Deep Space Instrumentation Facility.[20] Nor can the passive radio-astronomical observations measure to such accuracy.[21] But for an E.M. signal originating at $M/r \lesssim 0.1$ or nearer to a compact star, such a change of signal strength should be observable on earth. In principle, this effect may provide another test of classical electromagnetic theory and general relativity, or may be used to single out the particular amplitude correction caused by gravitation.

[20] Deep Space Instrumentation Facility, Jet Propulsion Lab. System specification, Code No. 23835, Spec. No. DOW-1389-DTL, Rev. A., 1970 (unpublished).

[21] Typical signal-strength measurement is about to two-digit accuracy [A. T. Moffet (private communication)].

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Electron Scattering from a Standing Light Wave

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The transition probability for electron scattering in the Kapitza-Dirac experiment is calculated using nonrelativistic theory as a function of the time of flight t of the electron through the light beam, and as a function of the intensity. It is found that for intensities up to approximately 10^7 W/cm^2 the probability equals $\sin^2 a_0 t$, where a_0 is proportional to the intensity. At intensities of the order of 10^9 W/cm^2, the dependence of the probability on the time is more complicated but it also oscillates between zero and nearly unity as the time increases. Both first- and second-order Bragg reflections are considered as well as conditions off the Bragg maximum.

I. INTRODUCTION

In 1933 Kapitza and Dirac[1] predicted that electrons of well-defined momentum could be reflected from a standing light wave provided that $\lambda_p = \lambda \cos\vartheta$, where λ_p is the de Broglie wavelength, λ is the radiation wavelength, and ϑ is the angle of incidence. They called this a first-order Bragg condition, the lattice spacing being $\frac{1}{2}\lambda$. The probability per electron that reflection would take place was shown to be proportional to the square of the light intensity. At the time an experiment was impractical, given the intensity of available sources, but recently several attempts have been made to demonstrate this effect using Q-switched lasers as a light source.[2] Electrons scattered by the light beam have been observed but the dependence of the probability of scattering on the angle of incidence and on the intensity of the light has not yet been determined experimentally.

The formula for the transition probability given by Kapitza and Dirac,[1] derived using the theory of stimulated processes, is not valid at intensities used in current experiments, for it would predict probabilities in excess of unity. It is therefore essential to reexamine the theory of this scattering problem. The first extensive published investigation is that of Fedorov.[3] The Schrödinger equation, used to describe the electron, is cast into the form of a Mathieu equation by neglecting the time dependence of the standing light wave, treated as a classical field. Solutions are found, however, only in the case of either very low intensity, or of very high intensity, and predictions are not made in the intensity range used in recent experiments. Ezawa and Namaizawa[4] have also investigated solutions to the Mathieu equation. Schoenebeck[5] uses a modified first-order perturbation method to obtain a solution to the problem.

In this article a somewhat different approach is taken. The nonrelativistic Green function for an electron in a standing-wave field is calculated using perturbation theory. By neglecting the rapidly time-varying part of the electron-field interaction, it is possible to sum the perturbation series completely and obtain exact scattering matrix elements between states of definite electron momenta. This permits the transition probability to be evaluated for practically arbitrary intensities and interaction times. It is possible in addition to examine the transition probability for electron momenta not satisfying the Bragg condition, and to calculate the probability of higher-order Bragg reflections.

II. THEORETICAL BACKGROUND

A. Preliminary Remarks

The procedure adopted in this article is to calculate the scattering amplitude via the nonrelativistic Green

[1] P. L. Kapitza and P. A. M. Dirac, Proc. Cambr. Phil. Soc. 29, 297 (1933).

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[3] M. V. Fedorov, Zh. Eksperim. i Teor. Fiz. 52, 1434 (1967) [Sov. Phys. JETP 25, 952 (1967)].

[4] H. Ezawa and H. Namaizawa, J. Phys. Soc. Japan 26, 458 (1969).

[5] H. Schoenebeck, Phys. Letters 27A, 286 (1968).

function of the electron interacting with the laser field, using momentum-time representation. This Green function is defined in general through the following equation:

$$\left(\frac{p^2}{2m}-i\hbar\frac{\partial}{\partial t}\right)G(\mathbf{p},t;\mathbf{p}_0,t_0)+N^{-1}\int d^3p'\langle\mathbf{p}|H_I(t)|\mathbf{p}'\rangle G(\mathbf{p}',t;\mathbf{p}_0,t_0)=-N\hbar\delta(\mathbf{p}-\mathbf{p}_0)\delta(t-t_0)\,,\quad(2.1)$$

where $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{p}_0$ are eigenvalues of the momentum operator, $H_I(t)$ is an interaction Hamiltonian, and $N=(2\pi\hbar)^3/V$, where V is a quantization volume. This is an integrodifferential equation in contrast to the one for the Green function in position representation, a differential equation. The exact solution may be expressed in terms of a zeroth-order Green function,

$$G_0(\mathbf{p},t;\mathbf{p}_0,t_0)=-iN\theta(t-t_0)\delta(\mathbf{p}-\mathbf{p}_0)\times\exp-i(p^2/2m\hbar)(t-t_0),\quad(2.2)$$

which satisfies the following equation:

$$\left(\frac{p^2}{2m}-i\hbar\frac{\partial}{\partial t}\right)G_0(\mathbf{p},t;\mathbf{p}_0,t_0)=-N\hbar\delta(\mathbf{p}-\mathbf{p}_0)\delta(t-t_0)\,.\quad(2.3)$$

It is readily shown that the exact Green function satisfies the equation

$$G(\mathbf{p},t;\mathbf{p}_0,t_0)=G_0(\mathbf{p},t;\mathbf{p}_0,t_0)+N^{-2}\hbar^{-1}\int\cdots\int d^3p'd^3p''dt'G_0(\mathbf{p},t;\mathbf{p}',t')\times\langle\mathbf{p}'|H_I(t')|\mathbf{p}''\rangle G(\mathbf{p}'',t';\mathbf{p}_0,t_0)\,.\quad(2.4)$$

The following iterative solution exists for the above Green function:

$$\begin{aligned}G(\mathbf{p},t;\mathbf{p}_0,t_0)&=G_0+\Delta G+\Delta G'+\cdots=G_0(\mathbf{p},t;\mathbf{p}_0,t_0)\\&+N^{-2}\hbar^{-1}\int\cdots\int d^3p'd^3p''dt'G_0(\mathbf{p},t;\mathbf{p}',t')\\&\times\langle\mathbf{p}'|H_I(t')|\mathbf{p}''\rangle G_0(\mathbf{p}'',t';\mathbf{p}_0,t_0)\\&+N^{-4}\hbar^{-2}\int\cdots\int d^3p'\cdots dt''G_0(\mathbf{p},t;\mathbf{p}',t')\\&\times\langle\mathbf{p}'|H_I(t')|\mathbf{p}''\rangle G_0(\mathbf{p}'',t';\mathbf{p}''',t'')\\&\times\langle\mathbf{p}'''|H_I(t'')|\mathbf{p}^{\mathrm{iv}}\rangle G_0(\mathbf{p}^{\mathrm{iv}},t'';\mathbf{p}_0,t_0)+\cdots.\end{aligned}\quad(2.5)$$

In terms of this Green function, any wave function in momentum space can be propagated forward in time using the following relation:

$$\Phi(\mathbf{p},t)=iN^{-1}\int d^3p'G(\mathbf{p},t;\mathbf{p}',t')\Psi(\mathbf{p}',t')\,,\quad t\geq t'.\quad(2.6)$$

It may be verified that $\Phi(\mathbf{p},t)$ is a solution of the Schrödinger equation in momentum representation,

$$\left(\frac{p^2}{2m}-i\hbar\frac{\partial}{\partial t}\right)\Phi(\mathbf{p},t)+N^{-1}\int d^3p'\langle\mathbf{p}|H_I(t)|\mathbf{p}'\rangle\Phi(\mathbf{p}',t)=0\,,\quad(2.7)$$

provided that $\Psi(\mathbf{p},t)$ is a solution.

The use of $G(\mathbf{p},t;\mathbf{p}_0,t_0)$ in a scattering problem may be summarized as follows. Suppose that at time t' the particle in question is known to be free and is described by a free-particle wave function, $\psi_i(\mathbf{p},t')$. Taking into account the interaction H_I, this free-particle state will evolve into a state

$$\Phi(\mathbf{p},t)=iN^{-1}\int d^3p'G(\mathbf{p},t;\mathbf{p}',t')\psi_i(\mathbf{p}',t')\,.\quad(2.8)$$

If we project this state on to a free-particle state at time t, we obtain the quantity

$$S_{fi}=iN^{-2}\int\int d^3p'd^3p\,\psi_f(\mathbf{p},t)^*G(\mathbf{p},t;\mathbf{p}',t')\psi_i(\mathbf{p}',t')\,.\quad(2.9)$$

Allowing t' to tend to $-\infty$ and t to tend to $+\infty$, one obtains the usual definition of the scattering amplitude. If, however, the interaction is known to be zero for times up to $t'=t_0$ and after times $t=t_f$, the scattering amplitude is given by Eq. (2.9) with $t'=t_0$ and $t=t_f$.

The latter situation corresponds rather closely to the Kapitza-Dirac experiment as it has been performed.[2] Experimentally, electrons are released from a gun and travel freely towards the standing light wave in the laser cavity. They suddenly enter a region where the radiation field is large, and just as suddenly leave, having passed through the laser beam. At the edges of the laser beam the electrons are free, and inside they are of course interacting with the laser field. Because the interaction considered will be independent of $v_\perp$, the velocity of the electron perpendicular to the direction of propagation of the laser beam, the duration of the interaction t_f-t_0 equals simply the width of the laser beam divided by $v_\perp$. In other words, the interaction may be thought of as deliberately switched on at time $t=t_0$, and switched off at time $t=t_f$.

Finally, one obtains the probability of scattering taking the absolute value squared of S_{fi} [Eq. (2.9)] and summing the result over a range of final states.

B. Choice of Hamiltonian and Zeroth-Order Green Function

The Hamiltonian of the electron in the presence of the standing wave is given by the usual nonrelativistic formula

$$\mathcal{H}=p^2/2m-e\mathbf{p}\cdot\mathbf{A}/mc+e^2A^2/2mc^2\,, \qquad (2.10)$$

where $\mathbf{A}$, the vector potential, may be written,

$$\mathbf{A}=\mathbf{A}_0\cos kz\cos\omega t\,. \qquad (2.11)$$

It is assumed that the wave is linearly polarized and monochromatic. For reasons of simplicity it will be further assumed that the plane containing the initial and final electron momenta, and the z axis, is perpendicular to $\mathbf{A}$. This permits one to neglect the term in $\mathcal{H}$ containing $\mathbf{p}\cdot\mathbf{A}$. It is convenient to write the Hamiltonian in the following way:

$$\mathcal{H}=H_0+H_1\,, \qquad (2.12a)$$

where

$$H_0=p^2/2m+a(1+\cos 2\omega t)\,, \qquad (2.12b)$$

$$H_1=a(1+\cos 2\omega t)\cos 2kz\,, \qquad (2.12c)$$

and

$$a=e^2A_0^2/8mc^2\,. \qquad (2.12d)$$

Because the potential H_1 does not depend on the coordinates x and y, the problem may be treated in one dimension; from now on the symbol p will refer to the z component of the momentum.

The zeroth-order Green function is chosen to be the solution of the equation,

$$(H_0-i\hbar\partial/\partial t)G_0'(p,t;\,p_0,t_0)=-N\hbar\delta(p-p_0)\delta(t-t_0)\,, \qquad (2.13)$$

that is,

$$G_0'(p,t;\,p_0,t_0)=-iN\theta(t-t_0)\delta(p-p_0)\times\exp\{-i[(p^2/2m+a)(t-t_0)+(a/2\omega)\times(\sin 2\omega t-\sin 2\omega t_0)]/\hbar\}. \qquad (2.14)$$

The reason for this choice, rather than the function G_0 already mentioned, is to take into exact account from the outset as much of the Hamiltonian as possible. This reduces the number of diagrams which must be considered in the perturbation calculation.

The matrix element of the interaction Hamiltonian, required in the perturbation calculation of $G(p,t;\,p_0,t_0)$, Eq. (2.5), is

$$\begin{aligned}\langle p'|H_1(t')|p''\rangle&\equiv H_1(p',p'';\,t')\\&=aV^{-\frac{1}{3}}(1+\cos 2\omega t')\int dz\,\cos 2kz\,e^{-iz(p'-p'')/\hbar}\\&=a(\tfrac{1}{2}N)(1+\cos 2\omega t')[\delta(-p'+2k\hbar+p'')\\&\qquad+\delta(-p'-2k\hbar+p'')]. \qquad (2.15)\end{aligned}$$

In order to establish certain mathematical procedures and to introduce a simplified notation we now discuss in detail the term ΔG in the expansion of the Green function, and we evaluate its contribution S_{fi}' to the scattering amplitude. It will be shown that the transition probability corresponding to S_{fi}' is equal to that deduced by Kapitza and Dirac.[1]

C. Scattering Amplitude in First Order

The first-order contribution to the Green function appearing in Eq. (2.5) is rewritten *mutatis mutandis*

$$\Delta G=N^{-2}\hbar^{-1}\int\int\int dt'dp'dp''G_0'(p,t;\,p',t')\times H_1(p',p'';\,t')G_0'(p'',t';\,p_0,t_0)\,. \qquad (2.16)$$

Introducing (2.14) and (2.15) into Eq. (2.16) and making use of the definition of the unit step function,

$$\theta(t)=\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega\,e^{i\omega t}}{\omega-i\epsilon}, \qquad (2.17)$$

we obtain the following expression for ΔG:

$$\begin{aligned}\Delta G=&\left(\frac{Na_0}{2\pi}\right)\exp\{-i[(\omega_p+2a_0)t\\&-(\omega_{p0}+2a_0)t_0+(a_0/\omega)(\sin 2\omega t-\sin 2\omega t_0)]\}\\&\times\int d\omega_1\frac{e^{i[\omega_1(t-t_0)+\Delta_{p0}t_0]}}{\omega_1-i\epsilon}\left\{\frac{1}{\omega_1-\Delta_{p0}-i\epsilon}\right.\\&\left.+\frac{e^{i2\omega t_0}}{2(\omega_1-2\omega-\Delta_{p0}-i\epsilon)}+\frac{e^{-i2\omega t_0}}{2(\omega_1+2\omega-\Delta_{p0}-i\epsilon)}\right\}\\&\times[\delta(-p+2k\hbar+p_0)+\delta(-p-2k\hbar+p_0)]. \qquad (2.18)\end{aligned}$$

In deducing (2.18) from (2.16), integrations have been performed over the intermediate momenta, the time, and one intermediate frequency resulting from the definition of $\theta(t)$. For convenience the following symbols have been introduced:

$$\omega_p=p^2/2m\hbar\,, \qquad (2.19a)$$

$$\omega_{p0}=p_0^2/2m\hbar\,, \qquad (2.19b)$$

$$\Delta_{p0}=\omega_p-\omega_{p0}\,, \qquad (2.19c)$$

$$a_0=a/2\hbar\,. \qquad (2.19d)$$

Before proceeding to carry out the remaining integration in expression (2.18), we examine the contribution of ΔG to the total scattering amplitude:

$$S_{fi}'=iN^{-2}\int\int dp\,dp_0\psi_f^*(p,t)\times\Delta G(p,t;\,p_0,t_0)\psi_i(p_0,t_0)\,. \qquad (2.20)$$

ψ_i and ψ_f are the initial- and final-state wave functions,

$$\psi_f^*(p,t)=C_f^*(p)e^{i\omega_p t}\,, \qquad (2.21a)$$

$$\psi_i(p_0,t_0)=C_i(p_0)e^{-i\omega_{p0}t_0}\,. \qquad (2.21b)$$

[We remind the reader at this point that the time difference $t-t_0$ contained in (2.20) must be interpreted as the duration of the interaction, that is, the time of flight of the electron through the laser beam in the real experiment. The symbol t_f previously introduced has been replaced by t for simplicity.] The momentum distributions C_f and C_i are normalized in the following way:

$$N^{-1}\int |C_f(p)|^2 dp = N^{-1}\int |C_i(p)|^2 dp = 1\,. \tag{2.22}$$

It follows immediately that

$$\begin{aligned} S_{fi}' = (2\pi N)^{-1} a_0 \exp\{-ia_0[2t+\omega^{-1}\sin 2\omega t]\} \\ \times \int dp\, C_f^*(p) \int d\omega_1 \frac{e^{i\omega_1 t}}{\omega_1 - i\epsilon} \\ \times\Bigg[C_i(p-2k\hbar)\bigg\{\frac{1}{\omega_1 - 2k(p-k\hbar)/m - i\epsilon} \\ + \frac{1}{2[\omega_1 - 2\omega - 2k(p-k\hbar)/m - i\epsilon]} \\ + \frac{1}{2[\omega_1 + 2\omega + 2k(p-k\hbar)/m - i\epsilon]}\bigg\} \\ + C_i(p+2k\hbar)\{k\to -k\}\Bigg], \end{aligned} \tag{2.23}$$

where the expression within the second curly brackets is identical with that within the first curly brackets, but with $k\to -k$, and where for simplicity t_0 has been put equal to zero.

For the calculation to correspond to the experiment originally proposed by Kapitza and Dirac,[1] it will be assumed that $C_i(p_0)$ is sharply peaked near $p_0 = -kh$. Other initial momentum states will be considered later. Because, as usual, the electrons will be detected incoherently, the final-state wave function may be chosen to be sharply peaked near $p = p_f$:

$$C_f^*(p) = (N\Delta p_f)^{1/2}\delta(p-p_f)\,. \tag{2.24}$$

Δp_f is an infinitesimal range of final momenta and $\delta(p-p_f)$ is a δ function defined so that (2.22) holds. An integration of the square of the scattering amplitude must of course be performed over p_f.

It will be assumed that only those electrons with momenta p_f near the value kh are detected. In this case, the second term of Eq. (2.23) may be omitted, and the scattering amplitude takes the form

$$\begin{aligned} S_{fi}' = (N^{-1}\Delta p_f)^{1/2}(a_0/2\pi)\exp\{-ia_0(2t+\omega^{-1}\sin 2\omega t)\} \\ \times C_i(p_f - 2k\hbar)\int d\omega_1 \frac{e^{i\omega_1 t}}{\omega_1 - i\epsilon}\bigg\{\frac{1}{\omega_1 - 2k(p_f - k\hbar)/m - i\epsilon} \\ + \frac{1}{2[\omega_1 - 2\omega - 2k(p_f-k\hbar)/m - i\epsilon]} \\ + \frac{1}{2[\omega_1 + 2\omega - 2k(p_f-k\hbar)/m - i\epsilon]}\bigg\}. \end{aligned} \tag{2.25}$$

The probability of detecting an electron may be written,

$$P' = \sum_f |S_{fi}'|^2 = N^{-1}a_0^2 \int dp_f |C_i(p_f - 2k\hbar)|^2 \times \left|\frac{1}{2\pi}\int d\omega_1 \cdots\right|^2. \tag{2.26}$$

The function $|C_i|^2$ in expression (2.26) can be considered as a δ function in the integration over p_f provided that it is considerably narrower than the function of p_f resulting from the integration over ω_1. The sharpness of the latter function depends on the time. For example, consider the function resulting from the integration of the first term in curly brackets in Eq. (2.25):

$$\begin{aligned} K(p_f,t) &= \frac{1}{2\pi}\int d\omega_1 \frac{e^{i\omega_1 t}}{(\omega_1 - i\epsilon)[\omega_1 - 2k(p_f - k\hbar)/m - i\epsilon]} \\ &= -e^{i(p_f - k\hbar)kt/m}\frac{\sin(p_f - k\hbar)kt/m}{(p_f - k\hbar)k/m}. \end{aligned} \tag{2.27}$$

$K(p_f,t)$ is peaked at $p_f = kh$ and has a width approximately equal to $\Delta = 2\pi m/kt$, which obviously decreases with increasing t. If $|C_i|^2$ is to be considered as a δ function, the momentum range in C_i must be less than Δ. This turns out to be a reasonable assumption in practice. Suppose that the initial electron wavepacket is localized in a region of space approximately 1 mm in size. Then the width of C_i must equal approximately $\Delta p = h/\Delta x = 10^{-31}$ kg m/sec. The requirement that Δ be greater than Δp implies that t must be less than 10^{-5} sec. This condition is satisfied in any experiment with a Q-switched laser, where the pulse duration is of the order of 10^{-8} sec.

With this assumption (and with the previously mentioned assumption that C_i is centered at $p_0 = -kh$), the probability (2.26) becomes simply

$$P' = \left|\frac{a_0}{2\pi}\int d\omega_1 \frac{e^{i\omega_1 t}}{\omega_1 - i\epsilon}\left[\frac{1}{\omega_1 - i\epsilon} + \frac{1}{2(\omega_1 - 2\omega - i\epsilon)} + \frac{1}{2(\omega_1 + 2\omega - i\epsilon)}\right]\right|^2 = a_0^2[t + (\sin 2\omega t)/2\omega]^2. \tag{2.28}$$

For times much greater than $1/\omega$, the second term in brackets in (2.28) is negligible compared to the first term. If a ruby laser is used, $\omega = 2.72\times 10^{15}$ sec^{-1}, and so for times greater than roughly 10^{-13} sec, the transition probability equals just $(a_0 t)^2$. We reiterate that t is the time of flight of the electron through the laser beam. Introducing the relation between the intensity of light and the quantity a_0, we obtain for the transition probability

$$P' = \left(\frac{\pi r_0 c I t}{\omega^2 \hbar}\right)^2, \tag{2.29}$$

where I is the sum of the intensities of the two traveling waves forming the standing wave, and where r_0 is the classical electron radius. This formula is equivalent to the one derived by Kapitza and Dirac.[1] If $I=10^8$ W/cm^2, a typical value in a Q-switched laser experiment, one calculates that P' equals unity for a time $t=0.28\times10^{-9}$ sec. This first-order treatment clearly cannot be trusted for times of flight of this order or greater.

D. Higher-Order Corrections to S_{fi}: Low-Intensity Case

The next correction to the Green function which will contribute to transitions from the state of momentum $p_0=-kh$ to the state of momentum $p=kh$ occurs in the third order of the perturbation series (2.5). This correction is

$$\Delta G''=Na_0^3(2\pi)^{-4}\exp\{-i[(\omega_p+2a_0)t+a_0\omega^{-1}\sin 2\omega t]\}$$
$$\times\int\cdots\int dt_1\cdots d\omega_4\frac{\exp\{i[\omega_1(t-t_3)+\omega_2(t_3-t_2)+\omega_3(t_2-t_1)+\omega_4 t_1]\}}{(\omega_1-i\epsilon)(\omega_2-i\epsilon)(\omega_3-i\epsilon)(\omega_4-i\epsilon)}(1+\cos 2\omega t_3)(1+\cos 2\omega t_2)(1+\cos 2\omega t_1)$$
$$\times\exp\{-i[(\omega_{p_5}-\omega_p)t_3+(\omega_{p_3}-\omega_{p_5})t_2+(\omega_{p_0}-\omega_{p_3})t_1]\}[\delta(-p+2k\hbar+p_5)+\delta(-p-2k\hbar+p_5)]$$
$$\times[\delta(p_5+2k\hbar+p_3)+\delta(-p_5-2k\hbar+p_3)][\delta(-p_3+2k\hbar+p_0)+\delta(-p_3-2k\hbar+p_0)]. \quad (2.30)$$

Explicit calculations, not given here, show that contributions to $\Delta G''$ coming from terms containing the factors $\cos 2\omega t_1$, $\cos 2\omega t_2$, and $\cos 2\omega t_3$ are negligible compared to the single term without such factors, for times much greater than ω^{-1}, as was the case in the first-order calculation just discussed. We henceforth neglect such terms completely, and the problem reduces to that of an electron in a stationary sinusoidal field; Fedorov[3] arrives at the same conclusion using a different argument.

Now performing the integrations over the intermediate times, and over the variables ω_2, ω_3, and ω_4, we obtain from Eq. (2.30)

$$\Delta G''=(Na_0^3/2\pi)\exp\{-i[(\omega_p+2a_0)t+a_0\omega^{-1}\sin 2\omega t]\}$$
$$\times\int\int\int dp_5dp_3d\omega_1\frac{e^{i\omega_1 t}}{(\omega_1-i\epsilon)(\omega_1-\Delta_{p_5}-i\epsilon)(\omega_1-\Delta_{p_3}-i\epsilon)(\omega_1-\Delta_{p_0}-i\epsilon)}$$
$$\times[\delta(\cdots)+\delta(\cdots)][\delta(\cdots)+\delta(\cdots)][\delta(\cdots)+\delta(\cdots)], \quad (2.31)$$

where $\Delta_{p_5}=\omega_p-\omega_{p_5}$, etc., and the product of δ functions is identical with the last factor of Eq. (2.30).

The result of the product of the various δ functions is conveniently visualized by means of diagrams. The quantities $2kh$ $(-2kh)$ occurring in the arguments of the δ functions are indicated by incoming (outgoing) horizontal lines, and the electron momenta p_3, p_5, etc., are indicated by vertical lines. The three diagrams corresponding to the part of $\Delta G''$ in which we are particularly interested at the moment are shown in Fig. 1. In Fig. 1(a), $p_3=p_0+2kh$, $p_5=p_3-2kh$, and $p=p_5+2kh$, which are the conditions dictated by one of the triplets of δ functions. If the electromagnetic field had been quantized, $2kh$ would be identified with the change in momentum of the radiation field, due to the absorption by the electron of a photon of momentum kh from one wave (the "incident" wave) and the emission by the electron of a photon of momentum $-kh$ into the other, stimulating, wave. This two-photon process is indicated in Fig. 1 by a single horizontal line and vertex. Vertical lines on the diagram are identified with factors in the Green function. In Fig. 1(a) the line p corresponds to the factor $(\omega_1-i\epsilon)^{-1}$, and the line p_5 corresponds to the factor $(\omega_1-\Delta_{p_5}-i\epsilon)^{-1}$, etc. These factors are called propagators. The quantity summed to $\omega_1-i\epsilon$ in any propagator is the difference between the corresponding intermediate electron energy, and the final energy, divided by h. Associated with every vertex in a diagram is a factor a_0 in the corresponding contribution to the Green function.

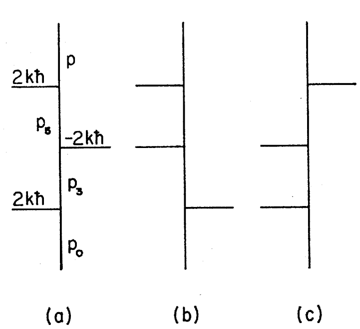

FIG. 1. Diagrams arising in third order.

Inspection of Eq. (2.28) shows that in first order, the probability of scattering reduces to the absolute value squared of a certain contour integral involving products of propagators; this is true in any order of perturbation theory. This contour integral will be called a transition amplitude. Corresponding to every scattering diagram j is a certain contour integral K_j and the sum of all integrals for all possible diagrams is the complete transition amplitude K, the square of which is the desired probability. We will henceforth concentrate on the transition amplitude rather than on the Green function itself; the latter may be obtained in an obvious way from K.

Using the rules described in the previous paragraphs, the amplitude K_j can be written immediately, given an arbitrary diagram. For example, the amplitude corresponding to the diagram of Fig. 1(a) is assuming $p_0=-kh$, and $p=kh$,

$$K_3=\frac{a_0{}^3}{2\pi}\int d\omega_1\frac{e^{i\omega_1 t}}{(\omega_1-i\epsilon)^4}=\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^4}=i\frac{(ia_0t)^3}{3!},\quad(2.32)$$

where we have made the change of variable $\eta=\omega_1/a_0$ and where ϵ/a_0 has been simply replaced by ϵ to avoid introducing a new symbol.[6] In fact, all those diagrams in which the intermediate electron momenta equals $\pm kh$ have an amplitude which is similar in form to (2.32). If $2n+1$ equals the order of perturbation, $n=0, 1, 2, \ldots$, we find[7]

$$K_{2n+1}=\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^{2n+2}}=i\frac{(ia_0t)^{2n+1}}{(2n+1)!}.\quad(2.33)$$

In each order of perturbation there is only one diagram of this type; the contribution to the total transition amplitude from this class of diagrams is

$$K'=\sum_{n=0}^{\infty}K_{2n+1}=i\sum_{n=0}^{\infty}\frac{(ia_0t)^{2n+1}}{(2n+1)!}=-\sin a_0t.\quad(2.34)$$

If these were the only diagrams of importance, one could assert that the transition probability is

$$P=\sin^2 a_0t\,,\quad(2.35)$$

an expression which is nearly the same as that derived by Fedorov[3] for the case of low-intensity fields. As will be discussed in more detail later, (2.35) holds for intensities less than roughly 10^7 W/cm^2. The Kapitza-Dirac formula (2.29) is the first term in the expansion of (2.35) in powers of a_0t. The Kapitza-Dirac formula gives an accurate prediction at these relatively low intensities (so far as Q-switched lasers are concerned), only if the product a_0t is of the order of 0.1 or 0.2. If the product a_0t equals unity, the probability given by (2.35) equals 0.7, not 1.0 as predicted by (2.29). Although this error would not seem serious, the situation is quite different if $a_0t=\pi$, when the probability equals zero according to (2.35). This illustrates one of the hidden dangers in attempting the Kapitza-Dirac experiment, not previously emphasized; it may very well happen that for the interaction time and intensity in a given experimental setup, the transition probability equals zero. For a total intensity of 10^7 W/cm^2 this would occur at a time $t=n(0.88\times10^{-8})$ sec, n integral.

Before proceeding to consider more general diagrams, we examine again the contribution to the transition amplitude from the "ladder" diagrams where the intermediate momenta equal $\pm kh$. This contribution is

$$K'=\sum_{n=0}^{\infty}\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^{2n+2}}.\quad(2.36)$$

In the region $|\eta-i\epsilon|>1$, the operations of summation and integration can clearly be exchanged; the integrand then contains the sum

$$(S\eta)=\sum_{n=0}^{\infty}\frac{1}{(\eta-i\epsilon)^{2n+2}}=\frac{1}{(\eta-i\epsilon)^2-1},\qquad|\eta-i\epsilon|>1.\quad(2.37)$$

The function $S(\eta)$ is now analytically continued by the last member of (2.37) over the entire complex plane, and we write for the transition amplitude

$$K'=\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^2-1}=-\sin a_0t,\quad(2.38)$$

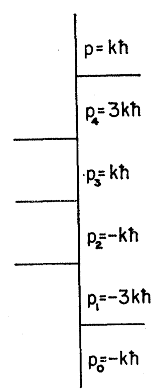

FIG. 2. Example of a fifth-order diagram.

[6] It may be remarked here that if the calculation were attempted using the standard rules for constructing scattering amplitudes from Feynman diagrams, Fig. 1(a) would give an infinite contribution because the two intermediate propagators, lines p_3 and p_5 in the diagram, blow up. These improper diagrams can be removed by renormalizing the wave functions, but this procedure is rather complex as may be seen in the article by Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964). The procedure here obviates this difficulty.

[7] Only odd order perturbation terms contribute to transitions from $p_0=-k\hbar$ to $p=k\hbar$.

which is identical to (2.34). This procedure of analytically continuing the perturbation series over the entire complex η plane will be extended to the case of more complex diagrams.

E. General Case

In a more general diagram, the intermediate momentum of the electron can naturally be greater than kh. Consider, for example, the fifth-order diagram shown in Fig. 2, and the corresponding amplitude,

$$K_5=\frac{a_0{}^5}{2\pi}\int d\omega_1\frac{e^{i\omega_1 t}}{(\omega_1-i\epsilon)(\omega_1-\Delta_{p1}-i\epsilon)(\omega_1-\Delta_{p2}-i\epsilon)(\omega_1-\Delta_{p3}-i\epsilon)(\omega_1-\Delta_{p4}-i\epsilon)(\omega_1-\Delta_{p0}-i\epsilon)}. \tag{2.39}$$

We have $p_0=-kh$, $p_1=-3kh$, $p_2=-kh$, $p_3=kh$, and $p_4=3kh$; hence $\Delta_{p_1}=\Delta_{p_4}=-4k^2h/m=\mu$ and $\Delta_{p_2}=\Delta_{p_3}=\Delta_{p_0}=0$. Introducing the new variables $\eta=\omega_1/a_0$ and $\alpha=\mu/a_0$, replacing ϵ/a_0 by ϵ as before, we obtain for (2.39)

$$K_5=\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^2[(\eta-i\epsilon)(\eta-\alpha-i\epsilon)]^2}. \tag{2.40}$$

The factors in the denominator have been grouped in the way shown to stress the fact that when a factor $(\eta-\alpha-i\epsilon)$ occurs, it must be accompanied by a factor $(\eta-i\epsilon)$, and that the remaining factors $(\eta-i\epsilon)$ occur in pairs. These statements hold in general, as may be verified by the inspection of an arbitrary diagram.

The general expression for the amplitude corresponding to diagrams in which the absolute value of the intermediate momentum can be no larger than $3kh$ is

$$K_N=\frac{1}{2\pi}\int d\eta\frac{e^{ia_0t\eta}}{(\eta-i\epsilon)^{2m_1}[(\eta-i\epsilon)(\eta-\alpha-i\epsilon)]^{m_2+n_2}}, \tag{2.41}$$

where m_1, m_2, and n_2 are integers. The number of times a propagator is found with momentum equal to kh, $3kh$, $-kh$, $-3kh$, equals m_1, m_2, m_1, n_2, respectively. In the example cited above (Fig. 2), $m_1=1$, $m_2=1$, and $n_2=1$. The order of perturbation is $N=2(m_1+m_2+n_2)-1$. The number of diagrams which correspond to the same integral K_N is[8]

$$n=\binom{m_1+m_2-1}{m_2}\binom{m_1+n_2-1}{n_2}. \tag{2.42}$$

The contribution to the total amplitude from these diagrams is

$$K'=\sum_{m_1=1}^{\infty}\sum_{m_2,n_2=0}^{\infty}nK_N. \tag{2.43}$$

As in the example already discussed [Eqs. (2.36)–(2.38)], the order of integration and summation may be interchanged in the region where the sum converges. Using twice the formula

$$\sum_{n=0}^{\infty}\binom{m+n-1}{n}x^n=\frac{1}{(1-x)^m}, \tag{2.44}$$

we obtain for the sum of the products of the propagators the expression

$$\begin{aligned}S(\eta)&=\sum_{m_1=1}^{\infty}\sum_{m_2,n_2=0}^{\infty}\frac{1}{(\eta-i\epsilon)^{2m_1}[(\eta-i\epsilon)(\eta-\alpha-i\epsilon)]^{m_2+n_2}}\binom{m_1+m_2-1}{m_2}\binom{m_1+n_2-1}{n_2}\\&=\frac{1}{2}\left[\frac{1}{1-\dfrac{1}{\eta-i\epsilon-\dfrac{1}{\eta-\alpha-i\epsilon}}}+\frac{1}{1+\dfrac{1}{\eta-i\epsilon-\dfrac{1}{\eta-\alpha-i\epsilon}}}\right]-1.\end{aligned} \tag{2.45}$$

As previously, the function S is extended over the entire complex plane by analytic continuation, and the amplitude K may be derived from the right-hand side of (2.45).

This procedure is now generalized to include diagrams with intermediate momenta of arbitrary magnitude. It is readily shown[8] that the sum S becomes a continued fraction,

$$S(z)=\frac{1}{2}\left(\frac{1}{1-}\,\frac{1}{z-}\,\frac{1}{z-\alpha-}\,\frac{1}{z-3\alpha-}\,\frac{1}{z-6\alpha-}\cdots+\frac{1}{1+}\,\frac{1}{z-}\,\frac{1}{z-\alpha-}\,\frac{1}{z-3\alpha-}\,\frac{1}{z-6\alpha-}\cdots\right)-1, \tag{2.46}$$

[8] The problem of counting diagrams of the type encountered here has been considered by Z. Fried and J. H. Eberly, Ref. 6, where a proof may be found for relation (2.42).

where for compactness $\eta - i\epsilon$ has been replaced by z. Expression (2.45) simply equals expression (2.46) terminated at the third level.

The evaluation of the scattering amplitude

$$K=\frac{1}{2\pi}\int d\eta\, S(\eta)e^{i\alpha_0 t\eta} \tag{2.47}$$

reduces to the determination of the poles and residues of the two continued fractions in (2.46). It is seen that K consists of the sum of a number of complex exponentials, one for each pole of S. The amplitude of each exponential equals the residue of S at the corresponding pole, and the argument is proportional to the product of the location of the pole, the time, and the intensity.

We have not found an expression in terms of known functions for $S(z)$, Eq. (2.46), and hence it is not possible to check analytically whether the solution satisfies the original equation. The continued fraction has the same form as those used in the solutions to the characteristic equation of the Mathieu equation; this is not surprising since the problem has formally a solution in terms of Mathieu functions. Because the continued fraction is readily investigated numerically, there is no pressing need to introduce these functions. We remark that a procedure similar to the one developed in this article has been used to obtain the Green function for an electron in a circularly polarized monochromatic traveling wave; in this case, the continued fraction playing the role of (2.46) can be expressed in terms of Bessel functions and the solution can be shown to satisfy the original Green function equation.[9] One may therefore have confidence that Eq. (2.46) correctly describes the behavior of the electron.

III. NUMERICAL RESULTS

A. Transitions from $p_0 = -k\hbar$ to $p = k\hbar$

The behavior of S [Eq. (2.46)] has been investigated numerically using a digital computer. Isolated poles are located near $1-1/\alpha$, $-1-1/\alpha$, and two poles close together are found at each of the locations α, 3α, 6α, etc. The value of the residue at a given pole depends strongly on α, which is related to the intensity in the following way:

$$\alpha=-8(\omega\hbar/mc^2)(r_0\lambda\omega^{-1}I/\omega\hbar)^{-1}\,. \tag{3.1}$$

For a ruby laser of total intensity $I=10^7$ W/cm^2 (an order of magnitude lower than the output of available Q-switched lasers), $\alpha=-100$. For α in this neighborhood, the residues at the poles near $1-1/\alpha$ and $-1-1/\alpha$ are nearly equal to $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and the residues at the poles near α, 3α, etc., are negligible. The transition amplitude is then given to an excellent approximation by expression (2.34). At higher intensities, where α is in the range from -1 to -10, the residues at the poles near α, 3α, and 6α must be included and the transition amplitude has a more complicated form. The sum of the residues of S equals zero, and the sum of the absolute values of the residues equals unity. This ensures that first of all the probability $P=|K|^2$ tends to zero as t tends to zero, and secondly that the probability cannot exceed unity.

In the numerical analysis the continued fraction is terminated at a certain level. This is equivalent to omitting contributions from diagrams which contain electron momenta greater than a certain value. For example, the transition amplitude obtained from the terminated fraction (2.45) does not contain contributions from diagrams in which the electron momentum is greater than $3k\hbar$. If the fraction were terminated at the explicitly written portion of (2.46), diagrams in which the momentum was greater than $7k\hbar$ would be omitted. It was found that even at the highest intensity considered ($I=10^9$ W/cm^2, $\alpha=-1$), diagrams in which the intermediate momentum is greater than approximately $15k\hbar$ contribute very little to the transition amplitude.

The transition probability has been evaluated numerically as a function of the time for a number of cases of practical interest. Graphs of this probability are shown in Figs. 3 and 4. For the lowest intensity [Fig. 3(a), solid curve], the probability varies practic-

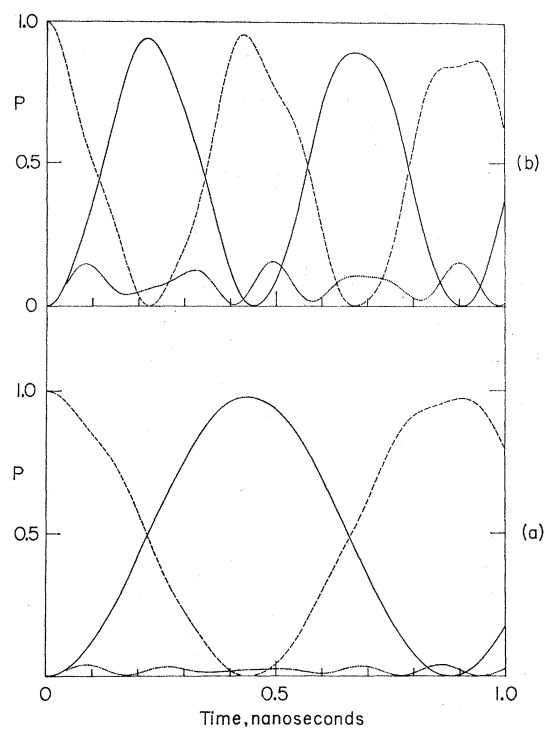

FIG. 3. Transition probability P as a function of the interaction time. The solid line is for transitions from $-k\hbar$ to $k\hbar$, the dashed line for transitions from $-k\hbar$ to $-k\hbar$, and the dotted line is the transition probability to all other states. The standing-wave intensity equals (a) 10^8 W/cm^2, (b) 2×10^8 W/cm^2.

[9] R. Gush and H. P. Gush (unpublished).

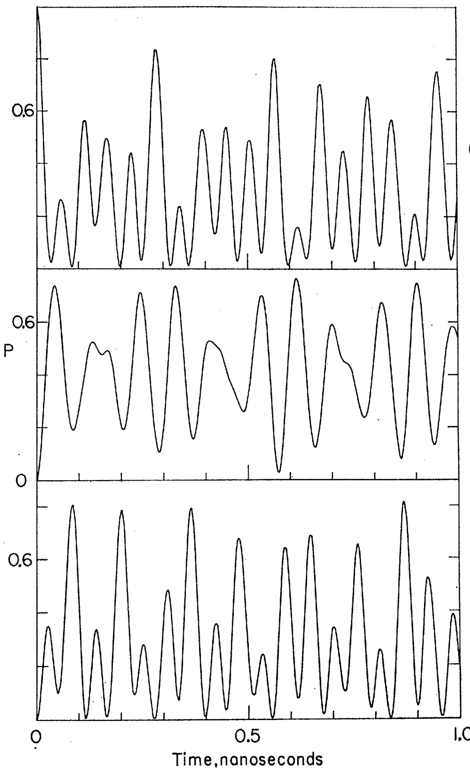

FIG. 4. Transition probability P as a function of the interaction time for a standing-wave intensity of 10^9 W/cm^2. (a) $-k\hbar$ to $k\hbar$; (b) transition probability to all other states; (c) $-k\hbar$ to $-k\hbar$.

ally sinusoidally, reaching nearly unity and decaying to zero. At high intensities the behavior is more complex and the probability does not reach such high values. The graph of Fig. 3(b) shows the probability for an intensity approximately equal to the maximum intensity employed by Bartell *et al.*[2] The time of flight of the electron through the laser beam in their experiment was equal to 0.47×10^{-9} sec. Referring to Fig. 3(b), one sees that it is quite possible that this experiment was conducted near a minimum in the transition probability. We reach a similar conclusion concerning the experiment of Pfeiffer[2]; this may explain why, in both of these experiments, difficulty was found in observing scattering. In any case it is evident that unless the experiment is carried out with good control over the laser intensity and over the interaction time, quite variable results could be expected, particularly at the higher intensities where the probability varies rapidly with time [Fig. 4(a)].

B. Transitions from $p_0=(-k+\delta)\hbar$ to $p=(k+\delta)\hbar$

So far we have concentrated on transitions from $p_0=-kh$ to $p=kh$. It is clearly possible to extend these calculations to include the case in which the electron momenta does not satisfy the Bragg condition. We have treated the case of transitions from $p_0=(-k+\delta)h$ to $p=(k+\delta)h$, with $\delta=\frac{1}{2}k$ and $\delta=k$. The continued fraction which is analogous to Eq. (2.46) is

$$S'(z)=\Bigg[\Bigg(z-\beta/2-\frac{1}{z-\alpha-\beta-}\,\frac{1}{z-3\alpha-\frac{3}{2}\beta-}\times\frac{1}{z-6\alpha-2\beta-}\cdots\Bigg)\Bigg(z-\frac{1}{z-\alpha+\beta/2-}\times\frac{1}{z-3\alpha+\beta-}\,\frac{1}{z-6\alpha+\frac{3}{2}\beta-}\cdots\Bigg)-1\Bigg]^{-1}, \quad (3.2)$$

where $\beta=-\alpha\delta/k$. One can show that when $\delta=0$, (3.2) reduces to (2.46). We have investigated the behavior of this continued fraction in the cases of $\alpha=-5$ and $\alpha=-10$. In Fig. 5 we show the results for $\alpha=-10$ only. Two features are obvious: (1) The transition probability goes down as one moves away from the exact Bragg condition, although not as rapidly as one would have expected on the basis of a first-order calculation, and (2) the period of oscillation of the transition probability changes substantially as δ is changed. In the case of $\alpha=-5$ (higher intensity), it was found that the off-Bragg-condition probability dropped even more slowly with increasing δ. It would appear that if the intensity is high enough, scattering takes place over a wide range of the incident electron momentum. This is probably the reason that Bartell *et al.*[2] did not find strong evidence that the Bragg condition was fulfilled.

C. Transitions from $p_0=-k\hbar$ to $p=-k\hbar$

The probability of no scattering has been calculated assuming the initial momentum to be equal to $-kh$. The continued fraction S has just the same form as (2.46), with the exception that the second fraction in the brackets is subtracted from the first rather than summed. The transition probability is plotted in Fig. 3 (dashed line), and Fig. 4(c), where it can be compared with the probability for transitions from $-kh$ to kh.

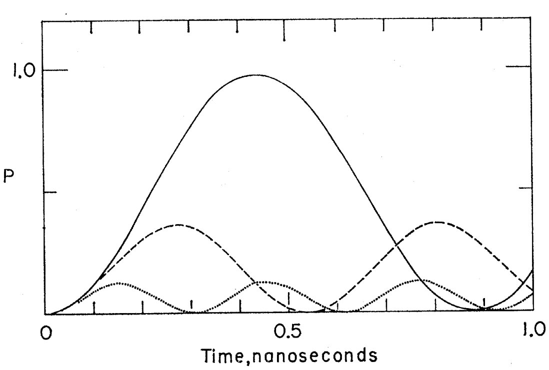

FIG. 5. Transition probability P as a function of the interaction time for transitions from $(-k+\delta)\hbar$ to $(k+\delta)\hbar$ for a standing-wave intensity of 10^8 W/cm^2. Solid line, $\delta=0$; dashed line, $\delta=\frac{1}{2}k$; dotted line, $\delta=k$.

Also plotted in these figures is the probability of transition to all other states except $k\hbar$ and $-k\hbar$, obtained by subtracting from unity the sum of the two probabilities explicitly calculated. At low intensities, transitions to other states are almost negligible but this is not the case at high intensities [Fig. 4(b)]. An electron beam passing through the laser field of intensity $I=10^9$ W/cm^2 would be scattered into a fan, as was predicted qualitatively by Fedorov.[3]

D. Transitions from $p_0=-2k\hbar$ to $p=2k\hbar$

In analogy with x-ray scattering, one would expect higher-order Bragg reflections to occur. These correspond in a quantized field description to the absorption of n photons from the primary beam and emission of n photons into the stimulating beam. We have examined the second-order Bragg reflection corresponding to $n=2$. The continued fraction may be shown to be

$$S''(z)=F^{-1}\left[\frac{1}{(z+\frac{1}{2}\alpha)F-2}\right], \qquad (3.3a)$$

with

$$F=z-\frac{1}{z-\frac{3}{2}\alpha-}\,\frac{1}{z-4\alpha-}\,\frac{1}{z-\frac{15}{2}\alpha-}\,\frac{1}{z-12\alpha-}\cdots. \qquad (3.3b)$$

The results for the second-order Bragg reflection are shown in Figs. 6 and 7 for two intensities, $\alpha=-10$ and $\alpha=-5$. The transition probability rises to a maximum

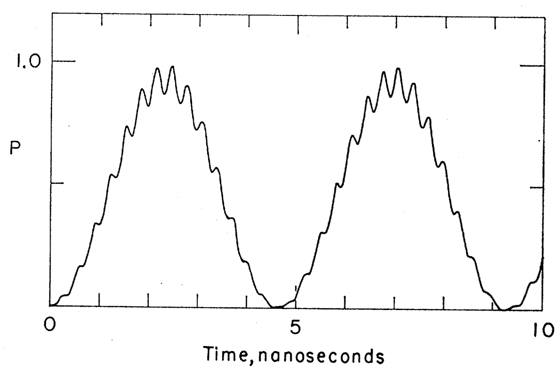

FIG. 6. Transition probability for a second-order Bragg reflection, $-2k\hbar$ to $2k\hbar$, for a standing-wave intensity of 10^8 W/cm^2.

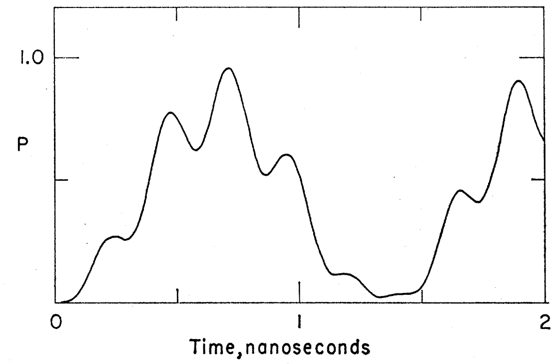

FIG. 7. Transition probability for second-order Bragg reflection, $-2k\hbar$ to $2k\hbar$, for a standing-wave intensity of 2×10^8 W/cm^2.

considerably later than for the case of first-order Bragg reflection, but it practically attains unity in the cases studied and oscillates with time. Presumably, higher-order reflections would behave similarly but these cases have not been worked out.

In conclusion we repeat the important result, with respect to experiments, that the transition probability in the Kapitza-Dirac effect oscillates rapidly with time of flight between zero and roughly unity and, as a consequence, certain experimental conditions are very unfavorable for a demonstration of the effect. At moderate laser intensities, the electron behaves like a two-state quantum-mechanical system driven in resonance, passing periodically from the initial state to the excited state, and back to the initial state. It has also been shown that, at high intensities, electrons are likely to be scattered into a large number of final states which would make quantitative analysis of a scattering experiment difficult. Lastly, it has been shown that the probability of high-order Bragg reflections can be nearly unity; this suggests that an experiment to detect such reflections should be no more difficult than the original experiment proposed by Kapitza and Dirac.

ACKNOWLEDGMENT

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