_\lambda^2/\lambda'^2, e_\lambda^2)]^{-1} \times m_\lambda s_M(p^2/\lambda'^2, m_\lambda^2/\lambda'^2, e_\lambda^2). \quad (4.34)$$

Using Eq. (2.52) for  $z_{2\lambda\lambda'}$ , setting  $p^2 = -\lambda^2$ , and using Eq. (4.31), one gets

$$m_\lambda = m_\lambda s_M(-\lambda^2/\lambda'^2, m_\lambda^2/\lambda'^2, e_\lambda^2). \quad (4.35)$$

Differentiate with respect to  $\lambda^2$ , then put  $\lambda'^2 = \lambda^2$ , and perform some further manipulation; one then has

$$\frac{d(m_\lambda/\lambda)}{d(\ln\lambda)^2} = \frac{m_\lambda}{\lambda} \phi_M\left(\frac{m_\lambda^2}{\lambda^2}, e_\lambda^2\right), \quad (4.36)$$

where

$$\phi_M\left(\frac{m_\lambda^2}{\lambda^2}, e_\lambda^2\right) = -0.5 + \left[ \frac{\partial s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2)}{\partial s} \right]_{s=1}. \quad (4.37)$$

The first term,  $-0.5$ , comes from differentiating the the factor  $\lambda^{-1}$  in  $m_\lambda/\lambda$ . The second term is of order  $e_\lambda^2$

since  $s_M$  has the form

$$s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2) = 1 + O(e_\lambda^2). \quad (4.38)$$

The function  $s_M(-s, m_\lambda^2/\lambda^2, e_\lambda^2)$  is finite for  $m_\lambda \rightarrow 0$ . The factor  $m_\lambda^{-1} B_\lambda(p^2)$  in the definition of  $s_M$  does not cause trouble in this limit, at least in perturbation theory, because every graph contributing to  $B_\lambda(p^2)$  contains a factor  $m_\lambda$ . The reason for this is that if  $m_\lambda$  is zero, the theory is  $\gamma_5$  invariant and in a  $\gamma_5$ -invariant theory,  $\Sigma_\lambda(p)$  can only contain terms proportional to  $\not{p}$ . Since  $s_M$  is finite for  $m_\lambda \rightarrow 0$ , the function  $\phi_M$  is also.

The basic equations of the renormalization group now consist of two coupled equations. If one writes  $x$  for  $e_\lambda^2$ ,  $y$  for  $m_\lambda/\lambda$ , and  $t$  for  $\ln\lambda^2$ , the coupled equations are

$$dx/dt = \psi_M(y^2, x), \quad (4.39)$$

$$dy/dt = y\phi_M(y^2, x). \quad (4.40)$$

The function  $\psi_M$  has the subscript to distinguish it from the function  $\psi$  of Secs. II and III. It is a different function because of the new subtraction method used in its definition. However, for zero mass the two subtraction methods should define the same theory, so it will be assumed that  $\psi_M(0, x) = \psi(0, x)$ . The differential equations for  $x$  and  $y$  have the same form as the renormalization-group equations for two coupling-constant theories in the absence of mass. However, because of the factor  $y$  multiplying  $\phi_M$ , any root  $x_n$  of  $\psi_M(0, x) = 0$  automatically defines a fixed point  $x = x_n$ ,  $y = 0$  of the coupled equations. The coupled equations will be discussed here only in the neighborhood of these fixed points: A general discussion is beyond the scope of this paper. Near the fixed point  $x = x_n$ ,  $y = 0$ , Eqs. (4.39) and (4.40) can be linearized, giving

$$dx/dt = a_n(x - x_n), \quad (4.41)$$

$$dy/dt = b_n y, \quad (4.42)$$

where  $a_n$  is given by Eq. (3.31) and

$$b_n = \phi_M(0, x_n). \quad (4.43)$$

The general solution of the linearized equations is

$$(x - x_n) = c_n e^{a_n t}, \quad (4.44)$$

$$y = d_n e^{b_n t}, \quad (4.45)$$

where  $c_n$  and  $d_n$  are arbitrary constants. Translated back in terms of  $e_\lambda^2$ , etc., these equations read

$$e_\lambda^2 = x_n + c_n \lambda^{2a_n}, \quad (4.46)$$

$$m_\lambda = d_n \lambda^{1+2b_n}. \quad (4.47)$$

The scale-invariance-breaking corrections to vacuum expectation values are easily determined by the method used previously. To order  $c_n$  and  $d_n$ , the amplitude  $T_{\mu\nu}$

is found to satisfy the scaling law

$$T_{\mu\nu}(sz_1, \dots, sz_4, m_1, e_1^2) = s^{2\sigma_n - 5} [1 + c_n(\tau a_n^{-1} - x_n^{-1})] \\ \times T_{\mu\nu n}(z_1, \dots, z_4) + c_n s^{2\sigma_n - 2a_n - 5} U_{\mu\nu n}'(z_1, \dots, z_4) \\ + d_n s^{2\sigma_n - 2b_n - 5} V_{\mu\nu n}(z_1, \dots, z_4), \quad (4.48)$$

where

$$V_{\mu\nu n}(z_1, \dots, z_4) = \left. \frac{\partial T_{\mu\nu}(z_1, \dots, z_4, y, x_n)}{\partial y} \right|_{y=0}. \quad (4.49)$$

There are corresponding formulas for any vacuum expectation value. If there is a term  $m_1 \mathcal{L}_{m_1}(z)$  in the Lagrangian density corresponding to the mass parameter  $m_1$ , then  $\mathcal{L}_{m_1}(z)$  has scale dimension  $4 + 2b_n$ ; this follows from the same argument that gave the dimension  $4 + 2a_n$  for  $\mathcal{L}_{e_1}(z)$ .

The constant  $b_n$  is known only for the fixed point  $x=y=0$  (call this  $n=0$ ) for which  $b_0 = -0.5$ ; in this case the interaction  $\mathcal{L}_{m_1}(z)$  is a generalized mass term. For the nonzero fixed points such as  $x_1$  and  $x_2$  the sign of  $b_n$  is not known, so  $\mathcal{L}_{m_1}(z)$  could be either a generalized mass term or a nonrenormalizable interaction.<sup>32</sup>

A peculiar situation arises if  $\mathcal{L}_{m_1}(z)$  is a nonrenormalizable interaction, i.e., if  $b_n > 0$ . First one notes that if  $\lambda = m$  then  $m_\lambda$  is of order  $m$ . The reason is this. The normalization condition on  $s_M$  [Eq. (4.31)] puts  $s_M(-1, m_\lambda^2/\lambda^2, e_\lambda^2) = 1$ . Therefore, barring exceptional circumstances,  $s_M(1, m_\lambda^2/\lambda^2, e_\lambda^2)$  should be of order 1. If  $\lambda = m$ , the mass condition is

$$m = m_m s_M(1, m_m^2/m^2, e_m^2). \quad (4.50)$$

With  $s_M$  of order 1, this equation requires  $m_m$  to be of order  $m$ , i.e.,  $m_\lambda/\lambda$  is of order 1 for  $\lambda = m$ . For large  $\lambda$ ,  $m_\lambda/\lambda$  must be small if finite mass corrections are to be small for large  $\lambda$ . This requires that  $m_\lambda/\lambda$  decrease as  $\lambda$  increases. But in the vicinity of the fixed point ( $e_\lambda^2 \simeq x_n$ ,  $m_\lambda/\lambda \simeq 0$ ),  $m_\lambda/\lambda$  decreases only if  $b_n$  is negative. More generally, whether or not  $e_\lambda^2$  is near a fixed point,  $m_\lambda^2/\lambda^2$  decreases as  $\lambda$  increases only if  $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$  is negative. There is no guarantee that  $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$  is negative except when  $e_\lambda^2$  is small. If  $m_\lambda^2/\lambda^2$  does not decrease as  $\lambda$  increases, then the asymptotic solution of Eqs. (4.39) and (4.40) for large  $t$  (i.e., large  $\ln \lambda^2$ ) will not be one of the fixed points with  $y=0$ ; instead it would be a fixed point with  $y \neq 0$  or a limit cycle or some other type of behavior. However, if the physical mass is zero, then there is a special solution of the renormalization-group equations with  $m_\lambda = 0$  for all  $\lambda$  and the analysis of previous sections applies to this special solution.

<sup>32</sup> There is one example known where a mass term in perturbation theory becomes nonrenormalizable in strong coupling. The example is the Thirring model: a spinor field in one space and one time dimension interacting via the Fermi interaction. The dimension of the mass term in the Thirring model is  $(1 - \lambda/2\pi)(1 + \lambda/2\pi)^{-1}$ , where  $\lambda$  is the coupling constant [this dimension is calculated in K. Wilson, Phys. Rev. D 2, 1473 (1970)]. For  $\lambda/2\pi < -\frac{1}{2}$  the dimension is greater than 2, which makes the mass term nonrenormalizable (in a two-dimensional world interactions with dimension  $> 2$  are nonrenormalizable).

A fixed point  $x_n$  with  $b_n < 0$  will be called "mass stable" while a fixed point with  $b_n > 0$  will be called "mass unstable." What we have shown is that only the mass-stable fixed points among the  $x_n$  will be relevant to finite-mass theories. This conclusion can be restated; the conclusion is that for finite-mass theories the interaction  $\mathcal{L}_{m_1}(z)$  must be a generalized mass term; if  $\mathcal{L}_{m_1}(z)$  is a nonrenormalizable interaction then only the zero-mass theory exists. This confirms the assumptions of Ref. 24. More generally, if there is a variable parameter in the low-energy behavior of the field theory (either a mass or a renormalized coupling constant), the corresponding interaction Lagrangian density must be a generalized mass term, not a nonrenormalizable interaction. The case of an ordinary renormalizable interaction (one with dimension four exactly) hopefully does not occur for nontrivial fixed points such as  $x_1$ ,  $x_2$ , or  $x_3$ , since it would seem unlikely that the constants  $a_n$  or  $b_n$  would be exactly zero.

The analysis of this section shows that the question of mass independence for large momenta is more complicated than the perturbation-theory calculations indicate. As long as  $e_\lambda^2$  is small, so that perturbation theory is valid,  $\phi_M(m_\lambda^2/\lambda^2, e_\lambda^2)$  is approximately  $-0.5$ , which means  $m_\lambda/\lambda$  decreases with  $\lambda$  and is small when  $\lambda \gg m$ . But when  $e_\lambda^2$  is of order 1,  $\phi_M$  may be positive in which case  $m_\lambda/\lambda$  increases with  $\lambda$ , and if so, no mass independence is possible for large  $\lambda$ .

## V. RENORMALIZATION GROUP AND STRONG INTERACTIONS

The purpose of this section is to discuss what form the renormalization group should take for strong interactions. It is assumed here that a renormalization group exists for strong interactions. The discussion is based on the work of Secs. III and IV. However, no particular model (such as the gluon model<sup>5</sup> or the  $\sigma$  model<sup>6</sup>) is assumed here.

Analysis of the renormalization group for electro-dynamics (see Sec. III G) shows that the  $\lambda$ -dependent charge  $e_\lambda$  increases with  $\lambda$ , eventually becoming of order 1.<sup>4</sup> By this is meant that no matter how small the renormalized charge  $e$  is,  $e_\lambda$  becomes of order some fixed number independent of  $e$  if  $\lambda$  is large enough. This suggests that there is a cutoff  $\Lambda$  beyond which radiative corrections to strong interactions are too large to be treated as a perturbation. So it will be assumed here that the theory of strong interactions in isolation is valid only below the cutoff  $\Lambda$ . For purposes of discussion it will be assumed that electro-dynamics, rather than weak interactions or some other interaction, is the cause of the cutoff.

It is evident from Sec. IV that including the mass parameter  $m_\lambda$  in the renormalization-group equations makes the renormalization-group method more powerful; so it will be assumed here that the renormalization group of strong interactions includes mass parameters

as well as coupling constants. Furthermore, the equations of the renormalization group have the same form for mass parameters as for coupling constants provided one replaces mass parameters such as  $m_\lambda$  by dimensionless parameters such as  $m_\lambda/\lambda$ . The parameters which are distinguished in the renormalization-group equations are those which break an internal symmetry. For example,  $m_\lambda/\lambda$  is a symmetry-breaking parameter in electrodynamics (it breaks  $\gamma_5$  symmetry), and as a result if  $m_\lambda/\lambda$  is zero for one value of  $\lambda$ , it is zero for all values of  $\lambda$ . More generally, the renormalization group respects the possible internal symmetries of a field theory in the sense that if the parameters which break the symmetry are zero for one value of  $\lambda$ , then these parameters will remain zero for all values of  $\lambda$ .

A solution of the renormalization-group equations for strong interactions should consist of a set of symmetry-preserving coupling constants  $g_{1\lambda}, g_{2\lambda}, \dots, g_{n\lambda}$  and a set of symmetry-violating constants  $h_{1\lambda}, h_{2\lambda}, \dots, h_{k\lambda}$ . For the purposes of this discussion mass parameters are divided by  $\lambda$  and included among these "coupling constants." All coupling constants are to be dimensionless. (Coupling constants of any super-renormalizable interactions, such as a  $\phi^3$  interaction of a scalar field, are also divided by  $\lambda$  to make them dimensionless, and then included in the list.) For purposes of discussion, the symmetries of strong interactions will be assumed to be  $P, C, T$ , and  $U(3) \times U(3)$ .<sup>33</sup> Experiment indicates that the symmetry-violating part of strong interactions break  $U(3) \times U(3)$ , leaving  $P, C, T$ , isospin, and strangeness intact; the theoretical discussion given here will allow for more arbitrary types of symmetry breaking. The number  $n$  of symmetry conserving constants and the number  $k$  of symmetry-violating constants will not be specified. To write the renormalization-group equations it is convenient to introduce an abstract notation. Let  $P_\lambda$  be the point in an  $n$ -dimensional space ( $S_1$ ) with coordinates  $(g_{1\lambda}, g_{2\lambda}, \dots, g_{n\lambda})$ , and let  $Q_\lambda$  be the point in a  $k$ -dimensional space ( $S_2$ ) with coordinates  $(h_{1\lambda}, \dots, h_{k\lambda})$ . Then the general form of the renormalization group equations is

$$dP_\lambda/d(\ln\lambda^2) = T_1(P_\lambda, Q_\lambda), \quad (5.1)$$

$$dQ_\lambda/d(\ln\lambda^2) = T_2(P_\lambda, Q_\lambda), \quad (5.2)$$

where  $T_1(P_\lambda, Q_\lambda)$  is itself a point in  $S_1$  and  $T_2(P_\lambda, Q_\lambda)$  is a point in  $S_2$ , i.e.,  $T_1$  has  $n$  components and  $T_2$  has  $k$  components. The point  $T_2(P_\lambda, Q_\lambda)$  is zero when  $Q_\lambda = 0$ ;  $T_1$  is completely unknown.

The renormalization group will be discussed assuming that the points  $P_\lambda, Q_\lambda$  go to a fixed point of the group when  $\lambda$  becomes large compared to a typical strong-interaction mass (i.e., for  $\lambda \gg 1$  GeV). As explained in Sec. III H, experiments on  $e^+e^-$  annihilation

at large momentum transfers can probably distinguish between fixed-point asymptotic behavior and other types of asymptotic behavior (such as a limit cycle). Until further experimental or theoretical information is available, it seems more sensible to discuss the fixed point than to try to discuss more general asymptotic behavior; for example, one does not even have a classification of the possible asymptotic forms for solutions of more than two simultaneous nonlinear equations. (See, however, Sec. VI.) It is commonly assumed that  $U(3) \times U(3)$  becomes an exact symmetry at small distances, i.e., large  $\lambda$ . Hence it will be assumed that the fixed point is of the form

$$P = P_f, \quad (5.3)$$

$$Q = 0. \quad (5.4)$$

For  $P_f$  to be a fixed point, one must have

$$T_1(P_f, 0) = 0. \quad (5.5)$$

According to the discussion of Sec. III E, the fixed point should be infrared stable due to the presence of the cutoff  $\Lambda$ . The argument was that  $P_\lambda - P_f$  and  $Q_\lambda$  are likely to be of order 1 when  $\lambda \sim \Lambda$  due to large radiative corrections. Therefore  $P_\lambda - P_f$  and  $Q_\lambda$  must decrease as  $\lambda$  decreases in order that  $P_\lambda \simeq P_f$  and  $Q_\lambda \simeq 0$  for  $1 \text{ GeV}/c \ll \lambda \ll \Lambda$ . But this is unlikely if the linearized equations for  $P_\lambda - P_f$  and  $Q_\lambda$  have solutions which increase as  $\lambda$  decreases. Unfortunately, this analysis leads to a nonsensical result. If  $Q_\lambda$  decreases as  $\lambda$  decreases, and is small for  $1 \text{ GeV}/c \ll \lambda$ , then  $Q_\lambda$  will be extremely small for  $\lambda \sim 1$  GeV or less. But this would mean that  $U(3) \times U(3)$ -symmetry breaking would be small at laboratory energies, whereas in fact  $U(3) \times U(3)$ -symmetry breaking is large at these energies. It is therefore necessary to presume that there are some  $U(3) \times U(3)$ -symmetry-breaking parameters which increase as  $\lambda$  decreases. In particular one expects there to be two parameters, say,  $h_{1\lambda}$  and  $h_{2\lambda}$ , which break  $SU(3) \times SU(3)$  symmetry according to the Glashow-Weinberg (Gell-Mann, Oakes, and Renner) theory.<sup>34</sup> We shall also assume there is a third parameter,  $h_{3\lambda}$ , which preserves  $SU(3) \times SU(3)$  but breaks  $U(3) \times U(3)$ .<sup>35</sup> These three parameters should increase as  $\lambda$  decreases, becoming of order 1 between 100 MeV and 1 GeV, where  $SU(3) \times SU(3)$  and  $U(3) \times U(3)$  are strongly broken.

Since  $h_{1\lambda}, h_{2\lambda}$ , and  $h_{3\lambda}$  are small for  $\lambda \gg 1$  GeV and decreasing as  $\lambda$  increases, they will be very small indeed when  $\lambda$  is of order  $\Lambda$ . This is possible only if there are no large radiative corrections to  $h_{1\lambda}, h_{2\lambda}$ , or  $h_{3\lambda}$  when  $\lambda \sim \Lambda$ . It is hard to see how this can come about unless these coupling constants also break an electrodynamic

<sup>34</sup> S. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

<sup>35</sup> S. Glashow, in *Hadrons and Their Interactions*, edited by A. Zichichi (Academic, New York, 1968), pp. 102-103; K. Wilson (Ref. 24); M. Gell-Mann (Ref. 7).

<sup>33</sup> The group  $U(3) \times U(3)$  consists of  $SU(3) \times SU(3)$  plus baryon number and an axial baryon number. It is assumed here that axial baryon number exists, although there is no experimental evidence for it.

symmetry. If they do break a symmetry of electro-dynamics, then electrodynamic corrections to  $h_{1\lambda}$ , etc., will be of order  $h_{1\lambda}e\lambda^2$ , etc., instead of  $e\lambda^2$  and will not be a problem. This means there must be a symmetry common to electro-dynamics and strong interactions which is broken by the couplings  $h_{1\lambda}$ ,  $h_{2\lambda}$ , and  $h_{3\lambda}$ ; a logical choice is axial baryon number since the usual electrodynamic Lagrangian for strong interactions preserves axial baryon number. This probably must be a symmetry of weak interactions also in order that weak corrections to  $h_{1\lambda}$ , etc., at large momenta not be large.

While the fixed point has to be infrared unstable with respect to the couplings  $h_{1\lambda}$ ,  $h_{2\lambda}$ , and  $h_{3\lambda}$ , it must be infrared stable to symmetry-breaking parameters which do get large radiative corrections for  $\lambda \sim \Lambda$ . For example,  $h_{4\lambda}$  might break  $SU(3) \times SU(3)$  without breaking axial baryon number or other (electrodynamic+strong) symmetries; then  $h_{4\lambda}$  will be large for  $\lambda \sim \Lambda$  and must decrease as  $\lambda$  decreases. Also, it seems likely that all coupling constants that preserve the symmetries of strong interactions, namely, the constants  $P_\lambda$ , will have large radiative corrections and therefore  $P_f$  must be infrared stable to perturbations of  $P_\lambda$  about  $P_f$ . Since the theory defined by the coupling constants ( $P_f, 0$ ) is scale invariant (by the analysis of Sec. III C), this means that the breaking of scale invariance at low momenta is due entirely to couplings which also break internal symmetries; in particular, all generalized mass terms must break an internal symmetry. A generalized mass term is any coupling which causes particles to have finite mass rather than zero mass. It is interesting to note that there are no weakly coupled scalar particles in nature; scalar particles are the only kind of free particles whose mass term does not break either an internal or a gauge symmetry.

This discussion can be summarized by saying that mass or symmetry-breaking terms must be "protected" from large corrections at large momenta due to various interactions (electromagnetic, weak, or strong). A symmetry-breaking term, such as  $h_{1\lambda}$ ,  $h_{2\lambda}$ , or  $h_{3\lambda}$ , is protected if, in the renormalization-group equation for  $h_{1\lambda}$ ,  $h_{2\lambda}$ , or  $h_{3\lambda}$ , the right-hand side is proportional to  $h_{1\lambda}$ ,  $h_{2\lambda}$ ,  $h_{3\lambda}$  or other small coupling constants even when high-order strong, electromagnetic, or weak corrections are taken into account. The mass terms for the electron and muon and the weak boson, if any, must also be protected. This requirement means that weak interactions cannot be mediated by scalar particles.<sup>36</sup>

One basic mystery remains from this analysis, namely, why the breaking of the axial baryon number is small when  $\lambda \sim \Lambda$ ; even if the mixing of electro-dynamics with strong interactions does not force the breaking to be large, it is strange that it is small without being zero.

According to the analysis of Sec. III E, all the renormalized coupling constants of strong interactions

could be computed by solving the renormalization-group equations. This is no longer true. There is no argument that can determine  $h_{1\lambda}$ ,  $h_{2\lambda}$ , or  $h_{3\lambda}$  for  $\lambda \sim 1$  GeV and the values of these constants for one value of  $\lambda$  must be determined from experiment. The renormalization group can then be used to fix the values of  $h_{1\lambda}$ ,  $h_{2\lambda}$ , and  $h_{3\lambda}$  for other values of  $\lambda$ . If there are other coupling constants which increase as  $\lambda$  decreases, these coupling constants must also be determined from experiment. Such coupling constants will be small for  $\lambda \sim \Lambda$  and therefore must also be protected. This presumably means these constants are also symmetry-breaking terms. Hopefully, the dominant symmetry-breaking terms are  $h_{1\lambda}$ ,  $h_{2\lambda}$ , and  $h_{3\lambda}$ ; then other symmetry-breaking terms, while surely present, are small for  $\lambda \sim 1$  GeV, and cannot increase further for  $\lambda \ll 1$  GeV because amplification ceases for  $\lambda$  less than the hadron masses (see Sec. III B).

The renormalization group for strong interactions contains mass terms and coupling constants for any super-renormalizable interactions. Should it include nonrenormalizable interactions? The answer is yes, for several reasons. It was shown in Sec. IV that the interaction Lagrangian density  $\mathcal{L}_{e1}(x)$  is a nonrenormalizable interaction in the neighborhood of an infrared-stable fixed point. This will also be true of the interactions associated with the non-symmetry-breaking couplings  $g_{1\lambda} \cdots g_{n\lambda}$  of strong interactions, since the fixed point  $P_f$  must be infrared stable except for symmetry breaking. So in effect some nonrenormalizable interactions are already present in the renormalization group. Conversely, there is no reason to suppose that a symmetry-breaking interaction which is nonrenormalizable in perturbation theory will stay nonrenormalizable near a fixed point with large coupling constants: for example, the  $U(3) \times U(3)$  breaking constant  $h_{3\lambda}$  might correspond to a nonrenormalizable interaction in perturbation theory [especially in the gluon model, where in perturbation theory there are no renormalizable interactions or mass terms which break  $U(3) \times U(3)$  without breaking  $SU(3) \times SU(3)$  also]. So it may be essential to include interactions which are nonrenormalizable in perturbation theory to find all the generalized mass terms near a strongly interacting fixed point. Furthermore, there has never been any fundamental physical distinction between nonrenormalizable interactions and renormalizable ones, so one would like to treat them on an equal footing. Finally, there is a model with a renormalization group which can be solved rigorously in strong coupling which necessarily includes nonrenormalizable interactions.<sup>37</sup>

If the renormalization group for strong interactions includes nonrenormalizable couplings, it will be difficult to construct it as a simple extension of the Gell-Mann-Low group, requiring that one start from scratch. It will also be considerably more complicated than the

<sup>36</sup> This rules out the models of W. Kummer and G. Segré, Nucl. Phys. 64, 585 (1965) and N. Christ, Phys. Rev. 176, 2086 (1968).

<sup>37</sup> K. Wilson, Phys. Rev. D 2, 1438 (1970).

Gell-Mann-Low group, since there are an infinite number of nonrenormalizable interactions. Whether the conclusions of this paper actually apply to such a group remains to be seen; but surely these conclusions are an indication of the kind of physics that can come out of such a group. The inclusion of nonrenormalizable interactions in the renormalization-group equations does *not* change the conclusion that scale-invariance breaking in strong interactions is due only to generalized mass terms, provided that the asymptotic solution of the renormalization group is a fixed point.

### VI. $\Delta I = \frac{1}{2}$ RULE

It has been shown elsewhere<sup>24</sup> that one can understand the  $\Delta I = \frac{1}{2}$  rule in nonleptonic weak interactions given that strong interactions are scale invariant at short distances and that dimensions of local fields in strong interactions are not as predicted from canonical field theory. The purpose of this section is to show that if the strong interactions have a renormalization group, then the  $\Delta I = \frac{1}{2}$  rule can be understood without assuming broken scale invariance. In other words, the  $\Delta I = \frac{1}{2}$  rule can be understood regardless of what kind of asymptotic solution the renormalization-group equations have (fixed point, limit cycle, or otherwise).

The first part of the analysis of Ref. 24 will be assumed here. According to this analysis, the current-current Lagrangian for nonleptonic weak interactions can be approximated by the form

$$\mathcal{L}_w(x) \simeq \sum_n G_n O_n(x), \quad (6.1)$$

where the fields  $O_n(x)$  are a set of local fields, and the  $G_n$  are constants. This form assumes that the weak boson mass or weak interaction cutoff is of order  $M$ , where  $M \gg (1 \text{ GeV})$ . The fields  $O_n(x)$  are assumed to belong to irreducible representations of  $SU(3) \times SU(3)$ .

Before discussing the consequences of the renormalization group for  $\mathcal{L}_w(x)$ , we shall try to explain what is meant by "understanding" the  $\Delta I = \frac{1}{2}$  rule. What will be argued in the following is that  $\mathcal{L}_w(x)$  is dominated by a single field out of the set  $\{O_n(x)\}$ , say,  $O_m(x)$ . The argument below gives no clue as to which field dominates. This means that  $\mathcal{L}_w(x)$ , to a good approximation, belongs to a single irreducible representation of  $SU(3) \times SU(3)$ , despite the fact that the current-current product contains several representations of  $SU(3) \times SU(3)$ . However, one cannot determine theoretically which representation of  $SU(3) \times SU(3)$  will dominate. The current-current product contains the following representations of  $SU(3) \times SU(3)$ :  $(1,1)$ ;  $(8,1) \oplus (1,8)$ ;  $(27,1) \oplus (1,27)$ ;  $(10,1) \oplus (1,10)$ ;  $(\bar{10},1) \oplus (1,\bar{10})$ ; and  $(8,8)$ . In addition, since  $SU(3) \times SU(3)$  is not an exact symmetry, there is some leakage into other representations as well.

The theoretical analysis given below predicts that one  $SU(3) \times SU(3)$  representation will dominate in

$\mathcal{L}_w(x)$ , and the dominance is by a power of  $M$  over all other representations. One has to look to experiment to find out which  $SU(3) \times SU(3)$  representation dominates. It is obvious from the  $K$ -decay amplitudes that the dominant  $SU(3) \times SU(3)$  representation contains no  $\Delta I = \frac{3}{2}$  term. The factor 20 that separates the  $\Delta I = \frac{1}{2}$  amplitude in  $K$  decay from the  $\Delta I = \frac{3}{2}$  amplitude is too large to be accounted for credibly without having  $\Delta I = \frac{1}{2}$  dominance in the effective Lagrangian  $\mathcal{L}_w(x)$ . This limits the possible dominant  $SU(3) \times SU(3)$  representations to 2:  $(8,1) \oplus (1,8)$  and  $(3,3) \oplus (\bar{3},3)$ . [The representation  $(8,1) \oplus (1,8)$  seems more likely since it does not require leakage in order to occur.]

One uses  $K$  decay to choose the dominant  $SU(3) \times SU(3)$  representation. The theoretical argument then predicts the following otherwise mysterious facts: (1) the large dominance of  $\Delta I = \frac{1}{2}$  over  $\Delta I = \frac{3}{2}$  in  $K$  decay; this is possible because the dominance is by a power of  $M$ , and this factor should be large provided that  $M$  is large compared to strong interaction masses (i.e.,  $M \gg 1 \text{ GeV}$ ); (2) large  $\Delta I = \frac{1}{2}$  dominance in baryon nonleptonic decays, i.e., universal  $\Delta I = \frac{1}{2}$  dominance.

The theoretical analysis is the following. In a renormalization-group analysis of  $\mathcal{L}_w$ , it is characterized by a set of  $\lambda$ -dependent coupling constants  $G_{n\lambda}$ . When  $\lambda$  is of order  $M$  these coupling constants are the same size for different  $n$ . In particular the coupling constants corresponding to different  $SU(3) \times SU(3)$  representations have the same order of magnitude. This is because there is nothing peculiar about two currents separated by a distance  $M^{-1}$  if it is in a matrix element where *all* fields are separated by of order  $M^{-1}$ . But the constants  $G_{n\lambda}$  for  $\lambda \sim M$  measure the size of  $\mathcal{L}_w(x)$  when sandwiched between fields separated by  $\sim M^{-1}$ . Thus the various  $SU(3) \times SU(3)$  representations should have roughly the same strength in  $\mathcal{L}_w(x)$  when  $\lambda \sim M$ . The coupling constants which determine decay rates are the  $G_{n\lambda}$  with  $\lambda \lesssim 1 \text{ GeV}$ . These must be determined by solving the renormalization-group equations between  $\lambda = 1 \text{ GeV}$  and  $\lambda = M$ . But as shown in Sec. III B, there can be large amplification or deamplification effects in this interval (assuming  $M \gg 1 \text{ GeV}$ ), and these effects can be different for different  $SU(3) \times SU(3)$  representations.

To be more precise, consider the form of the renormalization-group equations for  $G_{n\lambda}$ . Assume for convenience that each  $n$  refers to a different  $SU(3) \times SU(3)$  representation. The equations can be linearized with respect to the  $G_{n\lambda}$ ; the symmetry-violating constants  $h_{n\lambda}$  will be neglected (in the region above 1 GeV which is important for producing amplification or deamplification the  $h_{n\lambda}$  are small and will not change the analysis appreciably). Then the renormalization-group equations have the form

$$dG_{n\lambda}/d(\ln \lambda^2) = G_{n\lambda} U_n(P_\lambda), \quad (6.2)$$

where  $P_\lambda$  is the set of symmetry-conserving coupling



constants for momentum  $\lambda$ . The  $U_n$  are unknown functions of  $P_\lambda$ . There are no cross terms relating  $dG_{n\lambda}/d(\ln\lambda^2)$  to  $G_{l\lambda}$  with  $l \neq n$  because of symmetry requirements. The solution of these equations is

$$G_{n\lambda} = G_{nM} \exp \left[ \int_M^\lambda 2U_n(P_{\lambda'}) (\lambda')^{-1} d\lambda' \right]. \quad (6.3)$$

Equivalently, one can write

$$G_{n\lambda} = (\lambda^2/M^2)^{\alpha_n(\lambda)} G_{nM}, \quad (6.4)$$

where  $\alpha_n(\lambda)$  is

$$\alpha_n(\lambda) = (\ln\lambda^2 - \ln M^2)^{-1} \int_M^\lambda 2U_n(P_{\lambda'}) (\lambda')^{-1} d\lambda'. \quad (6.5)$$

If  $U_n$  were a constant independent of  $\lambda'$ , then  $\alpha_n(\lambda)$  would be equal to  $U_n$ . The constant  $\alpha_n(\lambda)$  is presumably of order 1, in the absence of more precise information, but should be different for different  $n$ . If the asymptotic solution of the renormalization-group equations is a fixed point, then  $U_n$  would be constant for  $\lambda' \gg 1$  GeV where  $P_{\lambda'}$  is near the fixed point. In this case  $\alpha_n$  would be related to the dimension of the field  $O_n$ ; as long as the fields  $O_n$  have different dimensions,  $\alpha_n$  would also be different for different  $n$ . Even if the renormalization group does not have fixed-point asymptotic behavior, there is an exponent  $\alpha_n$  (1 GeV) which defines an "effective dimension" for the fields  $O_n$  over the interval  $1 \text{ GeV} < \lambda' < M$ . The exponents  $\alpha_n$  (1 GeV) determine the amount of amplification or deamplification that results from going from  $G_{nM}$  to  $G_{n\lambda}$  with  $\lambda \sim 1$  GeV. As long as the  $\alpha_n$  are different for different  $n$ , the low-energy coupling constants  $G_{n\lambda}$  with  $\lambda \sim 1$  GeV will differ by powers of  $M$  for different  $n$ . This ensures the dominance of one  $SU(3) \times SU(3)$  representation in the phenomenological Lagrangian for nonleptonic decays.

The existence of the  $\Delta I = \frac{1}{2}$  rule experimentally is encouragement to believe that a renormalization group does exist for strong interactions, since all other explanations of the  $\Delta I = \frac{1}{2}$  rule are unsatisfactory for one reason or another.<sup>24,25</sup>

## VII. FINAL REMARKS

The application of the renormalization group to strong interactions leads to profound results; for example, the possibility that the short-distance behavior of strong interactions is described by a limit cycle. It is disturbing, therefore, that the renormalization-group results derive from a not-very-profound property of perturbation theory, namely, that renormalized amplitudes renormalized by the Gell-Mann-Low method have a zero-mass limit. Furthermore, the renormalization group is not involved in the standard procedures for solving field theory, yet the renormalization-group analysis makes important predictions about the nature of the solutions of field theory. One would be happier about these predictions if the renormalization group

itself were a more essential part of the structure of field theory than it appears to be from perturbation theory.

There is a model field theory described elsewhere<sup>37</sup> which suggests that the renormalization group is an essential part of understanding a strongly coupled field theory. The model is a truncated version of the charged scalar theory of pions coupled to a fixed source. In the truncated version of this theory the  $\pi$  mesons are restricted to discrete wave packet states centered on the source; the  $n$ th state has mean momentum  $m\Lambda^n$ , where  $m$  is the pion mass and  $\Lambda$  is a large number. The renormalization group for the model is defined as part of solving the model as an expansion in  $\Lambda^{-1}$ . The renormalization group of the model has the following properties:

- (1) The renormalization group is defined *before* the solution of the theory is known.
- (2) The renormalization-group equations of the model involve coupling constants of all possible non-renormalizable interactions in the model.
- (3) There is no way (known to the author, at least) of obtaining a complete solution of the model, except by solving the renormalization-group equations.

The author suspects these three properties will also be true of strongly interacting relativistic fields. This is because the reason for the importance of the renormalization group in the model is that it has an infinite number of disparate scales of energy, namely, the energy scales  $m$ ,  $m\Lambda$ ,  $m\Lambda^2$ , etc. The function of the renormalization-group transformation is to solve the part of the Hamiltonian involving energies of one scale, say,  $m\Lambda^n$ , assuming that parts of the Hamiltonian with energies  $\gg m\Lambda^n$  have already been solved. In order to solve the infinite number of energy scales which exceed energies of practical interest, one must iterate the renormalization-group transformation an infinite number of times, thus making asymptotic behavior of this transformation of crucial importance in solving the model. (For details, see Ref. 37.) All relativistic field theories also have this infinite sequence of energy scales; they arise due to the possibility of creating particles of any energy from  $m$  to  $\infty$ . The problem in relativistic theory is that one does not have only the discrete energy scales  $m$ ,  $m\Lambda$ , etc.; one has all energies in between these values, so a perturbation expansion in  $\Lambda^{-1}$  is impossible for relativistic theories. This does not mean one does not have disparate energy scales present. In the model, one can only solve one order of magnitude of energy at a time; it is hard to see how one can do more than this in a relativistic theory. So in a relativistic theory one should also look for a renormalization-group transformation which solves one order of magnitude of energies in the Hamiltonian.<sup>38,38a</sup>

<sup>38</sup> A rather similar approach was suggested many years ago by F. J. Dyson, Proc. Roy. Soc. (London) **A207**, 395 (1951).

<sup>38a</sup> Note added in proof. Such a transformation has recently been obtained from qualitative arguments; see K. Wilson, Cornell University Laboratory of Nuclear Studies Report No. CLNS-142 (unpublished).

This discussion will not be pursued here; its purpose is only to emphasize that the next step after reading this paper is to study the model of Ref. 37.

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### APPENDIX: REVIEW OF GELL-MANN-LOW THEORY

There are many reasons why the Gell-Mann-Low theory<sup>1</sup> is hard to understand. But perhaps the most puzzling feature of it is that the reasoning and assumptions which led Gell-Mann and Low to formulate the differential equations of the renormalization group seem totally irrelevant to the conclusions one derives from the differential equations. In Gell-Mann and Low's paper, the motivation for setting up the renormalization group was the observation that while the renormalized photon propagator  $d_c(k^2/m^2, e^2)$  is logarithmically divergent when  $m^2 \rightarrow 0$ , the cut-off but unrenormalized Feynman graphs for the propagator are not divergent in this limit (at least in low orders). Gell-Mann and Low therefore propose a generalization of the usual renormalization procedure which is not divergent in low orders when  $m^2 \rightarrow 0$ ; they then discuss properties of their renormalized theory which leads them to derive the differential equation (1.2). One's reaction to going through this analysis is that one is looking at rather trivial and technical aspects of renormalized perturbation theory which cannot be the basis for any very basic results. In the author's opinion the analysis of Gell-Mann and Low is in fact dealing with technical aspects of field theory, but by doing so they have found an equation which embodies very fundamental properties of quantum field theory. New ideas are often discovered for irrelevant reasons, so this opinion is not unreasonable. Also, perturbation theory is the one approximation method which can be computed without any understanding whatsoever of the equations one is solving by perturbation theory. What this means is that while qualitative features of the original equations will be reflected in qualitative features of its perturbation expansion, to understand these qualitative features one has to study the original equations and not just their perturbation expansion. So perhaps one should expect that a derivation of the renormalization-group equations from perturbation theory will not be very illuminating.

The Gell-Mann-Low theory will now be reviewed, much in the manner that Gell-Mann and Low pre-

sented it. The only major change is not to use Ward's method of renormalization<sup>39</sup> as a basis for modifying the usual renormalization program. The author has also benefited from the review of the theory in Bogoliubov and Shirkov<sup>4</sup>; however, they provide very little motivation for the calculations they describe.

Consider lowest-order vacuum polarization. If one computes this graph and renormalizes it in the conventional fashion it gives a contribution to vacuum polarization, denoted  $\Pi_{c\mu\nu}(k)$  ( $k$  is the photon momentum), which is

$$\Pi_{c\mu\nu}(k) = (e^2/4\pi^2)(g_{\mu\nu}k^2 - k_\mu k_\nu)I(k^2). \quad (A1)$$

The exact form of  $I(k^2)$  is given in Bogoliubov and Shirkov.<sup>40</sup> If  $m^2 \ll k^2$ ,  $I(k^2)$  is approximately

$$I(k^2) \simeq \frac{1}{3} \ln |k^2/m^2|. \quad (A2)$$

In the limit  $m \rightarrow 0$ ,  $I(k^2)$  is logarithmically divergent. Gell-Mann and Low point out that if one looks instead at the unrenormalized but cut-off vacuum polarization diagram, it does not diverge as  $m \rightarrow 0$ . It is worth showing this in a way that is generalized easily to higher-order diagrams. A cut-off version of the graph gives a function  $\Pi_{\Lambda\mu\nu}(k)$ :

$$\begin{aligned} \Pi_{\Lambda\mu\nu}(k) &= -ie^2 \int_p \text{Tr} \left\{ \gamma_\mu \frac{(\not{p}+m)\gamma_\nu(\not{p}-\not{k}+m)}{(\not{p}^2-m^2+i\epsilon)[(\not{p}-\not{k})^2-m^2+i\epsilon]} \right\} \\ &\quad \times \frac{\Lambda^4}{(\not{p}^2-\Lambda^2+i\epsilon)[(\not{p}-\not{k})^2-\Lambda^2+i\epsilon]}, \quad (A3) \end{aligned}$$

where  $\int_p$  is shorthand for  $(2\pi)^{-4} \int d^4p$  and  $\Lambda$  is the cutoff.<sup>41</sup>

It is convenient to consider spacelike  $k$  and choose a Lorentz frame in which the time component  $k_0$  is zero. In this frame, one can rotate the contour of integration in  $p_0$  from the real axis to the imaginary axis (no singularities of the integrand are crossed if the rotation is counterclockwise). This means one replaces  $p_0$  by  $ip_4$ , where  $p_4$  is real. The denominator of the integrand now has the form

$$(\not{p}^2+m^2)[(\not{p}-\not{k})^2+m^2](\not{p}^2+\Lambda^2)[(\not{p}-\not{k})^2+\Lambda^2],$$

where  $\not{p}^2$  now means  $\not{p}^2+p_4^2$ , and likewise for  $(\not{p}-\not{k})^2$  (i.e., one has a Euclidean metric instead of a Lorentz metric).

With cutoffs present the integral has no ultraviolet divergences, so the only way it can diverge is through vanishing denominators. If  $m^2$  is zero the denominators  $\not{p}^2$  and  $(\not{p}-\not{k})^2$  can vanish. Since the metric is Euclidean,  $\not{p}^2$  can vanish only if all four components of  $\not{p}$  vanish. Hence the integral is divergent only if the integrand is

<sup>39</sup> See Ref. 1.

<sup>40</sup> See Ref. 4, p. 399.

<sup>41</sup> For simplicity a non-gauge-invariant cutoff is used in Eq. (A3).

as singular as  $p^{-4}$  for  $p \rightarrow 0$  or as  $(p-k)^{-4}$  when  $(p-k) \rightarrow 0$ , which is not the case.

Even if  $k$  is zero, so the denominator behaves as  $(p^2)^2$  for  $p \rightarrow 0$ , the integral does not diverge because there are two powers of  $p$  from the numerator when  $k$  and  $m$  are both zero.

Why then does the renormalized graph diverge for  $m \rightarrow 0$ ? The reason lies in the way one defines the renormalized function  $\Pi_{c\mu\nu}(k)$ . The function  $\Pi_{\Lambda\mu\nu}(k)$  contains terms proportional to  $g_{\mu\nu}$ ,  $g_{\mu\nu}k^2$ , and  $k_\mu k_\nu$ , which are cutoff dependent and must be subtracted in order to give a finite result. The customary procedure is to subtract the expansion in  $k$  of  $\Pi_{\Lambda\mu\nu}(k)$  to order  $k^2$ , namely, to define

$$\Pi_{c\mu\nu}(k) = \Pi_{\Lambda\mu\nu}(k) - \Pi_{\Lambda\mu\nu}(0) - k^\pi k^\sigma \Pi_{\Lambda\mu\nu, \pi\sigma}(0), \quad (\text{A4})$$

where

$$\Pi_{\Lambda\mu\nu, \pi\sigma}(k) = \frac{\partial^2 \Pi_{\Lambda\mu\nu}(k)}{\partial k^\pi \partial k^\sigma}. \quad (\text{A5})$$

With this form of subtraction,  $\Pi_{c\mu\nu}(k)$  is of order  $k^4$  when  $k \rightarrow 0$ ; this means that the radiative correction to the photon propagator, which is proportional to  $(k^2)^{-2} \Pi_{c\mu\nu}(k)$ , is finite as  $k \rightarrow 0$ . Hence the pole of the exact photon propagator is the same as the pole term in the free propagator.

The divergence for  $m \rightarrow 0$  in the renormalized vacuum polarization comes from the subtraction  $\Pi_{\Lambda\mu\nu, k\sigma}(k)$ . One can see that differentiating  $\Pi_{\Lambda\mu\nu}(k)$  twice with respect to  $k$  and then setting  $k=0$  makes the integral in Eq. (A3) diverge: Each differentiation makes the integrand more singular by one power of  $p$  at  $p=0$ , and hence the integrand for  $\Pi_{\Lambda\mu\nu, k\sigma}(0)$  behaves as  $p^{-4}$  when  $m=0$ , causing a divergence.

Low-order diagrams for the electron propagator and the vertex function show a similar feature, namely, that the unrenormalized but cut-off Feynman integrals are finite for zero mass. One shows these results using the same method described above for the vacuum polarization graph. To get a divergence requires that several denominators vanish simultaneously and for low-order graphs it is trivial to check that this cannot happen. In the case of the vertex function, one needs some nonzero external momenta to prevent too many denominators from vanishing simultaneously (and likewise for four-point functions, five-point functions, etc.). In other words the external momenta provide an infrared cutoff for vertex-function diagrams. This was noted by Gell-Mann and Low. Also one must treat specially graphs with self-energy corrections on internal lines. Otherwise one can get a string of propagators depending on a single momentum  $p$ , say, and all diverging when  $p \rightarrow 0$ . One can sum up the self-energy corrections to internal lines to give exact propagators; an exact propagator has at worst a simple pole when  $p^2 \rightarrow 0$ . However, the condition for an exact propagator to be singular at momentum zero is that the physical

mass of the electron be zero, not that the bare mass of the electron be zero. This is why the problem of infrared divergences arises when the physical electron mass is zero and not necessarily when the bare mass of the electron is zero.

It is not trivial to test all higher-order graphs for divergences at zero mass, because there are so many diagrams to be checked.

A similar analysis can be made for the electron propagator with finite mass. There are infrared divergences due to the photon mass being zero in the renormalized electron propagator. However, unrenormalized but cut-off diagrams of low order do not show this divergence either. One shows this by considering the electron propagator with timelike momentum  $p$  in the rest frame of  $p$ . If  $|p_0| \leq m$ , one can again rotate contours of integration so that internal momenta are in a Euclidean metric. Consider an internal electron line in an electron self-energy graph carrying momentum  $p-k$ , where  $k$  is an internal momentum. After the rotation, the propagator for the line has the form

$$[\gamma_0(p_0 - ik_4) + \boldsymbol{\gamma} \cdot \mathbf{k} + m] \times [-2ip_0k_4 - k_4^2 - \mathbf{k}^2 - (m-p_0)(m+p_0)]^{-1}.$$

The propagator diverges if and only if  $p_0=m$  and all four components of  $k$  vanish. To produce a divergent integral requires several denominators to vanish simultaneously; to produce such a divergence in an electron self-energy graph, one must differentiate at least once with respect to  $p_0$  and then put  $p_0=m$ . This is in fact done as part of the conventional renormalization procedure for the electron propagator (see below), but if one does not renormalize the low-order graphs for the electron self-energy, they are finite for zero photon mass.

Since the zero-mass singularities of amplitudes seem to come in from the subtractions of the conventional renormalization program, Gell-Mann and Low propose that one set up an alternative method of renormalization which will not introduce such singularities. The basic idea can be illustrated with lowest-order vacuum polarization. Suppose the unrenormalized but cut-off vacuum polarization  $\Pi_{\Lambda\mu\nu}(k)$  is defined using Pauli-Villars regularization<sup>42</sup>; then  $\Pi_{\Lambda\mu\nu}(k)$  has the form

$$\Pi_{\Lambda\mu\nu}(k) = (e^2/4\pi^2)(g_{\mu\nu}k^2 - k_\mu k_\nu)I(k^2, \Lambda^2), \quad (\text{A6})$$

where  $I(k^2, \Lambda^2)$  will not be written down explicitly here. The cutoff dependence in  $I(k^2, \Lambda^2)$  is in a constant independent of  $k^2$ , and by custom it is removed by defining the renormalized function  $I(k^2)$  as

$$I(k^2) = I(k^2, \Lambda^2) - I(0, \Lambda^2). \quad (\text{A7})$$

The singularity can also be removed by subtracting  $I(-\lambda^2, \Lambda^2)$  from  $I(k^2, \Lambda^2)$ , where  $\lambda$  is arbitrary. (The subtraction is made at a spacelike momentum because

<sup>42</sup> W. Pauli and F. Villars, *Rev. Mod. Phys.* **21**, 434 (1949).

one wants the subtraction to be real; see below.) So one can define a whole set of possible renormalized functions  $I_\lambda(k^2)$  by

$$I_\lambda(k^2) = I(k^2, \Lambda^2) - I(-\lambda^2, \Lambda^2). \quad (\text{A8})$$

The function  $I(k^2, \Lambda^2)$  is finite for  $m \rightarrow 0$  provided  $k^2$  is nonzero. If  $k^2 = 0$  then  $I$  is proportional to a second derivative of  $\Pi_{\lambda\mu\nu}(k)$  with respect to  $k$ , which diverges logarithmically for  $m \rightarrow 0$ . This means that  $I_\lambda(k^2)$  is finite for  $m \rightarrow 0$  if  $\lambda$  and  $k^2$  are held fixed and neither is zero. So in summary  $I(k^2)$  diverges for  $m \rightarrow 0$  for any value of  $k^2$  (except  $k^2 = 0$ ), whereas  $I_\lambda(k^2)|_{m=0}$  is finite for  $k^2 \neq 0$  but diverges for the special value  $k^2 = 0$ .

In general the proposal of Gell-Mann and Low is that when making subtractions to remove divergences from the unrenormalized theory, these subtractions should be made at a subtraction momentum  $\lambda$ , rather than at momentum 0 or at the electron mass  $m$ . As a result one gets, in low orders at least, amplitudes which are finite at zero electron mass except for special values of the external momenta. The photon propagator of the zero-mass theory will not have just a simple pole singularity for  $k=0$ : It will have logarithmic singularities multiplying the pole. The same is true of the electron propagator on the mass shell. These results are no surprise because the same is true of the electron propagator of the finite-mass theory, due to photon infrared divergences.

The renormalization procedure with subtractions made at a momentum  $\lambda$  must obey the same restrictions as conventional renormalization. That is, the renormalized theory is allowed to differ only in the following respects from the unrenormalized cut-off theory:

(a) The renormalized electron and photon fields can differ from the unrenormalized fields by renormalization constants.

(b) The renormalized theory can be reparametrized in terms of a phenomenologically defined coupling constant and mass in place of the bare coupling constant and bare mass.

(c) One can make gauge transformations at will; a gauge transformation is accomplished by adding a term proportional to  $k_\mu k_\nu$  to the free-photon propagator.

The substitution of  $I_\lambda(k^2)$  for  $I(k^2)$  is equivalent to a renormalization and a change of gauge. To see this one must compare the photon propagators for the two cases. The standard renormalized propagator to order  $e^2$  is<sup>43</sup>

$$D_{c\mu\nu}(k) = (k^2)^{-1} [-g_{\mu\nu} + (e^2/4\pi)(k^2)^{-1} \times (k_\mu k_\nu - g_{\mu\nu} k^2) I(k^2)]. \quad (\text{A9})$$

The function  $I_\lambda(k^2)$  may be written

$$I_\lambda(k^2) = I(k^2) - I(-\lambda^2). \quad (\text{A10})$$

<sup>43</sup> This formula assumes the free propagator is in the Feynman gauge, i.e., has no  $k_\mu k_\nu$  term.

Hence the renormalized propagator  $D_{\lambda\mu\nu}(k)$  of Gell-Mann and Low is

$$D_{\lambda\mu\nu}(k) = D_{c\mu\nu}(k) - (e^2/4\pi)(k^2)^{-2} \times (k_\mu k_\nu - g_{\mu\nu} k^2) I(-\lambda^2). \quad (\text{A11})$$

One can replace  $e^2 g_{\mu\nu}(k^2)^{-1}$  by  $-e^2 D_{c\mu\nu}(k)$ , since the difference of these two expressions is of order  $e^4$ , which is being neglected anyway. So one has, neglecting terms of order  $e^4$ ,

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D_{c\mu\nu}(k) - (e^2/4\pi)(k^2)^{-2} k_\mu k_\nu I(-\lambda^2), \quad (\text{A12})$$

where

$$z_{3\lambda} = 1 - (e^2/4\pi) I(-\lambda^2). \quad (\text{A13})$$

The remaining  $k_\mu k_\nu$  term can be absorbed into a change of gauge of the conventionally renormalized propagator. If one adds  $-(e^2/4\pi)(k^2)^{-2} k_\mu k_\nu I(-\lambda^2)$  to the free propagator of the conventional theory, then the complete propagator of the conventional theory is  $D_{c\mu\nu}'(k)$  with

$$D_{c\mu\nu}'(k) = (k^2)^{-1} \{-g_{\mu\nu} - (e^2/4\pi)g_{\mu\nu} I(k^2) + (e^2/4\pi)k_\mu k_\nu (k^2)^{-1} [I(k^2) - I(-\lambda^2)]\} \quad (\text{A14})$$

and

$$D_{\lambda\mu\nu}(k) = z_{3\lambda} D_{c\mu\nu}'(k), \quad (\text{A15})$$

neglecting terms of order  $e^4$ .

Having to make a change of gauge when comparing the Gell-Mann-Low propagator with the conventional propagator is a nuisance. To simplify this problem, Gell-Mann and Low choose the gauge of the free propagator such that the exact propagator is in the Feynman gauge no matter which renormalization procedure is used. The Gell-Mann-Low prescription means that the exact propagator  $D_{c\mu\nu}(k)$  to order  $e^2$  is

$$D_{c\mu\nu}(k) = (k^2)^{-1} [-g_{\mu\nu} - (e^2/4\pi^2)g_{\mu\nu} I(k^2)] \quad (\text{A16})$$

and the corresponding free propagator is

$$(k^2)^{-1} [-g_{\mu\nu} - (e^2/4\pi^2)(k^2)^{-1} k_\mu k_\nu I(k^2)].$$

The propagator  $D_{\lambda\mu\nu}(k)$  is  $z_{3\lambda} D_{c\mu\nu}(k)$  and the corresponding free propagator is

$$(k^2)^{-1} [-g_{\mu\nu} - (e^2/4\pi^2)(k^2)^{-1} k_\mu k_\nu I_\lambda(k^2)].$$

Gell-Mann and Low calculate all Feynman graphs using the exact propagator for internal lines instead of the free propagator. As a result it is of no importance in practice that the free propagator undergoes a gauge change when the renormalization method is changed. Furthermore, to compute the exact propagator one computes only the  $g_{\mu\nu}$  term in vacuum polarization. In contrast, Bogoliubov and Shirkov do not allow the free propagator to depend on  $e^2$ ; instead they work mainly in the Landau gauge, or else make a change of gauge as the renormalization method is changed. Bogoliubov

and Shirkov claim that the Gell-Mann-Low treatment is wrong; I see nothing wrong with it and will use the Gell-Mann-Low approach in the following.

The subtraction momentum  $\lambda$  can be chosen arbitrarily. However, the renormalized theory will have an apparently nontrivial dependence on  $\lambda$ . Nevertheless, the physical consequences of the theory must be independent of  $\lambda$ . The transformations which connect the

renormalized theories with different values of  $\lambda$  are the renormalization-group transformations. They are discussed in Sec. II.

The above discussion should make clear the ideas involved in generalizing the usual renormalization procedure such that subtractions are made at a momentum  $\lambda$  rather than on the photon or electron mass shell.

## Renormalization of Chirally Invariant Lagrangians

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We discuss the renormalization problem for chirally invariant Lagrangians constructed from the pion isovector fields alone. We find that  $S$ -matrix elements have ultraviolet divergences even after the damping from resummation has been effected. The counterterms necessary to remove such divergences are shown to modify the original current commutation relations; it is not possible that these divergences cancel amongst themselves. We conclude that the current commutation relations are inconsistent with the Lagrangian formulation, unless they are changed or further particles are introduced.

### I. INTRODUCTION

IN this paper, we are interested in calculating higher-order corrections for scattering amplitudes arising from chirally invariant interactions. Physical results for such interactions have so far been calculated in the tree-graph approximation. Attempts have been made recently to extend this to include closed loops for non-polynomial Lagrangians without derivative interactions. There are two main difficulties in such problems, which are that the perturbation series in the minor coupling constants have in general zero radius of convergence, and that there are ultraviolet divergences of arbitrarily high order due to the highly unrenormalizable character of the interaction. The first of these has been tackled by various resummation techniques.<sup>1,2</sup> The second has been satisfactorily solved by addition of a suitable number (sometimes infinite) of counterterms in the Lagrangian.<sup>3,4</sup> There has been a start on extending this discussion to the more interesting case of chirally invariant Lagrangians.<sup>5</sup> This discussion has only been given up to second order in the major coupling constant. We wish here to extend it to all orders.

In Sec. II we set up the formalism for calculating  $S$ -matrix elements to each order in the major coupling

constant, by means of an extension of the concept of skeleton diagrams.<sup>3</sup> We then proceed, in Sec. III, to determine the over-all degree of ultraviolet divergence of the resulting amplitudes; we do this in coordinate space, though our results are the same as those obtained by using momentum-space methods.<sup>2</sup> In order to remove the divergences, we find it essential to add counterterms; it is not possible that the divergences are damped out by our resummation techniques. In Sec. IV we show that addition of chirally invariant counterterms destroys the current algebras. Since these were the original reason for setting up the Lagrangians, it would seem that an alternative to subtraction techniques is required. In Sec. V we investigate if it is possible that the ultraviolet divergences cancel among themselves. We find that this is not possible with mesons alone. We conclude with a discussion in Sec. VI of the implications of this for the general program of obtaining a Lagrangian realization of current algebras.

### II. CALCULATION OF $S$ -MATRIX ELEMENTS

We consider the pions alone.  $\pi = (\pi^1, \pi^2, \pi^3)$  are the fields and the interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = \frac{1}{2} f_{ab}(\pi) \partial_\mu \pi^a \partial_\mu \pi^b,$$

where  $f_{ab}$  belongs to a certain class of (nonpolynomial) functions such that Lagrangians corresponding to different functions are related by point transformations of the fields.<sup>6</sup>

<sup>1</sup> G. V. Efimov, CERN Report No. Th. 1087, 1969 (unpublished).

<sup>2</sup> R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969).

<sup>3</sup> B. W. Keck and J. G. Taylor, Proc. Phys. Soc. (London) (to be published).

<sup>4</sup> R. Delbourgo, K. Koller, and A. Salam, Imperial College Report No. ICTP/69/10, 1970 (unpublished).

<sup>5</sup> A. P. Hunt, K. Koller, and Q. Shafi, Phys. Rev. D **3**, 1327 (1971).

<sup>6</sup> See, e.g., K. J. Barnes and C. J. Isham, Nucl. Phys. **B15**, 333 (1970).