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Quantization of $m^2 < 0$ Field Equations*

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We construct a causal covariant quantum field theory which corresponds to a classical field with $m^2<0$. "Pseudotachyons" or "jelly" states lead to a causal screening of "tachyon" states.

I. INTRODUCTION

VARIOUS attempts to quantize a classical Klein-Gordon equation with $m^2=-\mu^2<0$ have been made by a number of physicists. Feinberg[1] considered a field whose Fourier transform is restricted to a spacelike hyperboloid, and introduced creation and annihilation operators related to the sign of the energy. This method leads to a noninvariant commutator function and was subsequently criticized by Arons and Sudarshan.[2] They proposed to construct a "tachyon" field by starting from Wigner's irreducible representations belonging to $m^2<0$ particles and introducing the Fock space in the way in which it is done in the ordinary $m^2>0$ case. Creation and annihilation operators are then introduced as ladder operators in the Fock space, and in this way one obtains a non-Hermitian field[3] which violates the Einstein causality even on a macroscopic level. Our main interest here is a "quantization" of the classical equation

$$(\partial_\mu\partial^\mu-\mu^2)\phi(x)=0\,, \tag{1}$$

and since this equation gives a causal (hyperbolic) propagation of a real field, it is clear that the Feinberg and the Arons-Sudarshan approach (whatever its other merits are) cannot be used as a quantization procedure. We will see that the correct quantum theory corresponding to (1) is quite different and much more complicated than the aforementioned proposals. We will see in particular that we are necessarily led to a theory in which no ground state exists. The state space of the quantized Hermitian field $A(x)$ corresponding to the classical $\phi(x)$ in (1) can be written as a direct product:

$$\mathcal{H}=\tilde{\mathcal{H}}\times\mathcal{H}_t\,, \tag{2}$$

where $\tilde{\mathcal{H}}$ corresponds either to states with complex energies and indefinite metric ("pseudotachyon") or to a purely continuous spectrum of the Hamiltonian ("jelly states"), whereas $\mathcal{H}_t$ contains particlelike states ("tachyon states"). Owing to the presence of the pseudotachyon or the jelly background (there is no normalizable time-translational-invariant jelly state), the tachyons propagate causally. The background can never be removed, and therefore, strictly speaking, it does not make sense to talk about tachyons in the sense of Arons and Sudarshan and Feinberg in this field theory. The causality is in correspondence to the propagation of the classical initial states at less than the velocity of light. It turns out that in the positive definite "jelly" representation the translation and the Lorentz transformation can be locally unitarily implemented for every compact space-time region; however, the global invariance is broken in the sense of Goldstone.[4] In contrast to the situation discussed by Ezawa and Swieca,[5] there are no Goldstone particles but just Goldstone excitations.

The interaction of such a field with an external potential does not lead to a catastrophic behavior and we think that an interaction with other quantized fields may lead to a well-defined theory. We also speculate that an interaction of ordinary fields may in certain cases lead to a state space of the type (2).

II. PROPERTIES OF CLASSICAL THEORY

The classical field (1) leads to an energy-momentum density

$$P_\mu(x)=\tfrac{1}{2}((\partial_t\phi)^2+\sum_i(\partial_i\phi)^2-\mu^2\phi^2;\,2\partial_t\phi\partial_i\phi)\,. \tag{3a}$$

The energy density is not positive definite. For mathematical purposes we may introduce a positive definite vector density

$$\hat{P}_\mu(x)=\tfrac{1}{2}((\partial_t\phi)^2+\sum_i(\partial_i\phi)^2+c^2\phi^2;\,2\partial_t\phi\partial_i\phi)\,, \tag{3b}$$

with

$$\hat{P}_0(x)\geqslant 0\,,\quad \hat{P}^\mu\hat{P}_\mu\geqslant 0\,.$$

The new vector is not conserved, but rather satisfies

$$\partial^\mu\hat{P}_\mu(x)=(c^2+\mu^2)(\partial_t\phi)\phi\,. \tag{4}$$

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[1] G. Feinberg, Phys. Rev. 159, 1089 (1967).

[2] M. E. Arons and E. C. G. Sudarshan, Phys. Rev. 173, 1622 (1968).

[3] J. Dhar and E. C. G. Sudarshan, Phys. Rev. 174, 1808 (1968). In this paper the authors attempt to introduce a Hermitian Lorentz-invariant field $\chi(x)$. However, the statement that their formula (1.8) defines an invariant function is wrong, and therefore their aim was not achieved. Their subsequent treatment of space-time properties of tachyons is misleading.

[4] J. Goldstone, Nuovo Cimento 19, 154 (1961).

[5] H. Ezawa and J. A. Swieca, Commun. Math. Phys. 5, 330 (1967).

Using a Sobolev[6] norm,

$$\|\phi\|_t^2=\int_{t=\text{const}}[\dot\phi^2+\sum_i(\partial_i\phi)^2+c^2\phi^2]\,d^3x,$$

one easily shows[7] the causality of the propagation.

For this free-field equation (1), one could of course have computed directly the propagation function

$$G(\xi)=i\frac{\mu^2}{4\pi}\frac{I_1(\mu\sqrt{\xi^2})}{\mu\sqrt{\xi^2}}\epsilon(\xi_0)\Theta(\xi^2)-\frac{1}{2\pi}i\epsilon(\xi_0)\delta(\xi^2)\,, \quad (5)$$

$$\phi(x,t)=\int_{t_y=0}iG(x-y)\overleftrightarrow{\partial}_0\Phi(y,t_y)\,. \quad (6)$$

But the aforementioned method to establish causality (and existence) of the propagation also works in more complicated cases, for example, if we add to (1) a potential term $V(x)\phi(x)$, where V is assumed to be bounded below.[7] The exponential increase of $G(\xi^2)$ is related to the increase of the nonconserved energy $\int\hat{P}_0\,d^3x$. However, there is nothing alarming about this: The physical energy $\int P_0\,d^3x$ [as well as the inner product $(\phi,\phi)=i\int\bar\phi\overleftrightarrow{\partial}_0\phi\,d^3x$ in the complex case] is strictly conserved, even though the classical fields may increase at a point in space. Similarly to Ref. 7, one can show the completeness of the eigenfunctions in the presence of a stationary potential $V(x)$. Note that the Fourier decomposition of the field ϕ is more complicated than in the $m^2>0$ case. We may write

$$\begin{aligned}\phi(\mathbf{x},t)=&\frac{1}{(2\pi)^{3/2}}\int_{\mathbf{k}^2>\mu^2}e^{-i\lambda t+i\mathbf{k}\cdot\mathbf{x}}\tilde\phi_1(\mathbf{k})\frac{d^3k}{(\mathbf{k}^2-\mu^2)^{1/2}}+\text{H.c.}\\&+\frac{1}{(2\pi)^{3/2}}\int_{\mathbf{k}^2<\mu^2}e^{\kappa t+i\mathbf{k}\cdot\mathbf{x}}\tilde\phi_2(\mathbf{k})\frac{d^3k}{(\mu^2-\mathbf{k}^2)^{1/2}}\\&+\frac{1}{(2\pi)^{3/2}}\int_{\mathbf{k}^2<\mu^2}e^{-\kappa t+i\mathbf{k}\cdot\mathbf{x}}\tilde\phi_3(\mathbf{k})\frac{d^3k}{(\mu^2-\mathbf{k}^2)^{1/2}},\\&\lambda=\lambda(\mathbf{k})=(\mathbf{k}^2-\mu^2)^{1/2},\quad\kappa=\kappa(\mathbf{k})=(\mu^2-\mathbf{k}^2)^{1/2}.\end{aligned} \quad (7)$$

The Fourier transforms $\tilde\phi_2$ and $\tilde\phi_3$ are necessary to solve a Cauchy initial-value problem with local initial data.

III. ALGEBRAIC ASPECT OF QUANTIZATION

The algebraic aspects of quantization are contained in the following formula (for $t=0$):

$$A(\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}}q(\mathbf{k})\,d^3k\,, \quad (8a)$$

$$\pi(\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}}p(\mathbf{k})\,d^3k\,, \quad (8b)$$

with

$$[q(\mathbf{k}),p(-\mathbf{k}')]=i\delta(\mathbf{k}-\mathbf{k}')\,,\qquad [q,q]=[p,p]=0\,, \quad (9)$$

and $q^\dagger(\mathbf{k})=q(-\mathbf{k})$, $p^\dagger(\mathbf{k})=p(-\mathbf{k})$ for Hermitian fields. In order to compute the field at a later time, we have to work out the Hamiltonian:

$$\begin{aligned}H=&\int\mathcal{H}(\mathbf{x})\,d^3x\\=&\tfrac12\int_{\mathbf{k}^2>\mu^2}[p^\dagger(\mathbf{k})p(\mathbf{k})+\lambda^2(\mathbf{k})q^\dagger(\mathbf{p})q(\mathbf{k})]\,d^3k\\&+\tfrac12\int_{\mathbf{k}^2<\mu^2}[p^\dagger(\mathbf{k})p(\mathbf{k})-\kappa^2(\mathbf{k})q^\dagger(\mathbf{k})q(\mathbf{k})]\,d^3k.\end{aligned} \quad (10)$$

Computing $A(\mathbf{x},t)=e^{iHt}A(\mathbf{x})e^{-iHt}$ by using (9), we obtain

$$\begin{aligned}A(\mathbf{x},t)=&\frac{1}{(2\pi)^{3/2}}\int e^{-i\lambda t+i\mathbf{k}\cdot\mathbf{x}}\frac12\left(q(\mathbf{k})-\frac1i\frac{p(\mathbf{k})}{\lambda}\right)d^3k\\&+\frac{1}{(2\pi)^{3/2}}\int e^{i\lambda t+i\mathbf{k}\cdot\mathbf{x}}\frac12\left(q(\mathbf{k})+\frac{1}{i\lambda}p(\mathbf{k})\right)d^3k\\&+\frac{1}{(2\pi)^{3/2}}\int e^{\kappa t+i\mathbf{k}\cdot\mathbf{x}}\left(q(\mathbf{k})+\frac1\kappa p(\mathbf{k})\right)\frac{d^3k}{2}\\&+\frac{1}{(2\pi)^{3/2}}\int e^{-\kappa t+i\mathbf{k}\cdot\mathbf{x}}\left(q(\mathbf{k})-\frac1\kappa p(\mathbf{k})\right)\frac{d^3k}{2}.\end{aligned} \quad (11)$$

Before we study the representation problem we will convince ourselves that the commutator is local and Lorentz invariant.

$$\begin{aligned}[A(x),A(y)]=&\frac{1}{(2\pi)^3}\int e^{-i\lambda\xi_0+i\mathbf{k}\cdot\xi}\frac{d^3k}{2\lambda}\\&-\frac{1}{(2\pi)^3}\int e^{i\lambda\xi_0-i\mathbf{k}\cdot\xi}\frac{d^3k}{2\lambda}+\frac{-i}{(2\pi)^3}\int e^{\kappa\xi_0+i\mathbf{k}\cdot\xi}\frac{d^3k}{2\kappa}\\&-\frac{-i}{(2\pi)^3}\int e^{-\kappa\xi_0-i\mathbf{k}\cdot\xi}\frac{d^3k}{2\kappa}.\end{aligned} \quad (12)$$

Every *single term* is *noncovariant*, but the *sum* of all four terms comes out *covariant*:

$$[A(x),A(y)]=\frac{\mu^2}{4\pi}\frac{I_1(\mu\sqrt{\xi^2})}{\mu\sqrt{\xi^2}}i\epsilon(\xi_0)\Theta(\xi^2)-\frac{1}{4\pi^2}i\epsilon(\xi_0)\delta(\xi^2)\,. \quad (13)$$

As quite often in relativistic quantum theory, formal covariance goes hand in hand with causality. Evidently (13) vanished for spacelike separations, whereas any

[6] K. Yoshida, *Functional Analysis* (Springer, New York, 1968), 2nd ed.

[7] B. Schroer and J. A. Swieca, Phys. Rev. D 2, 2938 (1970).

combination in (12) not including all four terms is not causal.

To see these statements, let us consider a special term, for example,

$$\int_0^\mu e^{\kappa\xi_0+i\mathbf{k}\cdot\xi}\frac{d^3k}{2\kappa}=2\pi\int_0^\mu e^{\xi_0\kappa+ikr\cos\theta}\sin\theta\, d\theta\frac{k^2dk}{2\kappa}$$
$$=-\frac{2\pi}{r}\frac{\partial}{\partial r}\int_{-\mu}^{\mu} e^{\kappa\xi_0+ikr}\frac{dk}{2\kappa}$$
$$=-\frac{\pi}{r}\frac{\partial}{\partial r}\int_0^\pi e^{\mu\xi_0\sin\rho-i\mu r\cos\phi}d\phi\,.$$

Introducing (for $\xi_0>r$) $\xi_0=\alpha\cosh\chi$, $r=\alpha\sinh\chi$, we obtain the integral

$$-\frac{\pi}{r}\frac{\partial}{\partial r}\int_0^\pi e^{\mu\alpha\sin(\phi-i\chi)}d\phi\,, \qquad (14)$$

which explicitly depends on the angle χ and hence is not invariant. It is straightforward to check that the four χ-dependent contributions add up to give an integral over a path *not* depending on χ. The result of this integration is (13).

The covariance also can be made plausible by applying the infinitesimal L transformation

$$N_i=p_i\frac{\partial}{\partial p_0}-p_0\frac{\partial}{\partial p_i}$$

to the momentum-space kernel $\epsilon(p_0)\delta(p^2+\mu^2)|_{\mathbf{p}^2>\mu^2}$ as well as to $\epsilon(p_0)\delta(p^2+\mu^2)|_{\mathbf{p}^2<\mu^2}$. The two new invariant contributions coming from the differentiation of the $\epsilon(p_0)$ function compensate.

Finally, we want to point out that although the Hamiltonian is diagonal in the p, q variables for $\mathbf{k}^2>\mu^2$ and $\mathbf{k}^2<\mu^2$, this is not true for the Lorentz transformation

$$\mathbf{N}=\int \mathbf{x}\mathcal{H}(\mathbf{x})\,d^3x\,,\quad t=0 \qquad (15)$$

which mixes the p and q belonging to the different regions.

IV. REPRESENTATION PROBLEM

The Hamiltonian (10) describes a collection of positive as well as inverted (repulsive) oscillators. There is not much of a problen as to what to do about the first type. We write

$$q(\mathbf{k})=\frac{1}{[2\lambda(\mathbf{k})]^{1/2}}[a^\dagger(-\mathbf{k})+a(\mathbf{k})]\,,$$
$$p(\mathbf{k})=i\frac{[\lambda(\mathbf{k})]^{1/2}}{\sqrt{2}}[a^\dagger(-\mathbf{k})-a(\mathbf{k})]\,. \qquad (16)$$

The first part of the Hamiltonian becomes

$$H_t=\tfrac{1}{2}\int\lambda(\mathbf{k})[a^\dagger(\mathbf{k})a(\mathbf{k})+a(\mathbf{k})a^\dagger(\mathbf{k})]\,d^3k\,. \qquad (17)$$

For the quantized H_t we take (subtracting a c number)

$$H_t=\int\lambda(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})\,d^3k \qquad (18)$$

in the Fock space of the creation and annihilation operators. Such an H_t would be positive definite. But at this point there is a certain ambiguity coming in. Since $\lambda(\mu)=0$, we could also have taken one of the infinitely many infrared-type representations.[8]

The Hamiltonian (18) is in some way the Hamiltonian of particle-type modes. Following Feinberg, we call these modes "tachyons."

The remaining Hamiltonian for $\mathbf{k}^2<\mu^2$ [Eq. (10)] cannot really be parametrized in terms of "particle" modes, since the spectrum of an inverted oscillator is purely continuous. We have discussed the problem of quantization of the degree of freedom related to an inverted oscillator

$$H_{\rm osc}=(p^\dagger p-\kappa^2q^\dagger q) \qquad (19)$$

in connection with bound states belonging to a complex classical energy.[7] Here we can apply the same considerations. We could introduce (non-Hermitian) "particle" modes by

$$q=\frac{1}{(2n)^{1/2}}(a+b)\,,$$
$$\mathbf{p}^\dagger=(\tfrac{1}{2}\kappa)^{1/2}(b^\dagger-a^\dagger)\,.$$

In this case we obtain

$$H_{\rm osc}=-\kappa(a^\dagger b+b^\dagger a)+(c\text{ number})\,,$$

but the commutation relations are

$$[a,b^\dagger]=i\,,\quad \text{all other }[\ ,\ ]=0\,.$$

This leads to particle modes with an indefinite metric:

$$a^\dagger|0\rangle\equiv|a\rangle\,,\quad b^\dagger|0\rangle=|b\rangle\,,$$
$$\langle a|a\rangle=0=\langle b|b\rangle\,,\quad\text{but}\quad\langle a|b\rangle=i\,. \qquad (20)$$

In that case $H_{\rm osc}$ is pseudoadjoint in a space with indefinite metric and the eigenvalues of H are imaginary:

$$H_{\rm osc}=i\kappa(N_b-N_a)\,, \qquad (21)$$

where $N_b=-ib^\dagger a$, $N_a=N_b{}^\dagger$, and N is a kind of "number operator":

$$N_b|nb\rangle=n|nb\rangle\,.$$

[8] S. Doplicher, Commun. Math. Phys. 3, 228 (1966).

Applying this to the $\mathbf{k}^2<\mu^2$ of (10), one obtains

$$H_{\rm pt}=i\int\kappa(\mathbf{k})(N_a-N_b)\,d^3k+R\,, \qquad (22)$$

where the rest R commutes with the field and its canonical conjugate. This is an indication that our representation of the field $A(x)$ will be reducible. It can be shown that there is no reduction if one wants to keep the Poincaré invariance of the Wightman functions.[9] Using the reducibility, we may change the Hamiltonian and eliminate R.

Owing to the presence of a gradient term in k space, the Lorentz transformation mixes the tachyon state with the pseudotachyon. The two-point function $\langle A(x)A(y)\rangle$ has tachyon and pseudotachyon contributions and is an invariant function which can be easily explicitly computed. The higher functions are products of the two-point function as in an ordinary free-field theory. The representation of the Poincaré group is "pseudo-unitary" (unitary with respect to the indefinite metric) and therefore does not appear in Wigner's classification scheme. Since an *indefinite Hilbert* space leads, however, to the *well-known difficulties* with the principles of *quantum theory*, we will not investigate this point further.

We now turn to a quantization in a *positive definite Hilbert* space. To achieve this, we write the $\mathbf{k}^2<\mu^2$ part of the Hamiltonian first for a quantization box with volume V $[\mathbf{k}_i=(2\pi/L)\mathbf{n}]$:

$$H_j{}^V=\tfrac{1}{4}\sum_i(p_{1i}{}^2-\kappa^2q_{1i}{}^2+p_{2i}{}^2-\kappa^2q_{2i}{}^2)\,, \qquad (23)$$

where we have introduced

$$p(\mathbf{k}_i)=\frac{p_{1i}+ip_{2i}}{\sqrt{2}},\quad q(\mathbf{k}_i)=\frac{q_{1i}+iq_{2i}}{\sqrt{2}}.$$

Because of the reality condition, we have to limit one momentum, for example, by assuming $n\mathbf{k}_{iz}>0$.

The eigenfunctions for an inverted oscillator are purely continuous; they can be chosen as eigenfunctions of the dilatation operator[7]

$$\psi_{\epsilon,m}(x,y)=\frac{(r)^{i\epsilon-1}}{(2\pi)^{1/2}}\frac{e^{im\theta}}{(2\pi)^{1/2}},$$

r, θ are polar coordinates of x, y. (24)

Here we have set

$$p_1=\left(\frac{\kappa}{2}\right)^{1/2}\left(x-i\frac{\partial}{\partial x}\right),\quad q_1=\left(\frac{1}{2\kappa}\right)^{1/2}\left(x+i\frac{\partial}{\partial x}\right),$$

$$p_2=\left(\frac{\kappa}{2}\right)^{1/2}\left(y-i\frac{\partial}{\partial y}\right),\quad q_2=\left(\frac{1}{2\kappa}\right)^{1/2}\left(y+i\frac{\partial}{\partial y}\right).$$

The sum in (23) is finite as long as the quantization volume is finite. In order to obtain a normalizable state, we have to form packets:

$$\int f(\epsilon)\psi_{\epsilon,m}(x,y)\,d\epsilon. \qquad (25)$$

The state space in which $H_j{}^V$ acts is then spanned by products of the type

$$\psi_V=\prod\int f_i(\epsilon)\psi_{\epsilon,m_i}(x_i,y_i)\,d\epsilon.$$

By construction these states are in the domain of H.

Let us consider a state of the total system of the following form:

$$|0_\psi\rangle_V=|0\rangle_V{}^t\otimes\psi_V\,, \qquad (26)$$

where we have taken the "tachyon vacuum" $|0\rangle_V{}^t$, and ψ_v is a state of the jelly. It is now easy to see that on a "relative homogeneous" state of the jelly background,

$$\psi_V=\prod_i\int f(\epsilon)\psi_{\epsilon,m=0}(x_i,y_i) \qquad (27)$$

is invariant against spatial translations. Therefore, the expectation values of fields at equal times will turn out to be translation invariant in the limit of $V\to\infty$.

As a typical case, we consider

$$\lim_{V\to\infty}\langle 0_\psi|A(\mathbf{x})A(\mathbf{y})|0_\psi\rangle_V$$
$$=\lim_{V\to\infty}\frac{1}{V}\sum_{n,n'}\langle 0|a(k_n)a^\dagger(k_n')|0\rangle\frac{e^{i\mathbf{k}_n\cdot\mathbf{x}+i\mathbf{k}_n'\cdot\mathbf{y}}}{2\lambda}$$
$$+\lim_{V\to\infty}\frac{1}{V}\sum_{n,n'}e^{i\mathbf{k}_n\cdot\mathbf{x}+i\mathbf{k}_n'\cdot\mathbf{y}}\langle\psi|q(k_n)q(k_n')|\psi\rangle. \qquad (28)$$

Again the first term has a trivial limit because we only obtain a contribution if $\mathbf{k}_n'=-\mathbf{k}_n$. In order to see the existence of the limit for the second term, we have to remember that $q(\mathbf{k}_n)$ only acts on the nth factor in the product wave function (26). On such a wave function (24), the action is

$$c=\frac{1}{(2\kappa)^{1/2}}e^{i\theta}\left(r+i\frac{\partial}{\partial r}-\frac{1}{r}\frac{\partial}{\partial\theta}\right). \qquad (29)$$

Since the expectation value of this expression vanishes between states (24), we only receive a contribution for $\mathbf{k}_n'=-\mathbf{k}_n$:

$$\frac{1}{(2\pi)^{3/2}}\int_{\mathbf{k}^2<\mu^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{d^3k}{2\kappa}(f,cc^\dagger f).$$

[9] This has been demonstrated by M. Gomes (private communication).

Hence altogether

$$\langle 0_\psi | A(\mathbf{x})A(\mathbf{y}) | 0_\psi \rangle = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{k}^2>\mu^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{d^3k}{2\lambda(\mathbf{k})} + \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{k}^2<\mu^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{d^3k}{2\kappa(\mathbf{k})} (f, cc^\dagger f) . \quad (30)$$

The "relative homogeneity" of the jelly state gives, therefore, a space-translation-invariant two-point function for equal times. The computation of $\langle 0_\psi | A(\mathbf{x})\pi(\mathbf{y}) \times | 0_\psi \rangle$ and the higher-point functions can be carried out similarly.

The lack of invariance of the state $|0_\psi\rangle$ under time translation shows up if we compute the expectation value for unequal times. A computation similar to the previous one leads to

$$\begin{aligned} \langle 0_\psi | A(x)A(y) | 0_\psi \rangle = {} & \frac{1}{(2\pi)^3} \int e^{-i\lambda\xi_0 + i\mathbf{k}\cdot\xi} \frac{d^3k}{2\lambda} \\ & + \frac{1}{(2\pi)^3} \frac{1}{2} \int e^{\kappa\xi_0 + i\mathbf{k}\cdot\xi} \frac{d^3k}{2\kappa} (\psi, c_1 r c_1^\dagger c_2^\dagger \psi) \\ & + \frac{1}{(2\pi)^3} \frac{1}{2} \int e^{-\kappa\xi_0 + i\mathbf{k}\cdot\xi} \frac{d^3k}{2\kappa} (\psi, c_1 c_2 c_1^\dagger r \psi) \\ & + \frac{1}{(2\pi)^3} \frac{1}{2} \int e^{\kappa(x_0+y_0) + i\mathbf{k}\cdot\xi} \frac{d^3k}{2\kappa} (\psi, c_1 c_1^\dagger r^2 \psi) \\ & + \frac{1}{(2\pi)^3} \frac{1}{2} \int e^{-\kappa(x_0+y_0) + i\mathbf{k}\cdot\xi} \frac{d^3k}{2\kappa} (\psi, c_1 c_2 c_1^\dagger c_2^\dagger \psi) , \end{aligned}$$

with

$$c_1 = e^{i\theta}, \quad c_2 = \left(i\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \theta} \right).$$

The two last terms clearly destroy the invariance against time translations. What is happening here? Clearly the (symmetric) energy-momentum tensor $T_{\mu\nu}(x)$ is conserved in the same manner as for an ordinary $m^2>0$ field theory. (This is independent on the structure of the representation space.) Hence the global Hamiltonian and the Lorentz generators are at least formally conserved quantities. Looking at the matrix elements of H and $\mathbf{N}$ (and the matrix elements of their squares), one realizes, however, that the jelly background gives a nonvanishing contribution which is proportional to the volume V. Alternatively, one may consider the "local Hamiltonian"[10]

$$H_0 = \int T_{00}(x) f_R(\mathbf{x}) f_T(t) d^3x dt ,$$

with

$$f_R(\mathbf{x}) = \begin{cases} 1 & \text{for} \quad |\mathbf{x}| < R \\ 0 & \text{for} \quad |\mathbf{x}| > R+L , \end{cases}$$

and

$$\operatorname{supp} f_T \in [-T, T], \quad f_T \geq 0$$

$$\int f_T(t) dt = 1 .$$

Choosing $\mathcal{O}$ to be the double cone with the base $R+T$ (at $t=0$), one easily checks that H_0 executes the correct time propagation inside $\mathcal{O}$. In other words, the time propagation can be implemented by a unitary operator (which is not unique) for every compact region $\mathcal{O}$. However, the limit $\mathcal{O} \to R^4$ does not exist in the representation, due to the long-range aspect of the reference state. The same holds for the Lorentz transformations. Hence those transformations are spontaneously broken in the sense of Goldstone.[4] The Goldstone particles, whose existence in the normal (positive-momentum spectrum) theory has been demonstrated by Ezawa and Swieca,[5] are absent in our model. Instead we have only "zero-energy excitation." The situation is analogous to the Galilei invariance in the idealized ($V \to \infty$) many-body problem. This invariance is known to be always spontaneously broken,[11] and the zero-energy excitations in the absence of long-range forces are just those of the nonrelativistic free Hamiltonian H_0.

We want to emphasize that a spontaneously broken invariance unlike an intrinsically broken symmetry is almost as good as an ordinary quantum-theoretical symmetry. For every finite space region the invariance is there, and only as a result of long-range (cosmological) features of the states does it get lost.

The local Lorentz transformation couples the dynamical variables of the tachyons to those of the jelly background and in this way lead to a "causal screening" of the Arons-Sudarshan–Feinberg tachyons. It also may be of interest to mention that this $m^2<0$ quantization can be used to obtain a causal infinite-component Majorana field.

V. CONCLUDING REMARKS

We have demonstrated that the quantization of an $m^2<0$ field equation leads to causal fields. We also constructed a representation (by V-limiting) of the free-field algebra.

The validity of causality means that tachyons cannot be used for propagating signals with a speed faster than the speed of light, a fact which unfortunately makes the name an *abusus linguae*. Whereas the state space has a quite unconventional structure (long-range correlations), this is not in contradiction with causality inasmuch as the Einstein-Podolsky-Rosen "paradox" is no repudiation of causality.

[10] D. Kastler, Derek W. Robinson, and A. Swieca, Commun. Math. Phys. 2, 108 (1966).

[11] J. A. Swieca, Commun. Math. Phys. 4, 1 (1967).

The crucial test of whether this theory is consistent should be the coupling to a field with positive mass. If such an interaction is stable, i.e., leads to a finite probability of scattering for a finite number of physical particles (and a slight rearrangement of the tachyon-jellyon states), then the idea of using quantized $m^2<0$ fields will be acceptable. It should be noted that because of the lack of boundedness from below, the Hamiltonian does not necessarily lead to an instability (counter-example: a quantum many-body system in the finite-temperature state space). There are indications that the situation with respect to stability is quite different from the well-known instability of the old Dirac electron theory (without the "filled sea"). The "Pandora's box" of interactions will not be opened here.

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APPENDIX

The Lorentz invariance of the two-point function $F(\xi)=\langle A(x)A(y)\rangle$ in the indefinite metric quantization can easily be demonstrated by parametrizing $\xi=x-y$ in terms of hyperbolic angles. With $\xi=(t,r)$, the result can be written in the form

$$F(\xi)\sim\frac{1}{r}\frac{\partial}{\partial r}\int_C e^{d\sin\Phi}d\Phi, \qquad \text{(A1)}$$

with

$$d=\begin{cases} i\mu(r^2-t^2)^{1/2}, & r^2>t^2 \\ \mu(t^2-r^2)^{1/2}, & t^2>r^2. \end{cases}$$

The path C is the path in the complex Φ plane running from $-\infty$ to $+\infty$.[12] The invariant function $F(\xi)$ can be expressed in terms of Bessel-Neumann functions. Since the higher-point functions are products of the two-point function, we have by defining

$$U(\Lambda)|0\rangle=|0\rangle$$

and

$$U(\Lambda)A(x)U^\dagger(\Lambda)=A(\Lambda x) \qquad \text{(A2)}$$

a Lorentz-covariant field. The action of $U(\Lambda)$ on the "one-mode" states (containing a mixture of tachyon and pseudotachyon)

$$|f,g\rangle=\left(\int A(\mathbf{x})f(\mathbf{x})+\int\pi(\mathbf{x})g(\mathbf{x})\right)|0\rangle \qquad \text{(A3)}$$

can easily be computed and we obtain a pseudounitary representation of the L group on the one-mode states. The representation can be rewritten in terms of momentum-space wave functions, but this is a bit cumbersome; the x-space language for these $m^2<0$ wave functions seems preferable.

The discussion of propagative properties should be done in terms of expectation values of physical observables and not in terms of space-time properties of wave packets. To illustrate this point, let us consider a state which is localized in the sense of Knight[13]:

$$|\Phi_{f,g}\rangle=e^{i[A(f)+\pi(g)]}|0\rangle, \qquad \text{(A4)}$$

with

$$\operatorname{supp}(f(\mathbf{x}),g(\mathbf{x}))\subset B.$$

These states form a dense set and have the property that if we consider the expectation value of a physical observable affiliated with the region R, then this expectation value vanishes if R and B have no overlap. To be specific, let us consider the expectation value of the local charge Q_R assuming that we have a charged field $A(x)$ of mass $m^2>0$.

Consider

$$E(\mathbf{v},t)=\langle\Phi_{f,g}|Q_R(\mathbf{v},t)|\Phi_{f,g}\rangle,$$

with

$$Q_R(\mathbf{v},t)=U(\mathbf{v}t,t)Q_RU^\dagger(\mathbf{v}t,t). \qquad \text{(A5)}$$

This function is only *different from zero* if the R translated in the timelike $\mathbf{v}$ direction *has an overlap* with the forward light cone of B. Inside this cone the leading time is easily computed to be

$$\lim_{t\to\infty}E(\mathbf{v},t)\sim\langle f,g|Q_R(\mathbf{v},t)|f,g\rangle\sim(c/t^3)|\tilde\phi(p_{\mathrm{cl}})|^2, \qquad \text{(A6)}$$

with

$$\tilde\phi(\mathbf{p})=\tilde f(\mathbf{p})+i\tilde g(\mathbf{p})\sqrt{\mathbf{p}^2+m^2}$$

and

$$\mathbf{p}_{\mathrm{cl}}=m\mathbf{v}/(1-v^2)^{1/2}. \qquad \text{(A7)}$$

Inside this cone this is the same leading term as that of the asymptotic expansion of the one-particle wave packet:

$$|\phi(\mathbf{v}t,t)|^2\sim(1/t^3)|\tilde\phi(p_{\mathrm{cl}})|^2, \qquad \text{(A8)}$$

with

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int e^{-ip\cdot x}\tilde\phi(p)\frac{d^3p}{2p_0}. \qquad \text{(A9)}$$

Note also that the center of the packet is inside the cone; the spread also reaches the region outside. As far as space-time causality is concerned, we have to take the description in terms of $E(\mathbf{v},t)$; the wave-packet picture *only agrees* with the correct causal description *inside the cone.*

If we now go over to $m^2<0$, we see that the description in terms of $E(\mathbf{v},t)$ preserves its strict causality,

[12] For the timelike separation, this was discussed in detail by B. Schroer, talk presented at the Conference on Special Topics in Quantum Field Theory at the University of Missouri, 1970, University of Pittsburgh Report No. NYO-3829-58 (unpublished).

[13] J. M. Knight, J. Math. Phys. 4, 1443 (1963).

whereas a wave-packet picture becomes quite useless. The problem of asymptotic observables and causality of tachyons and pseudotachyons is sufficiently complex to warrant a separate discussion. The causal, Poincaré-invariant quantization, which necessarily led us to an indefinite Hilbert space, will only be of any use if the invariant subspace of tachyons and pseudotachyons does not appear in the asymptotic states.[14]

Note added in proof. The earliest and in our opinion most relevant work on quantization of $m^2<0$ equations was done by Tanaka.[15] Due to lack of references we only became aware of that work after the completion of this paper. The negative metric quantization is, apart from minor modifications, already contained in Tanaka's work. The main difference between Tanaka's and our treatment is that we do not attribute physical significance to the noncausal and noninvariant pieces of the $m^2<0$ field. As indicated at the end of the Appendix we would formulate the interaction with physical particles similar to Ref. 14.

[14] Here we have in mind a mechanism similar to the one given by T. D. Lee and G. C. Wick, Phys. Rev. D 2, 1033 (1970).

[15] S. Tanaka, Progr. Theoret. Phys. (Kyoto) 24, 171 (1960).

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Feynman-Parameter Approach to N-Tower Exchange in ϕ^3 Theory*

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We investigate by Feynman-parameter techniques the asymptotic form of N-tower graphs in ϕ^3 theory. A detailed study of the two-tower case in the leading $\ln s$ approximation agrees with the second-order piece of the eikonal expansion. By using certain assumptions about the N-tower case, we develop an iterative scheme for summing all N-tower exchanges. This generates the eikonal form outside of the framework of momentum-space techniques. Furthermore, only a limited class of structures contribute to the asympotic form: the Mandelstam nests.

I. INTRODUCTION

Attempts to find simple functional descriptions of high-energy phenomena have recently led to studies of field-theoretic generalizations of the classic eikonal approximation.[1] Regge asymptotic behavior can be generated in a field theory by exchanging a generic ladder structure, called a tower, and performing a sum over tower rungs.[2] From an eikonal point of view this corresponds to taking a simple Regge amplitude for the potential. The eikonal expansion then generates a pole and a sequence of multiple pole exchanges which form cuts in the j plane. Trilinear coupling of spinless particles, the so-called ϕ^3 theory, is the simplest structure within which one can study tower-exchange models. One could argue that its use is unphysical since non-tower structures cannot be neglected, for at large energy s, tower structures are dominated by the Born term. However, if we adopt the view that a study of tower exchange here will give insight into quantum electrodynamics (QED), where $g^2\sim s$ and all tower-exchange graphs are of equal importance, then ϕ^3 becomes a useful tool for studying tower models. The simplest such model is to exchange N off-mass-shell towers between continuous end lines in all possible ways. Even if one desires only the high-energy limit of the summed amplitude, this scheme is a structure of great complexity. Some authors have studied such a model using recently developed momentum-space techniques.[3–5] The result they derive is simply the eikonal form discussed above, i.e.,

$$A(s,t)=2is\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}(e^{i\chi(b,s)/2s}-1)\,, \tag{1.1}$$

where

$$\chi(b,s)=g^2\int\frac{d^2k}{(2\pi)^2}e^{-i\mathbf{k}\cdot\mathbf{b}}s^{\alpha(-\mathbf{k}^2)}\,,$$

$$\alpha(t)=-1+K(t)\,,$$

$$K(t)=-\frac{g^2}{16\pi^2}\int_0^1\int_0^1\frac{d\gamma d\delta\,\delta(\gamma+\delta-1)}{d'(\gamma,\delta,t)}\,,$$

and d' is described in Sec. II.

Momentum-space methods have two unpleasant features which motivate our present work. The first is the

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[1] H. Cheng and T. T. Wu, Phys. Rev. 186, 1611 (1969); S. J. Chang and S. K. Ma, *ibid.* 188, 2385 (1969).

[2] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., Cambridge, England, 1966). This also contains references to the original papers.

[3] S. J. Chang and T. M. Yan, Phys. Rev. Letters 25, 1586 (1970).

[4] B. Hasslacher, D. K. Sinclair, G. M. Cicuta, and R. L. Sugar, Phys. Rev. Letters 25, 1591 (1970).

[5] G. M. Cicuta and R. L. Sugar, Phys. Rev. D 3, 970 (1971).