# Infinity Suppression in Gravity-Modified Quantum Electrodynamics

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Computations of gravity-modified quantum electrodynamics are performed using nonpolynominal Lagrangian field-theory techniques. The inverse of the gravitational constant appears as an effective cutoff mass, and, in particular, it is shown that to order $e^2$ the electron and photon self-energies are finite. The cutoff can be interpreted as if the electron had an intrinsic radius equal to its Schwarzschild radius. A central feature is the construction of the tensor gravity superpropagator.

## I. INTRODUCTION

It has been conjectured in the past that the universal and nonlinear coupling of gravitation to matter may provide a natural mechanism for the damping of ultraviolet infinities in field theory.[1] In a recent Letter[2] we revived this conjecture and pointed out that the newly developed techniques for computing with nonpolynomial Lagrangians would lend themselves to testing it. It is the purpose of this paper to show by actual computations to second order in the electromagnetic coupling that the conjectured damping indeed happens in gravity-modified electrodynamics. Our results are

$$\frac{\delta m}{m}\approx\frac{3\alpha}{4\pi}\ln\left(\frac{1}{\kappa m}\right)^2\approx\frac{2}{11},$$

$$\frac{\delta e}{e}\approx-\frac{\alpha}{6\pi}\ln\left(\frac{1}{\kappa m}\right)^2\approx-\frac{1}{25},$$

where $\kappa$ (the square root of $16\pi$ times the Newtonian constant $G$) equals $0.5\times10^{-18}$ GeV$^{-1}$. These results are of interest because in spite of the extreme smallness of the gravitational constant $\kappa$, the values obtained for $\delta m/m$ and $\delta e/e$ are a reasonable order of magnitude.[3,4] It is amusing that the effective cutoff appears to come at a length which equals the Schwarzschild radius of the electron $R_c=2m_e^2G$ (measured in units of $m_e^{-1}$). As pointed out recently,[4] it seems not unreasonable to hope that when higher-order terms in the effective perturbation parameter $\alpha\ln(\kappa m)^2$ are included, one will find, for example, that nearly all of the electron's mass can be explained as electromagnetic ($\delta m/m\approx1$).

This paper is in the nature of a report on how these results were arrived at and is planned as follows. In Sec. II we review Einstein's gravity theory—particularly as applied to electrons—and state the Feynman rules for graviton Lagrangians. The infinity-suppression mechanism is presented in Sec. III, which is devoted to the construction of the graviton superpropagator, and in Sec. IV, where the superpropagator is used in the explicit computation of the leading parts of the abovementioned quantities $\delta m/m$ and $\delta e/e$. In Sec. V we briefly discuss the question of gauge invariance and equivalence theorems. The details of the computations needed for Sec. IV are contained in the Appendix. Apart from the numerical results, the item of greatest interest in the paper may be the graviton superpropagator exhibited in Sec. III.

## II. GRAVITATIONAL LAGRANGIAN AND FREE PROPAGATOR

Einstein's theory of gravity is based upon the requirement that the equations of physics should be expressible in a form which is independent of the system of space-time coordinates to which they are referred. That is, these equations should be covariant under the group of general coordinate transformations; they should be derived by varying an action integral which is itself invariant under this group. To formulate the requirement one must first assign the variables of the problem to realizations of the transformation group. The following remarks are intended to sketch the main features of this program together with the steps necessary for treating quantum gravity.

The basic field upon which the group of general coordinate transformations are realized is the metric tensor $g_{\mu\nu}$ or, equivalently, the vierbein system $L_{\mu a}$. On the one hand, the metric tensor transforms according to the usual prescription

$$g_{\mu\nu}(x)\to\bar{g}_{\mu\nu}(\bar{x})=\frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}g_{\alpha\beta}(x)\,,\tag{2.1}$$

where $\partial\bar{x}^\mu/\partial x^\alpha$ denotes the Jacobian matrix of the transformation $x^\mu\to\bar{x}^\mu$. The vierbein components, on the other hand, transform according to a hybrid prescription

$$L_{\mu a}(x)\to\bar{L}_{\mu a}(\bar{x})=\frac{\partial x^\nu}{\partial\bar{x}^\mu}\Lambda_a{}^bL_{\nu b}(x)\,,\tag{2.2}$$

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[1] See, for example, S. Deser, in Proceedings of the Symposium on the Last Decade in Particle Theory, Center for Particle Theory, University of Texas, Austin, 1970 (unpublished).

[2] R. Delbourgo, Abdus Salam, and J. Strathdee, Nuovo Cimento Letters **2**, 354 (1969).

[3] Similar results have been obtained by F. Hoyle and V. Narlikar [Ann. Phys. (N. Y.) (to be published)] using different techniques, and by I. B. Khriplovich {Yadern. Fiz. **3**, 575 (1966) [Soviet J. Nucl. Phys. **3**, 415 (1966)]} using an integral equation for the mass operator. We are indebted to H. Pegis for bringing this work to our attention. See also B. S. DeWitt, Phys. Rev. Letters **13**, 114 (1964).

[4] Abdus Salam and J. Strathdee, Nuovo Cimento Letters **4**, 101 (1970).

where $\Lambda_a{}^b$ denotes a Lorentz matrix,

$$\Lambda_a{}^b\Lambda_c{}^d\eta_{bd}=\eta_{ac}\,, \tag{2.3}$$

with $\eta_{ab}=\mathrm{diag}\ (+1, -1, -1, -1)$. The metric tensor field can be expressed in terms of the vierbein field by means of the formula

$$g_{\mu\nu}=L_{\mu a}L_{\nu b}\eta^{ab}\,. \tag{2.4}$$

Conversely, the vierbein field can be expressed in a (locally) unique fashion in terms of the metric if the 16 components $L_{\mu a}$ are reduced to 10 independent ones by imposing the symmetry condition

$$L_{\mu a}=L_{a\mu}\,. \tag{2.5}$$

(In a pseudo-Euclidean notation where $\eta_{ab}$ is effectively replaced by the Kronecker symbol $\delta_{ab}$, the symmetric matrix $L$ is given by a square root of the symmetric matrix $g$.)

It is particularly important to realize that the symmetry condition (2.5) not only fixes $L$ in terms of $g$, but in addition serves to determine the Lorentz matrix $\Lambda$ of (2.2) as a function of $\partial\bar{x}/\partial x$ and $L$ itself. This results from the (local) uniqueness of the polar decomposition of the matrix $F_{\mu a}=\partial x^\nu/\partial\bar{x}^\mu L_{\nu a}$ into the product of a symmetric matrix $\bar{L}_{\mu b}$ and a (pseudo) orthogonal matrix $(\Lambda^{-1})_a{}^b$. Thus, corresponding to each coordinate transformation $x^\mu\rightarrow\bar{x}^\mu$, one has an associated Lorentz transformation $\Lambda$ of the vierbein system,

$$\Lambda_a{}^b=\Lambda_a{}^b(\partial\bar{x}/\partial x,g)\,.$$

The function indicated here is generally nonlinear. One has constructed in this way a nonlinear realization of the group of general coordinate transformations.[5]

Any local field which belongs to a representation of the Lorentz group can be made to carry a nonlinear realization of the general group. Thus

$$\psi(x)\rightarrow\bar{\psi}(\bar{x})=\left|\det\frac{\partial\bar{x}}{\partial x}\right|^w D(\Lambda)\psi(x)\,, \tag{2.6}$$

where $\Lambda$ denotes the Lorentz transformation determined above and $w$ denotes a new parameter, the *weight*, which must be assigned. (For the subgroup of Lorentz transformations on space-time we have $\Lambda=\partial\bar{x}/\partial x$ and $|\det\partial\bar{x}/\partial x|=1$, so that $w$ becomes irrelevant.) Nonlinear realizations based on Lorentz four-vectors and their products can be made over into linear representations of the conventional sort with the help of the vierbein. Thus, for example, if

$$\phi_a\rightarrow\bar{\phi}_a=\Lambda_a{}^b\phi_b\,,$$

then the combination $\phi_\mu=L_{\mu a}\eta^{ab}\phi_b$ transforms according to

$$\phi_\mu\rightarrow\bar{\phi}_\mu=\frac{\partial x^\nu}{\partial\bar{x}^\mu}\phi_\nu\,.$$

The spinorial nonlinear realizations cannot be linearized in this way. In order to treat the gravitational interactions of fermions it is essential to construct the vierbein components.

The scheme adopted here for introducing gravity by means of a nonlinear realization technique, which is well known to differential geometers, is not the usual one adopted in the physics literature. In the latter it is assumed that the vierbein field has 16 independent components $L_{\mu a}$ which transform under the coordinate group according to

$$L_{\mu a}\rightarrow\frac{\partial x^\nu}{\partial\bar{x}^\mu}L_{\nu a}$$

and under an *independent* "gauge group" according to

$$L_{\mu a}\rightarrow\Lambda_a{}^bL_{\mu b}\,,$$

where $\Lambda_a{}^b$ is not related to $\partial\bar{x}/\partial x$. In this view the field of, say, a Dirac particle would comprise a true Dirac spinor under the gauge group and a set of scalars under the coordinate group. This view is of course equivalent to the one we adopt.

In order to obtain an invariant action integral, one must construct the Lagrangian function so that it transforms as a scalar density,

$$d^4x\ \mathfrak{L}(\psi)=d^4\bar{x}\ \mathfrak{L}(\bar{\psi})\,,$$

that is,

$$\mathfrak{L}(\psi)\rightarrow\mathfrak{L}(\bar{\psi})=\left|\det\frac{\partial x}{\partial\bar{x}}\right|\mathfrak{L}(\psi)\,. \tag{2.7}$$

When expressed in terms of fields which belong to nonlinear realizations, $\mathfrak{L}(\psi)$ clearly must take the form of a Lorentz scalar with $w=-1$. Such a Lagrangian can be generated from any Lorentz invariant one by introducing $L_{\mu a}$ (or $g_{\mu\nu}$) and its derivative in accordance with two simple rules.

(i) Adjust the total weight of each term in $\mathfrak{L}$ to $-1$ by adjoining a factor $|\det L_{\mu a}|^{-w}=|\det g_{\mu\nu}|^{-w/2}$ which transforms as a Lorentz scalar with weight $w$.

(ii) Replace the ordinary derivative $\partial_\mu\psi=\psi_{,\mu}$ wherever it occurs by the covariant form

$$\psi_{;a}=L^\mu{}_a(\partial_\mu\psi-\tfrac{1}{2}iB_{\mu[bc]}S^{bc}\psi+wL^{\nu b}L_{\nu b,\mu}\psi)\,, \tag{2.8}$$

where $S^{bc}$ denotes the Lorentz spin matrix appropriate to $\psi$ and $B_{\mu[ab]}$, the nonlinear Riemannian connection, denotes the combination

$$B_{\mu[ab]}=\tfrac{1}{2}(L^\nu{}_a\partial_\mu L_{\nu b}-L^\nu{}_b\partial_\mu L_{\nu a})-\tfrac{1}{2}(L^\nu{}_a\partial_\nu L_{\mu b}-L^\nu{}_b\partial_\nu L_{\mu a})$$
$$-\tfrac{1}{2}L_{\mu c}(\partial_\lambda L_\nu{}^c-\partial_\nu L_\lambda{}^c)L^\lambda{}_aL^\nu{}_b\,. \tag{2.9}$$

The matrix reciprocal to $L_{\mu a}$ is here denoted by $L^{\mu a}$. This notation is consistent with the convention that

[5] A fuller discussion of this point of view is contained in C. J. Isham, Abdus Salam, and J. Strathdee, Ann. Phys. (N. Y.) (to be published). See also B. S. and C. DeWitt, Phys. Rev. **87**, 116 (1952); V. I. Ogievetskii and I. V. Polubarinov, Zh. Eksperim. i Teor. Fiz **48**, 1625 (1965) [Soviet Phys. JETP **21**, 1093 (1965)].

Greek indices are to be raised with $g^{\mu\nu}$ and lowered with $g_{\mu\nu}$ while Latin indices are raised and lowered with the Minkowskian tensor $\eta_{ab}=\eta^{ab}$. Thus we have

$$\begin{aligned} &g^{\mu\nu}g_{\nu\lambda}=\delta_\lambda{}^\mu,\\ &L^{\mu a}L_{\mu b}=\delta_b{}^a,\\ &L^{\mu a}L_{\nu a}=\delta_\nu{}^\mu,\\ &L^{\mu a}g_{\mu\nu}=L_\nu{}^a=L_{\nu b}\eta^{ba}, \text{ etc.}\end{aligned} \tag{2.10}$$

The connection $B_{\mu[ab]}$ is related to the Riemannian connection

$$L_{\mu\nu}{}^\lambda=\tfrac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu}+g_{\rho\nu,\mu}-g_{\mu\nu,\rho}) \tag{2.11}$$

by the formula

$$B_{\mu[ab]}=\Gamma_{\mu\nu}{}^\lambda L^\nu{}_a L_{\lambda b}-L^\nu{}_a L_{\nu b,\mu}. \tag{2.12}$$

To the Lagrangian generated by means of the rules (i) and (ii) it is of course necessary to add a purely gravitational term, viz.,

$$\mathfrak{L}_{\text{grav}}=(1/\kappa^2)(-\det g_{\mu\nu})^{1/2}R=(1/\kappa^2)(-\det L_{\mu a})R, \tag{2.13}$$

where $R$ denotes the scalar curvature which can be expressed in terms of either the metric tensor $g_{\mu\nu}$ or the vierbein components $L_{\mu a}$.

It is well known that the ten equations of motion which are obtained by varying $g_{\mu\nu}$ or the (symmetric) $L_{\mu a}$ are not all independent. They satisfy four identities as a consequence of general covariance. In other words, only six of the ten components of $g_{\mu\nu}$ can be determined by these equations. In order to pick out a unique solution it is necessary to supplement the equations of motion with a set of four "coordinate conditions" (which are analogs of the gauge condition of electrodynamics). For example, one could impose the Fock–de Donder conditions

$$\partial_\mu[(-\det g)^{1/2}g^{\mu\nu}]=0 \tag{2.14}$$

on the metric tensor or, alternatively,

$$\partial_\mu[(-\det L)^{1/2}L^{\mu a}]=0 \tag{2.15}$$

on the vierbein components. These classical coordinate conditions have a counterpart in the quantized theory which we shall now discuss.

In setting up the quantized theory one can allow for the fact that the vacuum expectation value of the metric tensor coincides with the flat space form by writing

$$g^{\mu\nu}=\eta^{\mu\nu}+\kappa h^{\mu\nu} \tag{2.16}$$

and treating $h^{\mu\nu}$ as the graviton field. This represents only one of various equivalent schemes. One could take instead of $g^{\mu\nu}$ any second-rank object made out of the metric tensor, its square root $L_{\mu a}$, or its inverse $g_{\mu\nu}$ multiplied into some power of $|\det g^{\mu\nu}|$. These different choices for the basic graviton field would yield different expressions for the Green's functions but presumably identical results for the on-mass-shell $S$-matrix elements. On the other hand, the perturbation developments of $S$-matrix elements may converge more rapidly for some choices than for others.

To construct the perturbation series one must separate from $L_{\text{grav}}$ the terms which are quadratic in $h$. These constitute the free Lagrangian $L_0$. Corresponding to the parametrization (2.16), one finds for $L_0$ the expression

$$\begin{aligned}\mathfrak{L}_0=\tfrac{1}{4}(&\partial_\mu h^{\lambda\rho}\partial_\mu h^{\lambda\rho}-2\partial_\rho h^{\lambda\mu}\partial_\lambda h^{\rho\mu}\\ &-\partial_\lambda h^{\mu\mu}\partial_\lambda h^{\rho\rho}+2\partial_\mu h^{\mu\lambda}\partial_\lambda h^{\rho\rho}),\end{aligned} \tag{2.17}$$

where contractions relative to the Minkowski metric are tacitly implied, e.g., $h^{\mu\mu}=h^{00}-h^{11}-h^{22}-h^{33}$. The remainder $L_{\text{grav}}-L_0$ is to be treated as an interaction Lagrangian.

The free Lagrangian (2.17) is degenerate (in the sense that it yields an underdetermined set of field equations) owing to the general covariance of the system. In order to define a bare-graviton propagator it is necessary to take account of this general covariance by imposing a set of coordinate conditions. One way to achieve this is by means of a Lagrange multiplier method originated by Fradkin.[6] To $L_{\text{grav}}$ one adds the noncovariant term

$$\begin{aligned}-\tfrac{1}{2}\bar{B}_\mu\bar{B}_\mu+\frac{1}{\kappa}\bar{B}_\nu\partial_\mu[(-g)^{1/2}g^{\mu\nu}]=&-\tfrac{1}{2}\bar{B}_\mu\bar{B}_\mu\\ &+\frac{1}{\kappa}\bar{B}_\nu\partial_\mu\left[\frac{\eta^{\mu\nu}+\kappa h^{\mu\nu}}{[-\det(\eta+\kappa h)]^{1/2}}\right],\end{aligned} \tag{2.18}$$

where $\bar{B}_\mu$ denotes the nonlocal expression

$$\bar{B}_\mu(x)=\int d_4y\;\mathfrak{D}_{\mu\nu}(x,y|h)B_\nu(y),$$

with a local field $B_\mu(x)$ to be varied independently of $h^{\mu\nu}$. The kernel $\mathfrak{D}_{\mu\nu}$ is determined by requiring that the equation

$$\partial^2 B_\mu(x)=0 \tag{2.19}$$

emerge as one of the equations of motion. Since this artificial field $B_\mu$ is free, one can pick out the space of physical states by means of the condition

$$B_\mu{}^{(+)}(x)|\;\rangle=0 \tag{2.20}$$

and be assured that the $S$ matrix is unitary in this space.

Having determined $\mathfrak{D}$, one can proceed to decouple the artificial field $B(x)$ by means of a field transformation

$$B_\mu\to B_\mu'=\mathfrak{D}_{\mu\nu}B_\nu-(1/\kappa)\eta_{\mu\alpha}\partial_\beta[(-g)^{1/2}g^{\alpha\beta}]. \tag{2.21}$$

Since this transformation is a nonlocal one, it yields a Jacobian factor with a nonlocal structure which must be taken into account in computing $S$-matrix elements.

[6] E. S. Fradkin and I. V. Tyutin, Phys. Rev. D **2**, 2841 (1970); Abdus Salam and J. Strathdee, *ibid.* **2**, 2869 (1970).

Thus

$$\left|\det\left(\frac{\delta B_\mu(x)}{\delta B_\nu'(x')}\right)\right| = \exp[-\mathrm{Tr}\ln \mathfrak{D}_{\mu\nu}(x,x'|h)], \quad (2.22)$$

which can be evaluated by perturbation methods. The perturbation development of this functional can be represented graphically by a set of diagrams in which external graviton lines are coupled to closed disjoint loops of a massless vector particle. We shall not go into the rather complicated details[7] here since in fact we shall have no occasion to include the fictitious particle loops in the processes to be considered in Sec. IV. This discussion was intended only to make plausible the effective gravitational action

$$\int dx \left\{ \mathfrak{L}_{\mathrm{grav}} + \frac{1}{2\kappa^2}[\partial_\mu((-g)^{1/2}g^{\mu\nu})]^2 - i\,\mathrm{Tr}\ln\mathfrak{D}\right\}, \quad (2.23)$$

from which one can separate the bilinear terms which define a nondegenerate free-graviton Lagrangian,

$$\mathfrak{L}_0 = \tfrac{1}{4}(\partial_\mu h^{\lambda\rho}\partial_\mu h^{\lambda\rho} - \tfrac{1}{2}\partial_\mu h^{\lambda\lambda}\partial_\mu h^{\rho\rho}). \quad (2.24)$$

The bare-graviton propagator which corresponds to this Lagrangian is simply

$$\langle T(h^{\lambda\mu}(x)h^{\nu\rho}(0))\rangle = \tfrac{1}{2}(\eta^{\lambda\nu}\eta^{\mu\rho}+\eta^{\lambda\rho}\eta^{\mu\nu}-\eta^{\lambda\mu}\eta^{\nu\rho})D(x), \quad (2.25)$$

where $D$ denotes the zero-mass causal propagator $(-4\pi^2x^2)^{-1}$.

Similar arguments can be applied when the graviton is interpolated by the vierbein field $L^{\mu a}$ rather than $g^{\mu\nu}$. One must substitute

$$L^{\mu a} = \eta^{\mu a} + \tfrac{1}{2}\kappa\phi^{\mu a} \quad (2.26)$$

and collect the terms bilinear in $h$ in order to define the bare-graviton propagator. Clearly these bilinear terms are the same as those obtained above, so that (2.25) remains the bare-graviton propagator. The interaction Lagrangian, which involves terms of the third and higher orders, will be different.

## III. GRAVITON SUPERPROPAGATOR

The Lagrangian of gravity-modified electrodynamics is given by

$$\mathfrak{L} = \mathfrak{L}_{\mathrm{grav}} + \mathfrak{L}_{\mathrm{matter}}, \quad (3.1)$$

where the purely gravitational part $\mathfrak{L}_{\mathrm{grav}}$ is the usual Einstein one,

$$\mathfrak{L}_{\mathrm{grav}} = \frac{1}{\kappa^2 \det L}R(L) = \frac{L^{\mu a}L^{\nu b}}{\kappa^2\det L}(B_{\mu ac}B_{\nu cb} - B_{\nu ac}B_{\mu cb}) + (\text{four-divergence}), \quad (3.2)$$

where the scalar curvature $R$ has been expressed as a function of the vierbein components. The vierbein connection $B_{\mu ab}$ is given by (2.9) as a function of $L$ and its first derivatives. The matter part of (3.1) is given by

$$\mathfrak{L}_{\mathrm{matter}} = (1/\det L)[\tfrac{1}{2}i(\bar\psi\gamma_a\psi_{;\mu} - \bar\psi_{;\mu}\gamma_a\psi)L^{\mu a} - m_0\bar\psi\psi + e_0\bar\psi\gamma_a A_\mu\psi L^{\mu a} - \tfrac{1}{4}L^{\mu a}L^{\nu a}L^{\kappa b}L^{\lambda b}F_{\mu\kappa}F_{\nu\lambda}] + 3i\delta^4(0)\ln|\det L|, \quad (3.3)$$

where

$$\psi_{;\mu} = \partial_\mu\psi - \tfrac{1}{4}iB_{\mu ab}\sigma_{ab}\psi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.4)$$

The general covariance of (3.3) and (3.4) is assured if both the electron and photon fields are assigned the weight $w=0$ [see Eq. (2.6)]. In the above expressions we are using the notational convention that repeated Latin indices are summed in the Minkowskian sense: $A_aB_a = \eta^{ab}A_aB_b = A^aB_a$, etc. The term proportional to $\delta^4(0)$ is due to the presence of a derivative of the electron field in the interaction. It can be justified by a canonical quantization procedure (Salam and Strathdee[6]). There could well be an analogous term in the purely gravitational Lagrangian but we have not been able to derive its form yet.

In order to complete the system of equations it is necessary, as outlined in Sec. II, to add to (3.1) a term which breaks the electromagnetic and gravitational gauge symmetries. Such a term is given by

$$\mathfrak{L}_{\mathrm{gauge}} = -\frac{1}{2\det L}(L^{\mu a}L^{\nu a}A_{\mu;\nu})^2 + \frac{2}{\kappa^2}\left\{\partial_\mu\left[\frac{L^{\mu a}}{(-\det L)^{1/2}}\right]\right\}^2. \quad (3.5)$$

This choice of symmetry-breaking term leads to the following expressions for the bare propagators of the photon and graviton, respectively:

$$\langle 0|T(A_\mu(x)A_\nu(0))|0\rangle = -\eta_{\mu\nu}D(x),$$
$$\langle 0|T(\phi^{\mu a}(x)\phi^{\nu b}(0))|0\rangle = \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab} - \eta^{\mu a}\eta^{\nu b} + \eta^{\mu\nu}\eta^{ab})D(x), \quad (3.6)$$

where $D(x)$ denotes the zero-mass causal propagator $[-4\pi^2(x^2-i0)]^{-1}$. The graviton field $\phi^{\mu a}$ is defined in terms of the vierbein components by

$$L^{\mu a} = \eta^{\mu a} + \tfrac{1}{2}\kappa\phi^{\mu a}. \quad (3.7)$$

With these definitions it is possible to proceed in the usual way to separate from $\mathfrak{L}+\mathfrak{L}_{\mathrm{gauge}}$ the interaction part and to construct a perturbation series. For our purposes, only one part of the interaction need be considered, viz.,

$$\mathfrak{L}_{\mathrm{em}} = e_0\frac{L^{\mu a}}{\det L}\bar\psi\gamma_a\psi A_\mu, \quad (3.8)$$

which will be used in Sec. IV to obtain a contribution of order $e_0^2$ to the electron and photon propagators.

[7] R. P. Feynman, Acta Phys. Polon. 24, 697 (1963); B. S. DeWitt, Phys. Rev. 162, 1195 (1967); L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967); S. Mandelstam, Phys. Rev. 175, 1604 (1968); E. S. Fradkin and I. V. Tyutin, Ref. 6.

There will of course be other contributions of this order due to gravitational couplings of the electron and photon fields. These, which are presumably of higher order in the gravitational constant $\kappa^2$, we shall disregard. From (3.8) one obtains for the electron self-energy part

$$(1/i)\Sigma(x)=e_0{}^2\mathfrak{D}^{\mu a,\nu b}(x)\gamma_a S(x)\gamma_b D_{\mu\nu}(x) \qquad (3.9)$$

and, for the photon self-energy part,

$$(1/i)\Pi_{\mu\nu}(x)=-e_0{}^2\mathfrak{D}^{\mu a,\nu b}(x)\times\mathrm{Tr}[\gamma_a S(x)\gamma_b S(-x)], \qquad (3.10)$$

where $S(x)$ and $D_{\mu\nu}(x)$ denote the electron and photon bare propagators, respectively. The graviton superpropagator $\mathfrak{D}^{\mu a,\nu b}(x)$ which appears in these expressions is defined by

$$\mathfrak{D}^{\mu a,\nu b}(x)=\langle 0|T\left(\frac{L^{\mu a}(x)}{\det L(x)}\frac{L^{\nu b}(0)}{\det L(0)}\right)|0\rangle, \qquad (3.11)$$

where $L^{\mu a}(x)$ is given in terms of the bare-graviton field $\phi^{\mu a}(x)$ by (3.7). The propagator of $\phi^{\mu a}$ is given by (3.6). The construction of the superpropagator from the bare one is a complicated algebraic operation, the main steps of which are sketched in the remainder of this section.

It is convenient to refer the graviton field to a Euclidean basis where the Minkowskian tensor $\eta^{\alpha\beta}$ is replaced by the Kronecker symbol $\delta^{\alpha\beta}$. At the end of the calculation one can transform back to the Minkowskian basis.

The basic step (see Delbourgo and Hunt[8]) in the simplification of (3.11) is to introduce for $(\det L)^{-1}$ the integral representation,

$$\frac{1}{\det L}=\frac{1}{\pi^4}\int d^4m d^4n\,\exp[-(m_\alpha m_\beta+n_\alpha n_\beta)L^{\alpha\beta}], \qquad (3.12)$$

where $m_\alpha$ and $n_\alpha$ are integrated over Euclidean four-space. This representation is useful because the chronological pairing of exponentials takes a simple form, viz.,

$$\begin{aligned}\langle 0|T(\exp[-a_{\alpha\beta}L^{\alpha\beta}(x)],\exp[-b_{\gamma\delta}L^{\gamma\delta}(0)])|0\rangle\\ =\exp(-a_{\alpha\alpha}-b_{\alpha\alpha})\exp[a_{\alpha\beta}\tfrac{1}{2}(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}-\delta^{\alpha\beta}\delta^{\gamma\delta})b_{\gamma\delta}\kappa^2D/4]\\ =\exp(-a_{\alpha\alpha}-b_{\alpha\alpha})\exp[(a_{\alpha\beta}b_{\alpha\beta}-\tfrac{1}{2}a_{\alpha\alpha}b_{\beta\beta})\kappa^2D/4],\end{aligned} \qquad (3.13)$$

which can be verified by expanding the exponentials in powers of $\kappa a_{\alpha\beta}\phi^{\alpha\beta}(x)$ and $\kappa b_{\alpha\beta}\phi^{\alpha\beta}(0)$, respectively, and applying (3.6). Differentiation of (3.13) with respect to $a_{\kappa\lambda}$ and $b_{\mu\nu}$ yields the formula

$$\begin{aligned}\langle 0|T(L^{\kappa\lambda}(x)\exp[-a_{\alpha\beta}L^{\alpha\beta}(x)],\ L^{\mu\nu}(0)\exp[-b_{\gamma\delta}L^{\gamma\delta}(0)])|0\rangle\\ =\{\tfrac{1}{2}(\delta^{\kappa\mu}\delta^{\lambda\nu}+\delta^{\kappa\nu}\delta^{\lambda\mu}-\delta^{\kappa\lambda}\delta^{\mu\nu})\kappa^2D/4\\ +[\delta^{\kappa\lambda}-(b^{\kappa\lambda}-\tfrac{1}{2}\delta^{\kappa\lambda}b^{\rho\rho})\kappa^2D/4]\\ \times[\delta^{\mu\nu}-(a^{\mu\nu}-\tfrac{1}{2}\delta^{\mu\nu}a^{\rho\rho})\kappa^2D/4]\}\\ \times\langle 0|T(\exp[-a_{\alpha\beta}L^{\alpha\beta}(x)],\exp[-b_{\gamma\delta}L^{\gamma\delta}(0)])|0\rangle,\end{aligned}$$

from which one obtains—by substituting $a_{\alpha\beta}=m_\alpha m_\beta+n_\alpha n_\beta$ and $b_{\alpha\beta}=m_\alpha' m_\beta'+n_\alpha' n_\beta'$ and then integrating over all 16 components—an expression for the superpropagator. It is convenient to define a pair of scalar amplitudes $\mathfrak{D}^{(0)}$ and $\mathfrak{D}^{(1)}$ by writing

$$\mathfrak{D}^{\kappa\lambda,\mu\nu}=\delta^{\kappa\lambda}\delta^{\mu\nu}\mathfrak{D}^{(0)}+\tfrac{1}{2}(\delta^{\kappa\mu}\delta^{\lambda\nu}+\delta^{\kappa\nu}\delta^{\lambda\mu}-\delta^{\kappa\lambda}\delta^{\mu\nu})\mathfrak{D}^{(1)}. \qquad (3.14)$$

The amplitudes $\mathfrak{D}^{(0)}$ and $\mathfrak{D}^{(1)}$ are now represented formally by 16-dimensional integrals:

$$\begin{aligned}\mathfrak{D}^{(0)}=\frac{1}{\pi^8}\int d^4md^4nd^4m'd^4n'\{1+\tfrac{1}{4}(m^2+n^2+m'^2+n'^2)\\ \times\kappa^2D/4+(1/18)(m^2+n^2)(m'^2+n'^2)(\kappa^2D/4)^2\\ +(1/36)[(m\cdot m')^2+(m\cdot n')^2+(n\cdot m')^2\\ +(n\cdot n')^2](\kappa^2D/4)^2\}e^F,\end{aligned} \qquad (3.15)$$

$$\begin{aligned}\mathfrak{D}^{(1)}=\frac{1}{\pi^8}\int d^4md^4nd^4m'd^4n'\{\kappa^2D/4\\ -(1/36)(m^2+n^2)(m'^2+n'^2)(\kappa^2D/4)^2\\ +\tfrac{1}{9}[(m\cdot m')^2+(m\cdot n')^2+(n\cdot m')^2\\ +(n\cdot n')^2](\kappa^2D/4)^2\}e^F,\end{aligned} \qquad (3.16)$$

where

$$\begin{aligned}F=-(m^2+n^2+m'^2+n'^2)\\ +[(m\cdot m')^2+(m\cdot n')^2+(n\cdot m')^2+(n\cdot n')^2\\ -\tfrac{1}{2}(m^2+n^2)(m'^2+n'^2)]\kappa^2D/4.\end{aligned} \qquad (3.17)$$

As they stand, the Riemann integrals (3.15) and (3.16) are divergent. This divergence reflects the fact that one is attempting to sum divergent series [the terms of which can be recovered by expanding the integrands of (3.15) and (3.16) in powers of $\kappa^2D$ and performing the resulting Gauss-type integrations]. In earlier references[8] the method of Borel summation was used to obtain amplitudes for which the divergent series correspond to asymptotic representations. An equivalent method, which is more convenient in the present case, is to obtain these amplitudes by means of analytic continuation[9] in a set of auxiliary parameters $\alpha,\beta,\gamma$. To this end, consider the integral

$$\begin{aligned}\mathfrak{D}(\alpha,\beta,\gamma)=\frac{1}{\pi^8}\int d^4md^4nd^4m'd^4n'\\ \times\exp\{-\alpha(m^2+n^2+m'^2+n'^2)\\ +\beta[(m\cdot m')^2+(m\cdot n')^2+(n\cdot m')^2+(n\cdot n')^2]\kappa^2D/4\\ -\tfrac{1}{2}\gamma(m^2+n^2)(m'^2+n'^2)\kappa^2D/4\},\end{aligned} \qquad (3.18)$$

which converges for real and positive values of $\kappa^2D$

[8] E. S. Fradkin, Nucl. Phys. **49**, 624 (1963); G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **44**, 2107 (1963) [Soviet Phys. JETP **17**, 1417 (1963)]; R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969); R. Delbourgo and A. Hunt, Imperial College, London, Report No. ICTP/69/8 (unpublished).

[9] This technique for defining an integral is a central feature of the method of Gel'fand and Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I. This method was employed particularly by M. K. Volkov, Ann. Phys. (N. Y.) **49**, 202 (1968); and also by J. J. Giambiagi and J. Tiomno, Nuovo Cimento Letters **2**, 674 (1969); and by Abdus Salam and J. Strathdee, Phys. Rev. D **1**, 3296 (1970).

provided that

$$\mathrm{Re}(\beta-\tfrac{1}{2}\gamma)<0 \quad\text{and}\quad \mathrm{Re}\alpha>0. \tag{3.19}$$

The amplitudes $\mathfrak{D}^{(0)}$ and $\mathfrak{D}^{(1)}$ are expressed in terms of $\mathfrak{D}(\alpha,\beta,\gamma)$ and its partial derivatives as follows:

$$\mathfrak{D}^{(0)}=\left[1+\frac{\kappa^2D}{4}\left(-\frac{1}{4}\frac{\partial}{\partial\alpha}+\frac{1}{36}\frac{\partial}{\partial\beta}-\frac{1}{9}\frac{\partial}{\partial\gamma}\right)\right]\times\mathfrak{D}(\alpha,\beta,\gamma)|_{\alpha=\beta=\gamma=1}, \tag{3.20}$$

$$\mathfrak{D}^{(1)}=\frac{\kappa^2D}{4}\left(1+\frac{1}{9}\frac{\partial}{\partial\beta}+\frac{1}{18}\frac{\partial}{\partial\gamma}\right)\mathfrak{D}(\alpha,\beta,\gamma)|_{\alpha=\beta=\gamma=1},$$

where the limit $\alpha=\beta=\gamma=1$ is to be taken in a sense to be specified below.

Many of the integrations in (3.18) are straightforward. Thus, for example, since the exponent in the integrand is a bilinear form in the eight components $(m'_\alpha,n'_\alpha)$, these integrations are Gaussian, yielding

$$\mathfrak{D}(\alpha,\beta,\gamma)=\frac{1}{\pi^4}\int d^4md^4n\{\exp[-\alpha(m^2+n^2)]\}\times\left[\alpha+\frac{\gamma}{2}(m^2+n^2)\frac{\kappa^2D}{4}\right]^{-2}\left[\alpha^2+\alpha(\gamma-\beta)(m^2+n^2)\frac{\kappa^2D}{4}+\frac{\gamma}{4}(\gamma-2\beta)(m^2+n^2)^2\left(\frac{\kappa^2D}{4}\right)^2+\beta^2[m^2n^2-(m\cdot n)^2]\left(\frac{\kappa^2D}{4}\right)^2\right]^{-1}.$$

Here the integrand depends only on the two combinations

$$\xi=m^2+n^2 \quad\text{and}\quad u^2=4\frac{m^2n^2-(m\cdot n)^2}{(m^2+n^2)^2}$$

and the integral takes the form

$$\mathfrak{D}(\alpha,\beta,\gamma)=\frac{1}{2}\int_0^\infty d\xi\,\xi^3e^{-\alpha\xi}[\alpha+\gamma\xi(\tfrac{1}{8}\kappa^2D)]^{-2}\times\int_0^1 du\,u^2\{\alpha^2+2\alpha(\gamma-\beta)\xi(\tfrac{1}{8}\kappa^2D)+[\gamma(\gamma-2\beta)+\beta^2u^2]\xi^2(\tfrac{1}{8}\kappa^2D)^2\}^{-1}. \tag{3.21}$$

It is useful to change the integration variables from $\xi$, $u$ to $\sigma$, $v$, which are defined by

$$v^2=1-u^2,\qquad \sigma=\frac{8\alpha+(\gamma-\beta)\xi\kappa^2D}{\beta\xi\kappa^2D}=8\alpha/\beta\xi\kappa^2D+\sigma_0, \tag{3.22}$$

where

$$\sigma_0=(\gamma-\beta)/\beta.$$

The parameter $\sigma_0$, which ultimately takes the value zero, must in the course of the integrations be kept in the half-plane

$$\mathrm{Re}\sigma_0>1.$$

The integral (3.21) now takes the form

$$\mathfrak{D}(\alpha,\beta,\gamma)=\frac{1}{2}\left(\frac{\kappa^2D}{8}\right)^{-4}\frac{1}{\beta^4}\times\int_{\sigma_0}^\infty\frac{d\sigma}{(1+\sigma)^2}\frac{\exp[-(8\alpha^2/\beta)(\kappa^2D(\sigma-\sigma_0))^{-1}]}{(\sigma-\sigma_0)}\times\frac{1}{2}\int_0^1 dv^2\frac{(1-v^2)^{1/2}}{\sigma^2-v^2}. \tag{3.23}$$

Of particular interest for the computations of Sec. IV is the Mellin transform of $\mathfrak{D}(\alpha,\beta,\gamma)$ defined by the integral representation

$$\mathfrak{D}(\alpha,\beta,\gamma)=\frac{1}{2\pi i}\int_{C_0}dz\,\Gamma(-z)\mathfrak{D}(\alpha,\beta,\gamma;z)(\kappa^2D)^z, \tag{3.24}$$

where the contour $C_0$ lies parallel to the imaginary axis with $-1<\mathrm{Re}z<0$. The amplitude $\mathfrak{D}$ is given by

$$\begin{aligned}\mathfrak{D}(\alpha,\beta,\gamma;z)&=\frac{1}{\Gamma(-z)}\int_0^\infty d(\kappa^2D)(\kappa^2D)^{-z-1}\mathfrak{D}(\alpha,\beta,\gamma)\\&=\frac{2^{10}}{\beta^4}\int_0^1dv^2\int_{\sigma_0}^\infty d\sigma\frac{(1-v^2)^{1/2}}{(1+\sigma)^2(\sigma-\sigma_0)(\sigma^2-v^2)}\frac{1}{\Gamma(-z)}\times\int_0^\infty d(\kappa^2D)(\kappa^2D)^{-z-5}\exp\left[-\frac{8}{\kappa^2D}\frac{\alpha^2/\beta}{\sigma-\sigma_0}\right]\\&=\frac{\Gamma(z+4)}{4\Gamma(-z)}\left(\frac{\beta}{8}\right)^z\left(\frac{1}{\alpha^2}\right)^{z+4}\int_0^1dv^2(1-v^2)^{1/2}\times\int_{\sigma_0}^\infty d\sigma\frac{(\sigma-\sigma_0)^{z+3}}{(1+\sigma)^2(\sigma^2-v^2)}.\end{aligned} \tag{3.25}$$

To evaluate the $\sigma$ integral, one can first replace it by a contour integral

$$\int_{\sigma_0}^\infty d\sigma\frac{(\sigma-\sigma_0)^{z-3}}{(1+\sigma)^2(\sigma^2-v^2)}=\frac{1}{2i\sin\pi z}\int_C d\sigma\frac{(\sigma_0-\sigma)^{z+3}}{(1+\sigma)^2(\sigma^2-v^2)},$$

where the contour $C$ is shown in Fig. 1. Since the integrand falls off for large $|\sigma|$ like $|\sigma|^{\mathrm{Re}z-1}$, it is possible to distort the contour so that it encircles only the singularities at $-1$ and $\pm v$, i.e., into the form $C'$ shown in

Fig. 1—provided $\mathrm{Re}z<0$. One finds

$$\frac{1}{\Gamma(-z)}\int_{\sigma_0}^{\infty}d\sigma\frac{(\sigma-\sigma_0)^{z+3}}{(1+\sigma)^2(\sigma^2-v^2)}$$
$$=-\frac{\Gamma(z+1)}{2\pi i}\int_{C'}d\sigma\frac{(\sigma_0-\sigma)^{z+3}}{(1+\sigma)^2(\sigma^2-v^2)}$$
$$=-\Gamma(z+1)\left[\frac{1}{2v}\frac{(\sigma_0-v)^{z+3}}{(1+v)^2}-\frac{1}{2v}\frac{(\sigma_0+v)^{z+3}}{(1-v)^2}\right.$$
$$\left.+\frac{2(\sigma_0+1)^{z+3}}{(1-v^2)^2}-\frac{(z+3)(\sigma_0+1)^{z+2}}{1-v^2}\right],$$

which can be substituted into (3.25) to give

$$\mathfrak{D}(\alpha,\beta,\gamma;z)$$
$$=\tfrac{1}{4}\Gamma(z+1)\Gamma(z+4)(\tfrac{1}{2}\beta)^z(1/\alpha^2)^{z+4}I(z,\sigma_0)\,, \quad (3.26)$$

where

$$I(z,\sigma_0)=-\int_0^1 dv(1-v^2)^{1/2}$$
$$\times\left[\frac{(\sigma_0-v)^{z+3}}{(1+v)^2}-\frac{(\sigma_0+v)^{z+3}}{(1-v)^2}+4(\sigma_0+1)^{z+3}\frac{v}{(1-v^2)^2}\right.$$
$$\left.-2(z+3)(\sigma_0+1)^{z+2}\frac{v}{1-v^2}\right], \quad (3.27)$$

which can be evaluated with the help of a table of integrals. However, for our purposes it will be sufficient to evaluate only the expression

$$I(z)\equiv\tfrac{1}{2}[I(z,+i0)+I(z,-i0)]$$
$$=\int_0^1 dv(1-v^2)^{1/2}\left[\frac{v^{z+3}}{(1+v)^2}\cos\pi z\right.$$
$$\left.+\frac{v^{z+3}}{(1-v)^2}-\frac{4v}{(1-v^2)^2}+\frac{2v(z+3)}{1-v^2}\right]$$
$$=\cos\pi z\, B(\tfrac{3}{2},z+4)F(\tfrac{3}{2},z+4;z+\tfrac{11}{2};-1)$$
$$+B(-\tfrac{1}{2},z+4)F(-\tfrac{1}{2},z+4;z+\tfrac{7}{2};-1)$$
$$-4B(-\tfrac{1}{2},2)F(\tfrac{3}{2},2;\tfrac{3}{2};-1)$$
$$+2(z+3)B(\tfrac{1}{2},2)F(\tfrac{1}{2},2;\tfrac{5}{2};-1)\,. \quad (3.28)$$

This expression corresponds to a particular limiting procedure. Thus, holding $\alpha$ and $\beta$ fixed at positive real values, the limit $\sigma_0\to\pm i0$ can be interpreted as $\gamma\to\beta\pm i0$. Considered as a function of the complex variable $\gamma$, the amplitude $\mathfrak{D}$ has a branch point at $\gamma=2\beta$ with the attached cut running to the left. We are taking the average of the values on the upper and lower sides of this cut in order to have a real amplitude. It may be remarked that the difference of the upper and lower

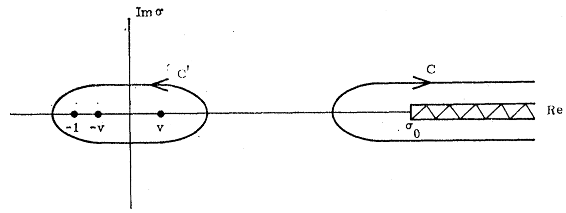


FIG. 1. $\sigma$-plane contours for the graviton superpropagator.

limits will vanish at integer values of $z$ and so must correspond to an entire function in momentum space.[10]

In order to construct the Mellin amplitudes corresponding to $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$, it is necessary to differentiate (3.26) with respect to the parameters $\alpha$, $\beta$, and $\gamma$, which are then set equal to unity. Using the identity

$$\partial I(z,\sigma_0)/\partial\sigma_0=(z+3)I(z-1,\sigma_0)\,, \quad (3.29)$$

which can be deduced from the integral representation (3.27), one finds

$$\frac{\partial}{\partial\alpha}\mathfrak{D}(z)=-2(z+4)\mathfrak{D}(z)\,,$$
$$\frac{\partial}{\partial\beta}\mathfrak{D}(z)=z\mathfrak{D}(z)-z(z+3)^2\mathfrak{D}(z-1)\,,$$
$$\frac{\partial}{\partial\gamma}\mathfrak{D}(z)=z(z+3)^2\mathfrak{D}(z-1)\,,$$

where

$$\mathfrak{D}(z)=(\tfrac{1}{2})^{z+2}\Gamma(z+1)\Gamma(z+4)I(z)\,.$$

Finally one arrives at the formula

$$\mathcal{D}^{\mu a,\nu b}(x)=\frac{1}{2\pi i}\int_{C_0}dz\;\Gamma(-z)[\eta^{\mu a}\eta^{\nu b}\mathfrak{D}^{(0)}(z)$$
$$+\tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab}+\eta^{\mu b}\eta^{\nu a}-\eta^{\mu a}\eta^{\nu b})\mathfrak{D}^{(1)}(z)](\kappa^2D)^z, \quad (3.30)$$

where the amplitudes $\mathfrak{D}^{(0)}(z)$ and $\mathfrak{D}^{(1)}(z)$ are given, respectively, by

$$\mathfrak{D}^{(0)}(z)=\mathfrak{D}(z)-(1/36)z(19z+53)\mathfrak{D}(z-1)$$
$$+(5/36)z(z-1)(z+2)^2\mathfrak{D}(z-2)\,,$$
$$\mathfrak{D}^{(1)}(z)=-\tfrac{1}{9}z(z+8)\mathfrak{D}(z-1)$$
$$+(1/18)z(z-1)(z+2)^2\mathfrak{D}(z-2)\,, \quad (3.31)$$

with $\mathfrak{D}(z)$ given by

$$\mathfrak{D}(z)=(\tfrac{1}{2})^{z+2}\Gamma(z+1)\Gamma(z+4)$$
$$\times[\cos\pi z\,B(\tfrac{3}{2},z+4)F(\tfrac{3}{2},z+4;z+\tfrac{11}{2};-1)$$
$$+B(-\tfrac{1}{2},z+4)F(-\tfrac{1}{2},z+4;z+\tfrac{7}{2};-1)$$
$$-4B(-\tfrac{1}{2},2)F(\tfrac{3}{2},2;\tfrac{3}{2};-1)$$
$$+2(z+3)B(\tfrac{1}{2},2)F(\tfrac{1}{2},2;\tfrac{5}{2};-1)]\,. \quad (3.32)$$

[10] The possibility of taking the limit $\sigma_0\to0$ in different ways is a reflection of the arbitrariness in defining $T$ products. See, for example, the discussions by M. K. Volkov (Ref. 9) and B. W. Lee and B. Zumino, Nucl. Phys. **13B**, 671 (1969).

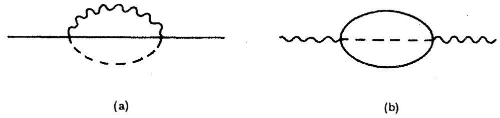


FIG. 2. (a) Electron self-energy $\Sigma$; (b) photon self-energy $\Pi_{\mu\nu}$.

Of particular importance for Sec. IV is the behavior near $z=0$, where

$$\begin{aligned}\mathfrak{D}^{(0)}(z)&=1+7.5z+O(z^2)\,,\\ \mathfrak{D}^{(1)}(z)&=-0.5z+O(z^2)\,.\end{aligned}\tag{3.33}$$

## IV. SECOND-ORDER COMPUTATIONS OF SELF-CHARGE AND SELF-MASS

We now apply the method outlined above to the computation of electromagnetic corrections to the electron and photon masses in the presence of a gravitational field. The graphs to be considered are of order $e^2$ and are shown in Figs. 2(a) and 2(b). In these figures the electron propagator is represented by the solid line, the photon propagator by the wavy line, and the multigraviton propagator by the dotted line. Corresponding to Fig. 1(a), the configuration space amplitude for electron self-energy is given by [using the approximation of Eq. (3.24)]

$$\begin{aligned}(1/i)\Sigma(x)&=e^2\gamma_\mu S(x)\gamma_\nu D_{\alpha\beta}(x)[\mathfrak{D}^{(0)}(x)\eta_{\mu\alpha}\eta_{\nu\beta}\\&\quad+\tfrac{1}{2}\mathfrak{D}^{(1)}(x)(\eta_{\mu\nu}\eta_{\alpha\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\alpha}\eta_{\nu\beta})]\\&=\frac{e^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}dz\,\Gamma(-z)[\mathfrak{D}^{(0)}(z)+2\mathfrak{D}^{(1)}(z)]\\&\qquad\times\gamma_\mu S(x)\gamma_\mu[\kappa^2D(x)]^z\,,\end{aligned}\tag{4.1}$$

where $\mathfrak{D}^{(0)}(z)$ and $\mathfrak{D}^{(1)}(z)$ are given by Eq. (3.31). Graphically, the pole $\Gamma(-z)$ at $z=0$ corresponds to the no-graviton exchange contribution, the pole at $z=1$ gives the one-graviton exchange contribution, and so on.

Since, as will be seen in the following, the dominant contribution to $\Sigma$ comes from the neighborhood of $z=0$, the important quantities are

$$\begin{aligned}&\mathfrak{D}^{(0)}(0)=1\,,\qquad\qquad \mathfrak{D}^{(1)}(0)=0\,,\\ &\frac{d\mathfrak{D}^{(0)}(0)}{dz}=-2\psi(1)+\frac{118\ln2+31}{18}\approx7.5\,,\\ &\qquad\qquad\frac{d\mathfrak{D}^{(1)}(0)}{dz}=\frac{52\ln2-40}{9}\approx-0.5\,,\end{aligned}\tag{4.2}$$

where $\psi(a)$ denotes the logarithmic derivative of the $\Gamma$ function.

Corresponding to Fig. 1(b), the configuration space amplitude for photon self-energy is given by

$$\begin{aligned}\frac{1}{i}\Pi_{\mu\nu}(x)&=\frac{e^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}dz\,\Gamma(-z)[\mathfrak{D}^{(0)}(z)+\tfrac{1}{2}\mathfrak{D}^{(1)}(z)]\\&\qquad\times\mathrm{Tr}(\gamma_\mu S(x)\gamma_\nu S(-x))(\kappa^2D)^z\,,\end{aligned}\tag{4.3}$$

where $c<0$.

The momentum space amplitudes $\Sigma(p)$ and $\Pi_{\mu\nu}(k)$ which correspond to (4.1) and (4.3) are given, according to the method of Ref. 9, by the respective contour integrals

$$\begin{aligned}\Sigma(p)&=\frac{1}{2\pi i}\int dz\,\Gamma(-z)[\mathfrak{D}^{(0)}(z)+2\mathfrak{D}^{(1)}(z)]\\&\qquad\times\Sigma(p,z)\,,\end{aligned}\tag{4.4}$$

$$\begin{aligned}\Pi_{\mu\nu}(k)&=\frac{1}{2\pi i}\int dz\,\Gamma(-z)[\mathfrak{D}^{(0)}(z)+\tfrac{1}{2}\mathfrak{D}^{(1)}(z)]\\&\qquad\times\Pi_{\mu\nu}(k,z)\,,\end{aligned}\tag{4.5}$$

where $\Sigma(p,z)$ and $\Pi_{\mu\nu}(k,z)$ are defined for sufficiently negative real values of $z$ by the convergent integrals

$$\frac{1}{i}\Sigma(p,z)=\int dx\,e^{ipx}e^2\gamma_\mu S(x)\gamma_\nu D_{\mu\nu}(x)[\kappa^2D(x)]^z\,,\tag{4.6}$$

$$\begin{aligned}\frac{1}{i}\Pi_{\mu\nu}(k,z)&=\int dx\,e^{ikx}e^2\,\mathrm{Tr}[\gamma_\mu S(x)\gamma_\nu S(-x)]\\&\qquad\times[\kappa^2D(x)]^z\,.\end{aligned}\tag{4.7}$$

The contour in (4.4) stands to the left of $z=0$ where, as will be seen below, $\Sigma(p,z)$ has a simple pole. The contour in (4.5) stands to the left of $z=-1$ where $\Pi_{\mu\nu}(k,z)$ has a simple pole.

Into the integrands of (4.6) and (4.7) one can substitute for the propagators as follows:

$$\begin{aligned}S(x)&=(i\gamma_\mu\partial_\mu+m)\frac{mK_1(m\sqrt{(-x^2)})}{4\pi^2\sqrt{(-x^2)}}\,,\\ D_{\mu\nu}(x)&=\eta_{\mu\nu}/4\pi^2x^2\,,\\ D(x)^z&=(-1/4\pi^2x^2)^z\,,\end{aligned}\tag{4.8}$$

and express $\Sigma$ and $\Pi_{\mu\nu}$ as integrals over the Feynman parameters $\alpha$, $\beta$, and $\gamma$. Alternatively, one could perform the integrals directly in configuration space. The results are as follows. Writing

$$\begin{aligned}\Sigma(p,z)&=\not{p}A(p^2,z)+mB(p^2,z)\,,\\ \Pi_{\mu\nu}(k,z)&=(k^2\eta_{\mu\nu}-k_\mu k_\nu)C(k^2,z)+\eta_{\mu\nu}D(k^2,z)\,,\end{aligned}\tag{4.9}$$

one finds (the details are given in the Appendix)

$$\begin{aligned}A(p^2,z)&=-\frac{\alpha}{4\pi}\left(\frac{\kappa m}{4\pi}\right)^{2z}\Gamma(2-z)\Gamma(-z)\\&\qquad\times{}_2F_1(2-z,\,-z;\,3;\,p^2/m^2)\,,\\ B(p^2,z)&=\frac{\alpha}{\pi}\left(\frac{\kappa m}{4\pi}\right)^{2z}\Gamma(1-z)\Gamma(-z)\\&\qquad\times{}_2F_1(1-z,\,-z;\,2;\,p^2/m^2)\,,\end{aligned}\tag{4.10}$$

$$C(k^2,z)=-\tfrac{1}{6}\alpha\pi^{3/2}\left(\frac{\kappa m}{2\pi}\right)^{2z+2}\frac{1}{z}\frac{[\Gamma(2-z)]^2\Gamma(4-z)}{\Gamma(\frac{5}{2}-z)}$$
$$\times{}_3F_2(4-z,\,2-z,\,-z;4,\tfrac{5}{2}-z;k^2/4m^2),\quad(4.11)$$
$$D(k^2,z)=-3\alpha\pi^{3/2}\left(\frac{\kappa m}{2\pi}\right)^{2z+2}\frac{1}{z+1}\frac{[\Gamma(1-z)]^2\Gamma(2-z)}{\Gamma(\frac{3}{2}-z)}$$
$$\times{}_3F_2(2-z,\,1-z,\,-1-z;3,\tfrac{3}{2}-z;k^2/4m^2),$$

where $\alpha=e^2/4\pi$ denotes the fine-structure constant. These results must be substituted into (4.4) and (4.5) to give the electron and photon self-energies. Since $\Sigma(p,z)$ and $\Pi_{\mu\nu}(k,z)$ contain the factor $(\kappa m)^{2z}$, one can obtain series developments in powers of $(\kappa m)^2$ by shifting the contour to the right.

The leading singularity in the integrand of (4.4) is a dipole at $z=0$ which occurs in both of the scalar amplitudes $A$ and $B$. The next singularity occurs at $z=1$: a dipole in $A$ and a tripole in $B$. The remaining singularities at $x=2, 3,\ldots$, are all tripoles. The contribution of the leading dipole takes the form

$$A(p^2)\approx-\frac{\alpha}{4\pi}\frac{\partial}{\partial z}\left[(\mathfrak{D}^{(0)}(z)+2\mathfrak{D}^{(1)}(z))\left(\frac{m}{4\pi}\right)^{2z}\times\frac{\Gamma(2-z)}{\Gamma(z+1)^2}{}_2F_1(2-z,\,-z;3;p^2/m^2)\right]_{z=0}$$
$$=\frac{-\alpha}{2\pi}\left[\ln\left(\frac{4\pi}{\kappa m}\right)+\frac{m^2-p^2}{2p^2}\times\left(1+\frac{m^2+p^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right)+O(1)\right],$$
$$B(p^2)\approx-\frac{\alpha}{\pi}\frac{\partial}{\partial z}\left[(\mathfrak{D}^{(0)}+2\mathfrak{D}^{(1)})\left(\frac{m}{4\pi}\right)^{2z}\frac{\Gamma(1-z)}{\Gamma(z+1)^2}\times{}_2F_1(1-z,\,-z;2;p^2/m^2)\right]_{z=0}\quad(4.12)$$
$$=\frac{2\alpha}{\pi}\left[\ln\left(\frac{4\pi}{\kappa m}\right)+\frac{m^2-p^2}{2p^2}\times\ln\left(\frac{m^2-p^2}{m^2}\right)+O(1)\right],$$

which clearly reproduces exactly the amplitudes of ordinary zero-gravity electrodynamics, except that here there remains a dependence on $\kappa$ in the form of an effective inbuilt cutoff:

$$\Lambda_{\text{cutoff}}\approx1/\kappa\approx2\times10^{18}\text{ GeV}.\quad(4.13)$$

In particular,[11] for the mass correction $\Sigma(p)|_{p=m}$ one obtains

$$\frac{\delta m}{m}=\frac{3\alpha}{4\pi}\ln\left(\frac{1}{\kappa m}\right)^2+O((\kappa m)^2\ln(\kappa m)^2).\quad(4.14)$$

Similar remarks apply to the photon self-energy.[11] That is, a dipole at $z=0$ in the transverse part $C$ yields the usual electrodynamic result subject to the effective cutoff (4.13),

$$C(k^2)\approx\tfrac{1}{6}\alpha\pi^{3/2}\frac{\partial}{\partial z}\left\{[\mathfrak{D}^{(0)}(z)+\tfrac{1}{2}\mathfrak{D}^{(1)}(z)]\left(\frac{m}{2\pi}\right)^{2z+2}\times\frac{\Gamma(2-z)^2\Gamma(4-z)}{\Gamma(z+1)\Gamma(\frac{5}{2}-z)}\times{}_3F_2(4-z,\,2-z,\,-z;4,\tfrac{5}{2}-z;k^2/4m^2)\right\}_{z=0}$$
$$=-\frac{\alpha}{3\pi}\left[-2\ln\left(\frac{2\pi}{\kappa m}\right)+\left(\frac{2\lambda-1}{2\lambda}\right)\left(\frac{1+\lambda}{\lambda}\right)^{1/2}\times\ln\left(\frac{(1+\lambda)^{1/2}+\sqrt{\lambda}}{(1+\lambda)^{1/2}-\sqrt{\lambda}}\right)+\frac{1}{2\lambda}+O(1)\right],\quad(4.15)$$

where $\lambda\equiv-k^2/4m^2$. Unfortunately, the longitudinal part $D(k^2)$, which should vanish in a gauge-invariant theory, also has a contribution from simple poles at $z=-1$ and $z=0$:

$$D(k^2)\approx-\alpha\left\{8\pi\,{}_3F_2(3,2,0;3,\tfrac{5}{2};k^2/4m^2)+\frac{3}{2}\frac{m^2}{\pi}\,{}_3F_2(2,\,1;\,-1;3,\,\tfrac{3}{2};k^2/4m^2)\right\}$$
$$=-(\alpha/2\pi)(8\pi^{3/4}/\kappa^2-3m^2+\tfrac{1}{3}k^2),$$

which reflects a gauge noninvariance of our procedure.[12] This will be discussed in Sec. V.

The charge renormalization may be computed from the transverse part $C(k^2)$ of the photon self-energy as

$$\frac{e_R^2}{e_0^2}=Z_3=1-\frac{2\alpha}{3\pi}\ln\frac{2\pi}{\kappa m}+O((\kappa m)^2\ln(\kappa m)^2).$$

The contributions of the tripoles at $z=1,\,2,\,\ldots$, to the electron self-energy will give terms like $(\ln\kappa)^2\kappa^{2n}$, $n\geqslant1$, and can of course be neglected. The important point about the results above is that the ultraviolet singularities of the conventional theory have disappeared via the mechanism of the intrinsic cutoff $1/\kappa$. They still leave their mark, however, as singularities of self-energies in the $\kappa$ *plane*, i.e., they reappear if the limit $\kappa\to0$ is taken.

[11] Similar formulas have been obtained for a "scalar gravity" model by P. Budini and G. Calucci, Nuovo Cimento **70A**, 419 (1970).

[12] Because of the absence of a $\ln k^2$ term, this non-gauge-invariant part can be removed by a *local* counterterm in the Lagrangian.

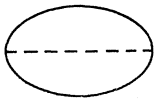

FIG. 3. Electron loop including gravitons.

## V. GAUGE INVARIANCE AND EQUIVALENCE THEOREMS

The Lagrangian of gravity-modified electrodynamics given in Sec. III is formally gauge independent. It must follow from this that the scattering amplitudes also are gauge independent provided the computational procedure employs a gauge-invariant regularization. Indeed, if these amplitudes could be expanded in powers of $e$ and $\kappa$ then they would be gauge independent in each order. However, it is one of the main arguments of this paper that, owing to the appearance of logarithmic singularities at $\kappa=0$, an expansion in powers of $\kappa$ cannot be made. The question therefore arises as to whether it is necessary to include *all* graphs of order $e^l$ (and these on general grounds must show gauge independence) or whether this collection can be subdivided into sets [possibly of order $e^l(\kappa^2)^m(\ln\kappa)^n$] which are themselves gauge independent. We are at present unable to decide this point and so must confine ourselves to some speculative remarks.

First, the prescription outlined in Sec. III for evaluating graphs with one superpropagator is not a gauge-independent one. This much can be seen from the fact that the vacuum polarization tensor $\Pi_{\mu\nu}(e^2)$ obtained in Sec. IV is not entirely transverse. Although the transverse part of $\pi_{\mu\nu}$ is satisfactory and defines a finite $Z_3$, the existence of a finite longitudinal component gives a clear indication of the breakdown of gauge symmetry.

If we follow the standard procedure of selecting a gauge-independent set of graphs in electrodynamics, the method is well known and has been formulated by Feynman and Ward. It consists of attaching photon lines in all possible ways to the basic graph of Fig. 3. This then leads to the set of four topologically distinct graphs of Fig. 4. Now, since some of the vertex pairs in these graphs [e.g., pairs 1-3 and 2-3 in Fig. 4(b)] are not connected by superpropagators, we cannot apply the nonpolynomial methods of this paper for computing these contributions. We are in a dilemma now. If we do connect these vertices by graviton superpropagators we can carry through the computation, but the Feynman-Ward procedure for securing gauge invariance would now demand that we include more complicated configurations with three or more superpropagators. For example, graphs depicted in Fig. 5 would have to be added to Fig. 4(b). The resulting graphs will themselves now need new superpropagators, and so on. This "leap-frogging" of the Feynman-Ward procedure for securing gauge invariance, and of joining vertices with superpropagators has only one limit—we must eventually reproduce the entire series $S(e^2)$. This complete series, as stated before, is definitely expected to be gauge invariant. The technical problem—which we have not yet been able to solve—is how to avoid this leap-frogging and to secure gauge invariance without having to sum the entire $S$-matrix series $S(e^2)$ to second order in $e^2$ but to all "orders" in $\kappa^2$.

Another problem which we have not resolved in this paper is the problem associated with different formulations of gravity theory. We have taken $g^{\mu\nu}$ as the basic field in terms of which tensors like $g_{\mu\nu}$ are to be expressed. One could equally well start with the tensor $g_{\mu\nu}$ as the basic interpolating field and then $g^{\mu\nu}$ would be expressed as a ratio of two polynomials in $g_{\mu\nu}$. If equivalence theorems hold for the type of theory we have been discussing (and these equivalence theorems state that on-mass-shell matrix elements are identical irrespective of which interpolating field we start from, provided that these fields possess the same one-particle asymptotic states and belong to the same locally commutative equivalence class) then the results obtained in this paper should stand, irrespective of which field $g^{\mu\nu}$ or $g_{\mu\nu}$ we choose as basic. The only difference between working with one or the other system of coordinates $g^{\mu\nu}$ or $g_{\mu\nu}$ would be the technical difference of ease in obtaining results more readily in one formulation of the theory relative to the other. The same remark applies to coordinate transformations considered by general relativists by means of which one can incorporate the factor $(\det g)^{-1/4}$ into definitions of matter fields, e.g., $\psi'=(\det g)^{-1/4}\psi$, and thus eliminate this factor from *some* of the terms in the full gravitational-matter Lagrangian. The on-mass-shell results we have obtained should be obtainable, if equivalence theorems hold, by using any set of suitable coordinates, provided that appropriate numbers of terms are added together to secure the equivalence.

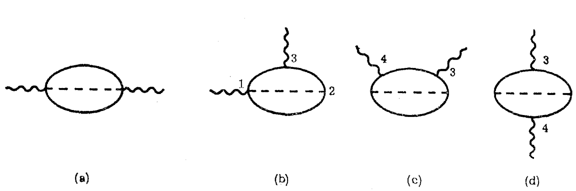


FIG. 4. Gauge-invariant set of graphs to order $e^2$.

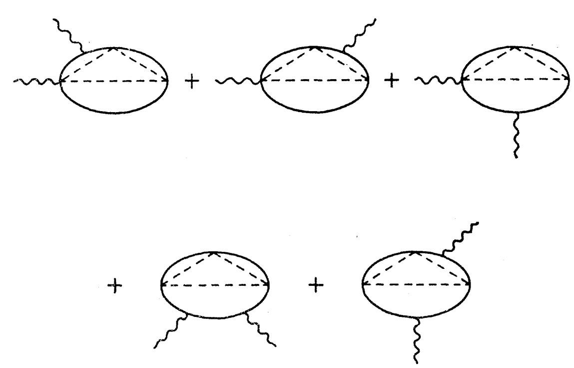


FIG. 5. Superpropagator modifications of Fig. 4.

In this connection it is worth remarking that rational nonpolynomial Lagrangians as a rule give rise to theories which are not strictly localizable (in the sense of Jaffe) when no derivatives appear in the interaction. The derivatives in the gravitational case could change this effect by introducing cancellations. It is, however, also perfectly possible to start with a strictly localizable formulation of gravity theory by parametrizing the vierbein with exponential coordinates, i.e., write, instead of (3.7),

$$L^{\mu a}=[\exp(\tfrac{1}{2}\kappa h)]^{\mu a}, \tag{5.1}$$

where the symmetrical matrix $h^{\mu\nu}$ interpolates the graviton. The exponential parametrization should give a strictly localizable version of gravity *ab initio.* This scheme simplifies some of the computations, e.g.,

$$\langle T\ \det L(x)^{-1}\det L(0)^{-1}\rangle=\exp[4\kappa^2 D(x)],$$

while making others more difficult. The computations of Sec. IV would be unaffected by this change since the leading terms $O(e^2\ln\kappa)$ are governed by the normalizing conditions at $z=0$ [cf. Eq. (3.33)],

$$\mathfrak{D}^{(0)}(0)=1,\quad \mathfrak{D}^{(1)}(0)=0,$$

which are unchanged. This equivalence of the two formulations of gravity theory in the lowest order in no way proves their general equivalence. So far as this paper is concerned, this problem is open.

Also open are all problems connected with normal ordering of the gravity-modified Lagrangian. In this paper we have followed a naive ordering prescription although it is known that normal ordering can drastically alter the local-vs-nonlocal properties of field theories, as well as their gauge properties.

*Note added in proof.* Since this paper was written two important developments have occurred. (1) H. Lehmann and K. Pohlmeyer (private communication) have established analyticity and unitarity for localizable nonpolynomial (and nonderivative) interactions. In addition, they have formulated an asymptotic condition which eliminates the usual distribution-theoretic ambiguities to all orders in the major coupling constant. (2) Assuming that equivalence theorems hold for localizable Lagrangian theories—such as gravity theory in an exponential parametrization Eq. (5.1)—we have succeeded in formulating the computational procedure in such a manner that gauge invariance is preserved to all orders. We have specifically checked that to the order $\alpha\ln k^2m^2$, there is no change in the numbers given in this paper.

## ACKNOWLEDGMENT

It is a pleasure to thank V. F. Weisskopf for a stimulating discussion concerning the significance of the Schwarzschild radius in the present context.

## APPENDIX

In this appendix the essential steps involved in the computation of the electron and photon self-energies will be outlined. From Eqs. (5.9) and (4.12) we have

$$A(p^2,z)=\frac{-2e^2}{p^2}\int d^4x\, e^{ip\cdot x}[D(x)]^{z+1}(ip^\mu\partial_\mu)\Delta(x), \tag{A1}$$

$$B(p^2,z)=4e^2\int d^4x\, e^{ip\cdot x}[D(x)]^{z+1}\Delta(x), \tag{A2}$$

where the integrals are taken over the Euclidean region in $x$-space and are defined initially for $p^2<0$. The angular integrations may be performed immediately using the polar variables

$$dx=4\pi r^2\sin^2\theta dr d\theta,$$
$$p\cdot x=-(-p^2)^{1/2}r\cos\theta,$$
$$(-x^2)^{1/2}=r.$$

We encounter the following integrals:

$$4\pi\int_0^\pi d\theta\,\sin^2\theta\,\exp[-i(-p^2)^{1/2}r\cos\theta]=4\pi^2\frac{J_1((-p^2)^{1/2}r)}{(-p^2)^{1/2}r}, \tag{A3}$$

$$4\pi\int_0^\pi d\theta\,\sin^2\theta\cos\theta\,\exp[-i(-p^2)^{1/2}r\cos\theta]=4\pi^2\frac{J_2((-p^2)^{1/2}r)}{i(-p^2)^{1/2}r}. \tag{A4}$$

Simplifying the derivative in $A(p^2,z)$ by writing

$$(ip^\mu\partial_\mu)\Delta=2ip^\mu x_\mu\frac{\partial\Delta}{\partial x^2}=2i(-p^2)^{1/2}\frac{\partial\Delta}{\partial r^2}r\cos\theta, \tag{A5}$$

we obtain the expressions

$$A(p^2,z)=16\pi^2e^2\left(\frac{-1}{p^2}\right)\int_0^\infty dr\, r^3J_2((-p^2)^{1/2}r)\times(D)^{z+1}\frac{\partial\Delta(r)}{\partial r^2}, \tag{A6}$$

$$B(p^2,z)=16\pi^2e^2\int_0^\infty dr\, r^3\frac{J_1((-p^2)^{1/2}r)}{(-p^2)^{1/2}}(D)^{z+1}\Delta(r), \tag{A7}$$

into which must be substituted the forms

$$D(r)=(1/4\pi^2)(1/r^2),$$
$$\Delta(r)=(m/4\pi^2)(K_1(mr)/r),$$
$$\frac{\partial\Delta(r)}{\partial r^2}=-\frac{m}{4\pi^2}\frac{K_2(mr)}{2r^2}.$$

The resulting integrals converge for $\text{Re}z<0$ and using the standard relations[13]

$$\int_0^\infty du\, u^{-2z-1}K_1(u)J_1(u(-p^2/m^2)^{1/2})$$
$$=(-p^2/m^2)^{-1/2}4^{-z-1}\Gamma(1-z)\Gamma(-z)$$
$$\times F(1-z,\,-z;\,2;\,p^2/m^2)\,,$$

$$\int_0^\infty du\, u^{-2z-1}K_2(u)J_2(u(-p^2/m^2)^{1/2})$$
$$=(-p^2/2m^2)4^{-z-1}\Gamma(2-z)\Gamma(-z)$$
$$\times F(2-z,\,-z;\,3;\,p^2/m^2)\,,$$

we arrive at Eqs. (4.10).

To evaluate the contribution of the double pole at $z=0$ in Eq. (4.4), it is necessary to evaluate the derivatives of these hypergeometric functions at $z=0$. This may be done by considering the power-series development

$$F(1-z,\,-z;\,2;\,x)=1+\frac{(1-z)(-z)}{2}x$$
$$+\frac{(2-z)(1-z)(1-z)(-z)}{2\times3}\frac{x^2}{2!}+\cdots,\quad |x|<1$$

from which

$$\left.\frac{\partial F(1-z,\,-z;\,2;\,x)}{\partial z}\right|_{z=0}=-\left(\frac{1\times x}{2}+\frac{2\times1\times x^2}{2\times3\times2!}+\cdots\right)$$
$$=-\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}.\quad \text{(A8)}$$

This latter sum may readily be shown to be

$$-\{1+[(1-x)/x]\ln(1-x)\}$$

and, substituting this result (with $x=p^2/m^2$), plus a similar one for $F(1-z,\,-z;\,3;\,p^2/m^2)$, into the Sommerfeld-Watson integral, we arrive at the final expression for the electron self-mass given in Eq. (4.12).

The computation of the photon self-energy is somewhat lengthier. The electron calculation was performed directly in configuration space. By way of contrast and also to illustrate the different technique involved, we shall compute the photon self-energy in momentum space using the usual Feynman $\alpha$-variables. In fact, as a consistency check we have computed the two self-energies both in configuration space and in momentum space.

The $\alpha$-parameter form of Eq. (4.7), written in terms of the function $C(k^2,z)$ and $D(k^2,z)$, defined in Eq. (4.9) is

$$\Pi_{\mu\nu}(k^2z)=(\eta_{\mu\nu}k^2-k_\mu k_\nu)C(k^2,z)+\eta_{\mu\nu}D(k^2,z)\,,$$

$$C(k^2,z)=\frac{e^2A(z)}{8}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2$$
$$\times\int_0^\infty d\alpha_3\frac{\alpha_3^{3-z}}{\Gamma(2-z)}\frac{\alpha_1\alpha_2}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^4}$$
$$\times\exp\left(k^2\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}-\alpha_1m^2-\alpha_2m^2\right),$$

$$D(k^2,z)=\frac{e^2A(z)}{16}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2$$
$$\times\int_0^\infty d\alpha_3\frac{\alpha_3^{1-z}}{\Gamma(2-z)}\frac{1}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2}$$
$$\times\left[\frac{\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}-k^2\frac{\alpha_1\alpha_2\alpha_3^2}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)^2}+m^2\right]$$
$$\times\exp\left(k^2\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}-\alpha_1m^2-\alpha_2m^2\right),$$

where $A(z)=(4\pi)^{-2z}\Gamma(2-z)/\Gamma(z)$. These iterated integrals are defined initially for $k^2<0$ and a certain range of $z$ values. The final answer may be analytically continued to other values.

We shall compute $D(k^2,z)$ first and start by defining new integration variables $x^2=1/\alpha_3$, $y^2=1/\alpha_2$, $z^2=1/\alpha_1$ and then converting to polar coordinates, $x=r\cos\theta$; $y=r\sin\theta\cos\varphi$, $z=r\sin\theta\sin\varphi$. This leads to

$$D(k^2,z)=\frac{e^2A(z)\pi^2}{2\Gamma(2-z)}\int_0^{\pi/2}d\varphi\int_0^{\pi/2}d\theta$$
$$\times\int_0^\infty dr\, r^{2z-1}\sin\varphi\cos\varphi\sin^3\theta(\cos\theta)^{2z-1}$$
$$\times(r^2\sin^4\theta\cos^2\varphi\sin^2\varphi-k^2\sin^4\theta\cos^2\varphi\sin^2\varphi+m^2)$$
$$\times\exp\left(\frac{k^2}{r^2}-\frac{m^2}{r^2\sin^2\theta\cos^2\varphi\sin^2\varphi}\right).$$

The $r$ integration is performed by defining $u=1/r^2$ and then consulting a standard text[13] on Laplace transforms. The result is

$$D(k^2,z)=\frac{e^2A(z)\Gamma(-z-1)}{4\Gamma(2-z)}\int_0^{\pi/2}d\varphi\int_0^{\pi/2}d\theta$$
$$\times\sin\varphi\cos\varphi\sin^3\theta(\cos\theta)^{2z-1}\left(\frac{m^2}{\sin^2\theta\sin^2\varphi\cos^2\varphi}-k^2\right)^z$$
$$\times[\sin^4\theta\sin^2\varphi\cos^2\varphi zk^2-m^2(z+1-\sin^2\theta)],$$

where $k^2<0$ and $\text{Re}z<1$.

[13] W. Erdélyi *et al.*, Bateman Manuscript Project, *Tables of Integral Transforms* (McGraw Hill, New York, 1954), Vols. I and II.

The $\theta$ integration may be carried out, using $\sin^2\theta$ as the variable, from the relation[13]

$$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}(x+\alpha)^\lambda dx=\alpha^\lambda \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \times {}_2F_1(-\lambda, \nu; \mu+\nu, -1/\alpha), \qquad \mathrm{Re}\mu>0,\ \mathrm{Re}\nu>0$$

leading to

$$D(k^2,z)=\frac{e^2A(z)\Gamma(-z-1)\Gamma(z)}{32\Gamma(2-z)}(4m^2)^z\int_0^1 dt(1-t)^{-1/2} \times\left\{\frac{z\Gamma(4-z)}{4\Gamma(4)}k^2t^{-z-1}\,{}_2F_1\left(-z,4-z;4;\frac{tk^2}{4m^2}\right) + m^2\frac{\Gamma(3-z)}{\Gamma(3)}t^{-z}\,{}_2F_1\left(-z,3-z;3;\frac{tk^2}{4m^2}\right) - m^2(z+1)\frac{\Gamma(2-z)}{\Gamma(2)}t^{-z}\,{}_2F_1\left(-z,2-z;2;\frac{tk^2}{4m^2}\right)\right\},$$

where $t=\sin^2 2\varphi$ and $0<\mathrm{Re}z<1$.

Using the standard integral[13]

$$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\,{}_2F_1(a_1,a_2;b;\alpha x)=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \times {}_3F_2(\nu,a_1,a_2;\mu+\nu,\nu,b_1;\alpha), \qquad |\alpha|<1$$

we obtain an expression for $D(k^2,z)$ in terms of ${}_3F_2$ functions. This may be usefully converted into one expression in terms of Meyer $G$-functions, giving

$$D(k^2,z)=\frac{e^2A(z)}{128}\frac{\Gamma(z)}{\Gamma(2-z)}\Gamma(\tfrac{1}{2})\times(-k^2)^{z+1} \times\left\{G_{33}{}^{13}\left(\frac{-k^2}{4m^2}\middle|\begin{matrix}-1, & 1, & -3\\ -z, & -\frac{3}{2}, & -3-z\end{matrix}\right) + G_{33}{}^{13}\left(\frac{-k^2}{4m^2}\middle|\begin{matrix}-1, & 1, & -3\\ -1-z, & -\frac{3}{2}, & -3-z\end{matrix}\right) + G_{33}{}^{13}\left(\frac{-k^2}{4m^2}\middle|\begin{matrix}-1, & 0, & -2\\ -1-z, & -\frac{3}{2}, & -2-z\end{matrix}\right)\right\}.$$

These $G$-functions may be combined together to give

$$-3zG_{33}{}^{13}\left(\frac{-k^2}{4m^2}\middle|\begin{matrix}-2, & -1, & 1\\ -z-1, & -z-3, & -\frac{3}{2}\end{matrix}\right).$$

It was for this reason that the Meyer functions were introduced. The final answer, expressed in terms of the more tractable ${}_3F_2$ function, is

$$D(k^2,z) = \frac{3e^2}{8\pi^{2+2z}}\frac{(m^2)^{z+1}}{4^z}\frac{\Gamma(\frac{1}{2})\Gamma(1-z)\Gamma(1-z)\Gamma(1-z)}{\Gamma(z-1)\Gamma(3)\Gamma(\frac{3}{2}-z)}\frac{\Gamma(z)}{(1+z)} \times {}_3F_2(2-z,1-z,-1-z;3,\tfrac{3}{2}-z;k^2/4m^2).$$

The same manipulations for $C(k^2,z)$ yield

$$C(k^2,z) = \frac{e^2(m^2)^{z+1}}{16\pi^{2+2z}4^z}\frac{\Gamma(\frac{1}{2})\Gamma(1-z)\Gamma(2-z)\Gamma(4-z)}{\Gamma(z-1)\Gamma(4)\Gamma(\frac{5}{2}-z)}\frac{\Gamma(z)}{z} \times {}_3F_2(4-z,2-z,-z;4,\tfrac{5}{2}-z;k^2/4m^2),$$

which simplifies slightly to give Eqs. (4.11). The contribution of the double pole at $z=0$ is obtained from the power-series expansion of the ${}_3F_2$ functions as in the electron self-energy calculation.