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Quantum Electrodynamics on Null Planes and Applications to Lasers*

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The conventional formulation of quantum electrodynamics in which the system develops from one spacelike hyperplane to the next is here replaced by one in which the development proceeds over null hyperplanes. For detailed study a quantized electromagnetic field A^μ is chosen to interact with a quantized spin-0 particle field Φ in an unquantized electromagnetic field $A_{\rm ext}{}^\mu$ as background. If the latter is chosen to be a laser field, the Φ-$A_{\rm ext}{}^\mu$ interaction permits exact closed-form solutions (Volkov) and allows the construction of wave packets which cannot be done in the usual formulation. The perturbation solution for the S matrix is therefore conveniently based on the Furry picture. The null-plane formulation has various advantages. In particular, the gauge problem which causes difficulties in the usual theory is absent in the null-plane gauge chosen here. Since there are only two dynamically independent components of A^μ, the commutation relations, field equations, gauge conditions, and vacuum definition are all mutually consistent. A natural null-plane gauge is used. Similarities and differences between this and the conventional theory are pointed out. As an application the Compton scattering of a charged particle with a laser beam is shown to lead to an intensity-dependent frequency shift. The controversy on this issue is settled here without divergent phase factors, because our wave-packet description permits a clean separation of the particle beam from the laser.

I. INTRODUCTION AND SUMMARY

In the present paper, quantum electrodynamics will be studied in terms of a special coordinate system in Minkowski space, characterized by two families of null planes, u=const and v=const. The physical system is then regarded as developing from one null plane to the next, i.e., from $u=u_0$ to $u=u_1$. The null coordinate u will therefore play a role similar to time, giving one of the two null-plane families a preferred status.

One may well inquire into the reasons for such an odd and physically seemingly meaningless description where spacelike planes are replaced by null planes, the time coordinate is replaced by a null coordinate, and initial conditions would have to be given on a null plane. Surprisingly enough there are three quite different motivations, each one being by itself sufficiently important to warrant such a treatment.

(1) One motivation for studying null-plane field theory arose during the period when one of us (R. A. N.) was engaged in describing the scattering of electrons by a laser beam.[1] Such a beam, when described as a finite-length wave train with plane-wave front, appears in Minkowski space as coherent electromagnetic radiation bounded by two null planes. This suggests null coordinates as the natural coordinate system. It then emerges that the difficulties previously encountered in this problem disappear, as we shall explain below. This work remained unpublished until now.

(2) Meanwhile papers began to appear[2–4] which indicated that the infinite-momentum limit needed in current-algebra techniques is very conveniently obtained by the use of null coordinates in Feynman diagrams and other expressions. (The naive infinite-momentum limit obtained by a Lorentz transformation with infinite velocity—thereby formally changing a spacelike plane into a null plane—does not exist.) The null coordinates lead naturally to null planes which are therefore sometimes referred to as the "infinite-momentum frame." Current-algebra calculations thus become not only more rigorous but also much simpler.[5]

(3) From the point of view of general quantum field theory and its S matrix, there is interest in a null-plane formulation simply as an alternative to the more customary theory, with the implicit hope that some of the usual difficulties might be ameliorated. As we shall see, this hope is richly rewarded in quantum electrodynamics.

With these three motivations in mind, we want to concentrate primarily on (1) and (3) in view of the various papers already published in connection with (2). To be sure, null-surface coordinates are not new to field theory since general relativists have discovered their usefulness some time ago, especially the use of null-cone coordinates in the context of gravitational radiation.[6] But there exist some essential differences between null cones and null planes which make the latter much more interesting for our purpose.

Returning to item (1), we note that the discovery of the laser has aroused considerable interest in the possibility of intensity-dependent effects in quantum electrodynamics. Sengupta[7] first predicted that in the Compton scattering of an electron by a strong external radiation field the emerging photon would have a frequency shift which depends on the intensity of the

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external field. Various authors have corroborated this prediction,[8–11] while others have contested it.[12–14]

In a very illuminating paper Kibble[15] resolved this discrepancy. He showed that it is essential to have a free electron before and after its interaction with the laser beam. This is not the case in the latter papers.[12–14] In contradistinction, the intensity-dependent frequency shift in Compton scattering is obtained in calculations such as Ref. 8 where the laser beam is represented by a wave train of *finite* length.

Unfortunately, these calculations are not entirely satisfactory and do not resolve the discrepancy beyond objections. The reason is as follows. Let the laser beam be a finite wave train moving in the positive z direction. Then the incident and the outgoing electron wave must have an upper and a lower bound in z (for fixed x and y), in order that the electron will be outside the laser beam before a finite time t_0 and after a finite time t_1. Thus, the electrons must be described by a wave packet which is compact in z. This is easy to do for the region outside the laser beam, but then requires a difficult matching to the solution inside the laser beam, unless one can construct a wave-packet solution in the z direction which is valid both *inside and outside* the laser beam. Neither the matching nor the complete wave-packet solution has ever been given. In fact, in the usual coordinate system a closed-form wave-packet solution *inside* the laser beam does not seem possible.

The null-plane coordinates resolve this difficulty. Compactness in z is equivalent to compactness in $t+z$ for fixed $t-z$, and this can easily be done also for the solutions *inside* the laser beam as will be shown. The wave-packet description can thus be carried consistently from before to after the collision.

While this resolves completely the separation of the electron from the laser beam by insuring a finite interaction time when looked at in the t-z plane, this is not so in the t-x or t-y plane when the laser beam and the electron have infinite wave fronts, unless these wave fronts are parallel, i.e., the electron moves in the (negative) z direction only (head-on collision). In all other cases the wave fronts of the laser beam and the electron would intersect at all finite times if one only goes far enough out in the x and/or y direction. Thus, compactness in x and y is in general also necessary if one wants a finite interaction time.

The complete wave-packet description in x, y, and $t+z$, both inside and outside the laser beam, will be given in Sec. III. In this way the important stipulation by Kibble is clearly and fully realized. The description in which the electron is separated from the laser beam only asymptotically[8] can be obtained from this wave-packet formulation in a suitable limit.

In this way a mathematically clean calculation can be carried out. Otherwise divergent phase factors arise.[8] The difference between our work and previous work can be summarized mathematically by saying that our wave functions are L_2 functions while previous work used plane waves which are not in a Hilbert space.

The results of our calculations confirm the intensity-dependent frequency shift as computed by Brown and Kibble.[8] This will be shown in Sec. VIII.

This useful application indicates the importance of the general formulation of null-plane field theory. The general case involves quantized charged particles and photons in interaction with one another and with an external field. A reasonable idealization of the laser field permits an exact closed-form solution of the charged-particle field in the presence of the laser field.[16] Section II is devoted to this solution for the plane-wave case and Sec. III for the wave-packet case. The associated Green functions are given in Sec. IV.

The general formulation of null-plane quantum electrodynamics is begun in Sec. V. The preliminary work for it was done in an earlier publication.[17] There, the Cauchy initial-value problem on null planes was studied both classically and in the quantum case for spin-0 and spin-$\frac{1}{2}$ fields. Here, this approach is applied to the interacting system of a spin-0 field Φ, the electromagnetic field $F_{\mu\nu}$, and the external field $F_{\mu\nu}{}^{e}$. The infinitesimal generators of translation along the spatial and null directions are constructed formally in the Heisenberg picture and appropriate commutation relations for the fields are postulated which were previously derived from the noninteracting case. These formal generators in the Heisenberg picture and the associated field equations receive a well-defined meaning by a transformation from the Furry picture (Sec. VI).

That picture is characterized by the field equations in which the quantized electromagnetic field is not coupled to the rest of the system and for which we have found exact solutions in Secs. II–IV. In this picture, normal ordering is meaningful and the generators are well defined. The transformation to the Heisenberg picture can then be carried out in a perturbation expansion analogous to standard methods. It is characterized by the null translation operator of the interaction in the Furry picture $P_{\mu F}{}^{(1)}$ which plays the role of the usual interaction Hamiltonian in that picture.

The formal solution for this transformation yields the Dyson form of the S matrix. This involves an operator ordering by the null coordinate u (rather than the time coordinate) but is in many ways analogous to the usual case.

The apparent disadvantage of null-plane coordinates seems to be the unphysical initial condition on a null plane which corresponds to the specification of the

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plane-wave front of an electromagnetic wave for all times. This disadvantage disappears when one takes the initial null plane $u=u_0$ to $u_0\to-\infty$, as is the case in the S-matrix description.

The advantages of the null-plane formulation are enormous. The most important of these is perhaps the gauge problem. This problem takes on major proportions in the usual formulation of quantum electrodynamics in terms of spacelike planes, leading to such unfortunate artifices as indefinite-metric Hilbert spaces, complicated limiting procedures, or other devices. At the same time, there is little or no physics in these artifices.

The null-plane formulation eliminates the gauge problem in a most elegant and unexpected way. It leads to a natural choice of gauge which we call the *null-plane gauge* (Sec. VII). In this gauge one of the four inhomogeneous Maxwell equations does not occur explicitly but is indirectly ensured to be satisfied. Another one of these four equations contains no time derivative and therefore plays the role of a constraint. This leaves only two dynamically independent equations and, correspondingly, only two dynamically independent components of the potential A^μ. As an irreducible representation of spin 1 and mass 0 of the Poincaré group, the free photon is known to have only two independent components. It is therefore most satisfactory to have the same situation also in the interacting case if one works in the null-plane formulation.

If one chooses the two transverse components A_1 and A_2 as the dynamically independent components, then the null-plane gauge $A_v=0$ states physically that the Coulomb potential and the longitudinal one are equal, and that the constraint equation expresses the Coulomb potential in terms of A_1, A_2, and the "static" source (i.e., the current density component that plays the role of the charge density on a null plane). In this way it becomes intuitively understandable why the null-plane gauge leads to no contradictions between the commutation relations, the gauge condition, the field equations, and the definition of the vacuum.

Near the conclusion of our work we became aware of a recent report by Kogut and Soper[18] which deals with a closely related subject and is in many respects similar to our work. However, their work differs from ours in several essential ways. First, we deal with spin-0 charged particles while they are treating spin-$\frac{1}{2}$ particles. Second, they use the "infinite-momentum gauge," which is equivalent to our null-plane gauge, but they do not pay attention to the fourth Maxwell equation, which does not occur explicitly, and they do not show the consistency of that gauge with the definition of the vacuum. We also differ in the treatment of the Green functions and in the external field which is not considered in Ref. 18, where no application to laser physics is made. Last but not least, we paid attention to the need for a careful treatment of various operators (in particular $P_{uF}^{(1)}$) without which the derivation of the field equations from the commutation relations would not be unambiguous.

Our work also shows that quantum field theory in the null-plane formulation is not just Feynman-Dyson theory with a particular choice of coordinates, as seems to be implied in Ref. 4. It suffices to point to the very different treatment of the gauge problem in the two instances to establish this fact. But if one satisfies the constraint equation (6.9) in each order of the perturbation expansion, the Feynman-Dyson techniques can be taken over with little change.

Finally, to avoid any misunderstanding, we note that our null-plane formulation of quantum electrodynamics will not lead to any predictions different from those obtained in the conventional spacelike-surface formulation. But we feel that the formulation here proposed does have distinctive advantages, as explained above.

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II. PLANE-WAVE SOLUTIONS

Exact solutions $\phi(x,p)$ to the equation (we use the Minkowski metric with signature $+2$)

$$\{[\partial-ieA^e(x)]^2-m^2\}\phi(x,p)=0 \tag{2.1}$$

can be found in closed form provided the c-number field $A_\mu{}^e$ satisfies certain conditions. One particular set of conditions on $A_\mu{}^e$ which allows a solution is the following:

(i) It describes a free radiation field, i.e., $\partial\cdot\partial A_\mu{}^e=0$;
(ii) it satisfies (for convenience) the Lorentz gauge condition $\partial\cdot A^e=0$;
(iii) it propagates in a single direction given by the null vector $n^\mu=(1,0,0,1)/\sqrt{2}$;
(iv) it depends on only the single null parameter $u=-n\cdot x=(t-x_3)/\sqrt{2}$.

These conditions are sufficient; in addition we shall assume that $A_\mu{}^e$ is real and transverse, and nonzero only for a finite range of u, i.e.,

$$A_\mu{}^e(u)=A_\mu{}^{e*}(u)\,,\quad A_\mu{}^e(u)=(0,A_1{}^e(u),A_2{}^e(u),0)\,,$$

and

$$A_\mu{}^e(u)=0\quad\text{for}\quad |u|>|u_0|\,.$$

Physically, this means that the external field is a finite-length, plane-wave-fronted pulse; this work is thereby differentiated from certain preceding treatments[12–14] in which the external field is an infinitely long plane-wave train.

The external field so defined will be used as the prototype of a laser beam, and we shall refer to it as the laser field. That a laser field can be described classically in excellent approximation has been known for some time.[15,19,20]

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As is evident from Fig. 1, the geometry of the field suggests the introduction of another null parameter $v^\mu \equiv -m\cdot x=(t+x_3)/\sqrt{2}$, where m^μ is the null vector $m^\mu \equiv (1,0,0,-1)/\sqrt{2}$. With these definitions of n^μ and m^μ and the notation $m\cdot A\equiv -A_u$, $n\cdot A\equiv -A_v$, $\mathbf{A}=(A_1,A_2)$ of Ref. 17, the product of two four-vectors becomes

$$A\cdot B=\mathbf{A}\cdot\mathbf{B}-A_uB_v-A_vB_u. \qquad (2.2)$$

In particular,

$$x\cdot p=\mathbf{x}\cdot\mathbf{p}-vp_v-up_u, \qquad (2.3)$$

and hence, the momentum components p_u and p_v are the variables canonically conjugate to u and v, respectively.

We also note that the indices u and v denote invariants and not components so that there is no distinction between their use as subscripts or superscripts. Of particular importance shall be the quantities $\partial_u=-m\cdot\partial=-\partial/\partial u$ and $\partial_v=-n\cdot\partial=-\partial/\partial v$.

In this notation, Eq. (2.1) becomes

$$K_x{}^A\phi(x,p)\equiv[\boldsymbol{\partial}^2-2ie\mathbf{A}_e\cdot\boldsymbol{\partial}-e^2\mathbf{A}_e{}^2-m^2-2\partial_u\partial_v]\phi(x,p)=0, \qquad (2.4)$$

where $\boldsymbol{\partial}=(\partial/\partial x_1,\partial/\partial x_2)$ and p is a label to be specified below.

Examination of Fig. 1 yields an important property of the solutions to this equation. Since $A_\mu{}^e$ depends only on u, the system possesses translational invariance in the three coordinates x_1, x_2, and v. This implies conservation of the conjugate momentum components p_1, p_2, and p_v, respectively. On the other hand, the system is not invariant under translations in t, x_3, and u, and consequently the momentum components p^0, p_3, and p_u are not conserved. This observation, although simple, is crucial to the development of the following formalism.

The plane-wave solutions to (2.4), with the initial condition

$$\phi(x,p)\,|_{u<-u_0}=e^{ip\cdot x}\,|_{p^2+m^2=0}, \qquad (2.5)$$

are essentially the spin-0 counterparts of the Volkov solutions[16]

$$\phi(x,p)=\exp i\left(\mathbf{p}\cdot\mathbf{x}-p_vv-\int_{ui}^{u}\frac{\{[\mathbf{p}-e\mathbf{A}^e(\eta)]^2+m^2\}d\eta}{2p_v}-p_u{}^fu_i\Bigg|\right)_{ui<-u0} \qquad (2.6)$$

with $p_u{}^f=(\mathbf{p}^2+m^2)/(2p_v)$. Note that p_v can never vanish for finite p. The solution is independent of the parameter u_i.

In the same way as the free plane-wave solutions $f(x,p)$ are considered to be on the mass shell (m.s.),

$$f_\epsilon(x,p)=e^{ix\cdot p}\,|_{\text{m.s. }\epsilon}, \qquad (2.7)$$

where

$$\text{m.s. }\epsilon\text{ means }p^2+m^2=0,\quad p^0=\epsilon|p^0|, \qquad (2.8)$$

solutions (2.6) can be thought of as being on the positive- or negative-energy part of a new "interacting

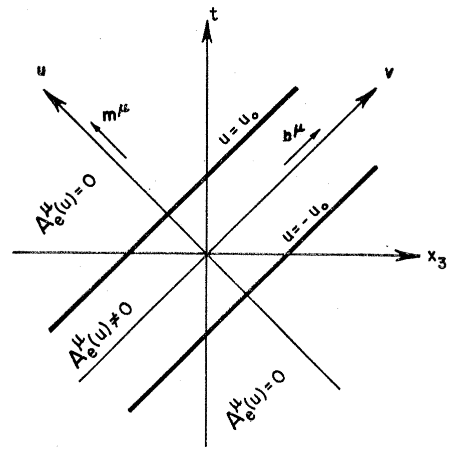

FIG. 1. Null-plane geometry and the laser field.

mass shell" (i.m.s.)

$$\phi(x,p)=\exp[i\mathbf{p}\cdot\mathbf{x}-ip_vv-ip_u(u-u_i)-ip_u{}^fu_i]\,|_{\text{i.m.s.}}, \qquad (2.9)$$

where[21]

$$\text{i.m.s. means }(u-u_i)^{-1}\times\int_{ui}^{u}\{[p-eA^e(\eta)]^2+m^2\}d\eta=0, \qquad (2.10)$$

and where the correct initial condition (2.5) has been incorporated into (2.9). Although it is conventional to implement the m.s. condition (2.8) by imposing on the p^0 component the condition $p^0=\pm(\mathbf{p}^2+p_3{}^2+m^2)^{1/2}$, it is possible to specify instead any one of a number of components, for example, $p_u=(\mathbf{p}^2+m^2)/(2p_v)$. This freedom is not available in the interacting case. From the previous discussion of translation invariance, it is seen that, of the four components p_u, p_1, p_2, and p_v, only p_u should depend on u. Therefore (2.10) is solved for p_u:

$$p_u=(u-u_i)^{-1}\int_{ui}^{u}\frac{\{[\mathbf{p}-e\mathbf{A}^e(\eta)]^2+m^2\}d\eta}{2p_v}. \qquad (2.11)$$

This value for p_u inserted into (2.9) yields the Volkov solution (2.6).

Since it will be necessary later to separate the solutions into positive- and negative-energy solutions, it is of interest to consider whether this can be accomplished by specifying the value of p_u. From (2.11) with the definitions of n^μ and m^μ, one finds that p^0 is given by

$$p^0=\frac{p_v(1+p_u/p_v)}{\sqrt{2}}. \qquad (2.12)$$

[21] Equation (2.10) for all u is of course equivalent to $(p-eA^e)^2+m^2=0$, but if one wishes to solve (2.4) by a plane wave with a suitable mass-shell restriction, then the average (2.10) must be used as the argument of the δ function. Compare Eqs. (3.1) and (3.2) below.

Since the term p_u/p_v is positive definite for real $A_e{}^\mu$, p^0 and p_v always have the same sign. It then follows from (2.11) that p^0 and p_u likewise have the same sign for all u. Moreover, the conservation of p_v implies that a positive- (negative-) energy solution remains a positive- (negative-) energy solution throughout its development with respect to u. A separation of solutions (2.6) into two sets, according to the sign of the energy, is possible and is made explicit by writing

$$\phi_\epsilon(x,p)=[2(2\pi)^3]^{-1/2} \times\exp\left\{i\epsilon\left[\bar{p}\cdot\bar{x}-\int_{ui}^{u} q_u{}^{(\epsilon)}(\eta)d\eta-|p_u{}^f|u_i\right]\right\}. \quad (2.13)$$

Here we have added a suitable normalization. We use the notation

$$\bar{p}\equiv(p_1,p_2,|p_v|),\quad \bar{x}\equiv(x_1,x_2,v),$$
$$\bar{p}\cdot\bar{x}=\mathbf{p}\cdot\mathbf{x}-|p_v|v, \quad (2.14)$$

$$q_u{}^{(\epsilon)}(\eta)\equiv\{[\mathbf{p}-\epsilon e\mathbf{A}^e(\eta)]^2+m^2\}/(2|p_v|), \quad (2.15)$$

$$\epsilon=\begin{cases}+1 & \text{for positive-energy solutions}\\ -1 & \text{for negative-energy solutions.}\end{cases} \quad (2.16)$$

III. WAVE-PACKET SOLUTIONS

In the construction of a complete orthonormal set of wave packets of Volkov solutions, we follow a method analogous to that employed in the free case. Free wave-packet solutions are formed by an integration over momentum four-space of the plane wave $e^{ip\cdot x}$ (p^μ as yet unspecified), with a momentum-space factor $g(p)$ restricted to the mass shell

$$f_\alpha(x)=(2\pi)^{-3/2}\int g_\alpha(p)\delta(p^2+m^2)e^{ip\cdot x}d^4p. \quad (3.1)$$

Volkov wave packets are formed similarly by replacing the m.s. condition by i.m.s. condition (2.10):

$$\phi_\alpha(x)=\sqrt{2}(2\pi)^{-3/2}\int d^4p\, h_\alpha(p_1p_2,p_v) \times\delta\left((u-u_i)^{-1}\int_{ui}^{u}\{[p-eA^e(\eta)]^2+m^2\}d\eta\right) \times\exp[i\mathbf{p}\cdot\mathbf{x}-ip_vv-ip_u(u-u_i)-ip_u{}^fu_i]. \quad (3.2)$$

The shape factor $h_\alpha(p_1,p_2,p_v)$ can depend only on p_1, p_2, and p_v because the system is translation invariant only in the directions x_1, x_2, and v, conjugate to these variables, and not in u which is conjugate to p_u. Since we do not have translation invariance in x_3 and x^0, the choice of the usual variables would permit construction of a wave packet only in p_1 and p_2 which is insufficient to produce a finite intersection time. The use of null coordinates eliminates this difficulty. At the same time, (3.2) is chosen to preserve the initial condition (2.5). That the i.m.s. condition must be incorporated by imposing the condition (2.11) on p_u (rather than on p^0 as is conventional in the free case) has already been seen. This requires that we rewrite the i.m.s. δ function as

$$\delta\left((u-u_i)^{-1}\int_{ui}^{u}[(p-eA^e)^2+m^2]d\eta\right) =(2|p_v|)^{-1}\sum_{\epsilon=\pm1}\theta(\epsilon p_v) \times\delta\left(p_u-\frac{\epsilon}{u-u_i}\int_{ui}^{u}\frac{[(p-eA^e)^2+m^2]d\eta}{2|p_v|}\right). \quad (3.3)$$

Here θ is the step function ($\theta=1$ and 0 for positive and negative arguments). Integrating (3.2) over p_u and separating into positive- and negative-energy solutions, one obtains

$$\phi_{\alpha\epsilon}(x)=\int\frac{h_{\alpha\epsilon}(\bar{p})\phi_\epsilon(x,p)d^3\bar{p}}{2|p_v|}, \quad (3.4)$$

where

$$d^3\bar{p}=dp_1dp_2dp_v,\quad h_{\alpha\epsilon}(\bar{p})\equiv h_\alpha(\epsilon p_1,\epsilon p_2,\epsilon|p_v|)=h_\alpha(\epsilon\bar{p}).$$

The momentum space functions $h_{\alpha\epsilon}(\bar{p})$, being functions of the three conserved momentum components $(p_1,p_2,|p_v|)$, are now chosen to form an orthonormal basis in an L_2 space with invariant measure $d^3\bar{p}/2|p_v|$

$$(h_{\alpha\epsilon}(\bar{p}),h_{\beta\epsilon}(\bar{p})) \equiv\int\frac{h_{\alpha\epsilon}{}^*(\bar{p})h_{\beta\epsilon}(\bar{p})d^3\bar{p}}{2|p_v|}=\delta_{\alpha\beta},\quad \epsilon=\pm1. \quad (3.5)$$

They satisfy the completeness relation

$$\sum_\alpha h_{\alpha\epsilon}(\bar{p})h_{\alpha\epsilon}{}^*(\bar{p}')=|p_v|\delta_3(\bar{p}-\bar{p}'),\quad \epsilon=\pm1 \quad (3.6)$$

with $\delta_3(\bar{p}-\bar{p}')\equiv\delta_2(\mathbf{p}-\mathbf{p}')\delta(|p_v|-|p_v'|)$. We note that this method results in Volkov wave packets which fall off fast enough as x_1, x_2, and v approach infinity, whereas the conventional construction for free solutions gives packets that do so as x_1, x_2, and x_3 approach infinity.

As a consequence of (3.5) and (3.6), the Volkov wave-packet solutions $\phi_{\alpha\epsilon}$ are orthonormal with respect to the inner product on the null plane $u=$const,

$$\langle\phi_{\alpha\epsilon}(x),\phi_{\beta\epsilon'}(x)\rangle=-i\int d^3\bar{x}\,\phi_{\alpha\epsilon}{}^*(x)\overleftrightarrow{\partial}_v\phi_{\beta\epsilon'}(x) =\epsilon\delta_{\epsilon\epsilon'}\delta_{\alpha\beta}, \quad (3.7)$$

where $d^3\bar{x}=dx_1dx_2dv$, $\overleftrightarrow{\partial}_v=\overrightarrow{\partial}_v-\overleftarrow{\partial}_v$. They satisfy the completeness relation

$$\sum_\alpha\phi_{\alpha\epsilon}(x)\phi_{\alpha\epsilon}{}^*(x')=-i\epsilon\Delta_\epsilon(x,x';A^e). \quad (3.8a)$$

The distribution $\Delta_\epsilon(x,x';A^e)$ projects on the space of solutions $\{\phi_{\alpha\epsilon}\}$. It follows that

$$\Delta(x,x';A^e)=\Delta_+(x,x';A^e)+\Delta_-(x,x';A^e) \quad (3.8b)$$

is the identity "operator" on the complete space of positive- and negative-energy solutions, i.e., by (3.7) and (3.8)

$$\Delta(x,x';A^e)\otimes\phi_{\beta\epsilon'}(x') \equiv -\int\Delta(x,x';A^e)\overleftrightarrow{\partial}_{v'}\phi_{\beta\epsilon'}(x')d^3\bar{x}'=\phi_{\beta\epsilon'}(x). \quad (3.9)$$

The integrand in (3.9) is the v component of the divergence-free four-vector

$$\Delta(x,x';A^e)\overleftrightarrow{\partial}_\mu'\phi_{\beta\epsilon'}(x') -2ieA_\mu^e(u')\Delta(x,x';A^e)\phi_{\beta\epsilon'}(x'). \quad (3.10)$$

The resultant integral in (3.9) is formally identical to the corresponding expression for the free case [Eq. (3.7) of Ref. 17]. Because the wave packets $\phi_{\alpha\epsilon}$ satisfy the asymptotic limit $\lim\phi_{\alpha\epsilon}(x)=0$, $|v|\to\infty$, we have by Theorem 2 of Ref. 17 that the specification of $\phi_{\alpha\epsilon}$ on the $u=u_0$ surface yields, via (3.9), a unique solution to the differential equation (2.4).

In the plane-wave limit, i.e., when $\phi_{\alpha\epsilon}$ becomes a Volkov plane wave,

$$h_{\alpha\epsilon}(\bar{p})\to|p_v|\delta_3(\bar{p}-\bar{p}_\alpha), \quad \phi_{\alpha\epsilon}(x)\to\phi_\epsilon(x,p_\alpha), \quad (3.11)$$

the inner product (3.7) of two solutions becomes

$$\langle\phi_{\alpha\epsilon}(x),\phi_{\beta\epsilon'}(x)\rangle\to -i\int d^3\bar{x}\,\phi_\epsilon^*(x,p_\alpha)\overleftrightarrow{\partial}_v\phi_{\epsilon'}(x,p_\beta) =\epsilon\delta_{\epsilon\epsilon'}|p_{v\alpha}|\delta_3(\bar{p}_\alpha-\bar{p}_\beta), \quad (3.12)$$

which is a time-independent expression of the orthogonality of the Volkov solutions with respect to the conserved momentum components p_1, p_2, and p_v.

IV. GREEN FUNCTIONS

The Green functions are classified as homogeneous or inhomogeneous depending on which of the Eqs. (4.1) or (4.2) they satisfy:

$$K_x{}^A\Delta_H(x,x';A^e)=0, \quad (4.1a)$$

$$K_{x'}{}^{A*}\Delta_H(x,x';A^e)=0, \quad (4.1b)$$

$$K_x{}^A\Delta_I(x,x';A^e)=-\delta_4(x-x'), \quad (4.2a)$$

$$K_{x'}{}^{A*}\Delta_I(x,x';A^e)=-\delta_4(x-x'). \quad (4.2b)$$

The Klein-Gordon operator with coupling to the laser field is defined by (2.4), and $K_{x'}{}^{A*}$ is the complex conjugate of $K_{x'}{}^A$. There are Green functions of this type corresponding to all the usual free Green functions, specified in this case by their initial values on the $u=u'$ null hyperplane.

By virtue of their definitions in (3.8), $\Delta_\epsilon(x,x';A^e)$ are homogeneous Green functions. The completeness relations (3.6) and (3.8) coupled with the explicit expressions of $\phi_{\alpha\epsilon}(x)$ and $\phi_\epsilon(x,p)$ yield representations of $\Delta_\epsilon(x,x';A^e)$ that can be conveniently written, by means of $q_u{}^{(e)}$ defined in (2.15), as follows:

$$\Delta_\epsilon(x,x';A^e)=i\epsilon(2\pi)^{-3}\int\theta(\epsilon p_v) \times\exp\left\{i\epsilon\left[\bar{p}\cdot(\bar{x}-\bar{x}')-\int_{u'}^{u}q_u{}^{(e)}(\eta)d\eta\right]\right\}\frac{d^3\bar{p}}{2|p_v|}. \quad (4.3)$$

Considered as functions of $A_\mu{}^e$, the $\Delta_\epsilon(x,x';A^e)$ satisfy the boundary conditions

$$\Delta_\epsilon(x,x';A^e)|_{A^e\to 0}=\Delta_\epsilon(x-x'). \quad (4.4)$$

This is a direct result of the initial condition (2.5) imposed on the plane-wave solutions.

In analogy to the free case, one can represent $\Delta_\epsilon(x,x';A^e)$ both as integrals over the i.m.s. δ function and as contour integrals in the complex p_u plane:

$$\Delta_\epsilon(x,x';A^e)=i\epsilon(2\pi)^{-3}\int d^4p\,\theta(\epsilon p_v)e^{ip\cdot(x-x')} \times\delta\left((u-u')^{-1}\int_{u'}^{u}\{[p-eA^e(\eta)]^2+m^2\}d\eta\right) \quad (4.5)$$

and

$$\Delta_\epsilon(x,x';A^e)=(2\pi)^{-4}\int d^3\bar{p}\int_{C_\epsilon}dp_u\,e^{ip\cdot(x-x')} \times\left((u-u')^{-1}\int_{u'}^{u}\{[p-eA^e(\eta)]^2+m^2\}d\eta\right)^{-1}. \quad (4.6)$$

The contours C_ϵ are formally identical to those of the free Green functions[22] with p^0 replaced by p_u. Indeed, any of the Green functions $\Delta_\Gamma(x,x';A^e)$, homogeneous or inhomogeneous, can be obtained simply by replacing the contour C_ϵ in the integral (4.6) by the contour C_Γ in complex p_u space, C_Γ being formally the same as the contour in p^0 space that gives the free Green function[22] $\Delta_\Gamma(x-x')$. As a result of the identically related contours, these Green functions $\Delta_\Gamma(x,x';A^e)$ satisfy the same linear relationships among themselves as do the free Green functions, the only modification being the replacement of x^0 by u. They do not, however, retain all the symmetry and reality properties of the free Green functions. They do satisfy

$$\Delta_\epsilon^*(x,x';A^e)=-\Delta_\epsilon(x',x;A^e) \quad (\epsilon=\pm), \quad (4.7)$$

$$\Delta^*(x,x';A^e)=-\Delta(x',x;A^e), \quad (4.8)$$

$$\Delta_R^*(x,x';A^e)=\Delta_A(x',x;A^e), \quad (4.9)$$

[22] J. M. Jauch and R. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1959), 2nd printing, Appendix 1.

$$\Delta_P^*(x,x';A^e)=\Delta_P(x',x;A^e)\,, \tag{4.10}$$

$$\Delta_{1R}^*(x,x';A^e)=\Delta_{1A}(x',x;A^e)\,. \tag{4.11}$$

We want to point out[1] that a solution $\Delta_\Gamma(x,x';A^e)$ to (4.1) or (4.2) combined with the condition $\Delta_\Gamma(x,x';A^e)=\Delta_\Gamma(x-x')$ for $A^e\to 0$ results in a unique $\Delta_\Gamma(x,x';A^e)$ coinciding with that given by the contour integral over C_Γ.

In practice we need the $u=u'$ values of $\Delta(x,x';A^e)$ and $\partial_v\Delta(x,x';A^e)$. As is clear from expressions (4.3), these are the same as for the free case[17]

$$\Delta(x,x';A^e)\,|_{u=u'}=\tfrac{1}{4}\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\,, \tag{4.12a}$$

$$\partial_v\Delta(x,x';A^e)\,|_{u=u'}=-\tfrac{1}{2}\delta_3(\tilde{x}-\tilde{x}')\,, \tag{4.12b}$$

where $\varepsilon(v)=1$ (-1) for $v>0(<0)$.

Since the derivative ∂_v normal to the null plane $u=u'$ is within that plane, (4.12) constitutes one initial condition, not two. This is another manifestation of the fact that only $\phi_{\alpha\epsilon}$ need be specified on the initial-value null plane for uniqueness.[17]

The initial values given in (4.12) enable one to demonstrate that the inhomogeneous Green functions, expressed formally in terms of $\Delta_+(x,x';A^e)$ and $\Delta_-(x,x';A^e)$ by the free-case relations with x^0 replaced by u, do indeed satisfy (4.2). For example, let us verify this for $\Delta_{1R}=\theta(u-u')\Delta_+-\theta(u'-u)\Delta_-$,

$$\begin{aligned}K_x{}^A\Delta_{1R}(x,x';A^e)&=-2\partial_u\theta(u-u')\partial_v\Delta_+(x,x';A^e)\\&\quad+2\partial_u\theta(u'-u)\partial_v\Delta_-(x,x';A^e)\\&=2\delta(u-u')\partial_v\Delta(x,x';A^e)\,.\end{aligned}$$

Combined with (4.12b), one finds the desired result

$$K_x{}^A\Delta_{1R}(x,x';A^e)=-\delta_4(x-x')\,. \tag{4.13}$$

Similarly, $\Delta_{1R}(x,x';A^e)$ satisfies (4.2b).

V. HEISENBERG PICTURE

In Secs. II–IV we restricted the discussions to charged particles in a laser field without the (quantized) radiation field of these charges. The generalization necessary to include the radiation field requires suitable field equations and commutation relations. In order to derive the former, we start with the usual heuristic classical situation.

The gauge-invariant Lagrangian density is

$$\mathfrak{L}=-(D_\mu\Phi)^*D^\mu\Phi-m^2\Phi^*\Phi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{5.1}$$

with

$$D_\mu\equiv\partial_\mu-ie(A_\mu+A_\mu{}^e)\,, \tag{5.2}$$

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{5.3}$$

In the standard way based on Noether's theorem, one derives the field equations, and from translation invariance the canonical energy tensor. The field equations written in null coordinates are

$$(D_uD_v+D_vD_u)\Phi=(\mathbf{D}^2-m^2)\Phi \tag{5.4}$$

and

$$\partial^\nu F_{\nu\mu}=\partial_jF_{j\mu}-\partial_uF_{vu}-\partial_vF_{u\mu}=-J_\mu\,, \tag{5.5}$$

$$\begin{aligned}-J_\mu\equiv{}&ie(\Phi^*\overleftrightarrow{\partial}_\mu\Phi)+ie(\Phi^*\overleftrightarrow{\partial}_v\Phi)m_\mu\\&+ie(\Phi^*\overleftrightarrow{\partial}_u\Phi)n_\mu+2e^2(\mathbf{A}_\mu+\mathbf{A}_\mu{}^e)\Phi^*\Phi\,.\end{aligned}$$

The index j (or k) takes on the values 1 and 2 only, and the summation convention is assumed also for it. The homogeneous Maxwell equations are identically satisfied because of (5.3).

Now these field equations are hyperbolic differential equations of the form

$$2\partial_u\partial_v f=Lf\,. \tag{5.6}$$

In a previous publication[17] we discussed in some detail the considerable simplifications which arise when the linear differential operator L is independent of ∂_u. Now (5.4) can be rewritten as ($A^{\rm tot}\equiv A+A^e$),

$$\begin{aligned}2\partial_u\partial_v\Phi={}&[(\boldsymbol{\partial}^2-m^2)+2ie(A_u{}^{\rm tot}\partial_v+A_v{}^{\rm tot}\partial_u-\mathbf{A}^{\rm tot}\cdot\boldsymbol{\partial})\\&+ie(\partial_uA_v{}^{\rm tot}+\partial_vA_u{}^{\rm tot}-\boldsymbol{\partial}\cdot\mathbf{A}^{\rm tot})\\&+e^2(2A_u{}^{\rm tot}A_v{}^{\rm tot}-\mathbf{A}^{\rm tot}\cdot\mathbf{A}^{\rm tot})\Phi\,.\end{aligned}$$

This shows that L for this equation can be made independent of ∂_u if one chooses $A_v{}^{\rm tot}=0$ or, since $A_v{}^e=0$ holds anyway for our external field,

$$A_v=0\,. \tag{5.7}$$

This choice is always possible by working in a suitable gauge: Given A_μ' in an arbitrary gauge, the transformation $A_\mu'\to A_\mu=A_\mu'+\partial_\mu\Lambda$ can be so chosen that $A_v=-n\cdot A=0$. In view of the importance of this choice of gauge in connection with null-plane coordinates—and for lack of a better name—we shall call the gauge characterized by (5.7) the *null-plane gauge.* In Sec. VII we shall see that it has most interesting properties in the quantized case. This gauge was also used by Kogut and Soper.[18]

When the null-plane gauge is used *in the Lagrangian* (5.1), the resulting field equations are (5.4) and (5.5) with various terms missing. Noting the properties of our external field $A_\mu{}^e$ we find

$$\begin{aligned}2\partial_u\partial_v\Phi={}&(\boldsymbol{\partial}^2-m^2)\Phi+2ie[A_u\partial_v-(\mathbf{A}+\mathbf{A}^e)\cdot\boldsymbol{\partial}]\Phi\\&+ie(\partial_vA_u-\boldsymbol{\partial}\cdot\mathbf{A})\Phi-e^2(\mathbf{A}+\mathbf{A}^e)^2\Phi\,,\end{aligned} \tag{5.8}$$

$$(2\partial_u\partial_v-\boldsymbol{\partial}^2)A_j-\partial_j(\partial_vA_u-\boldsymbol{\partial}\cdot\mathbf{A})=J_j\,, \tag{5.9}$$

$$\partial_v(\partial_vA_u-\boldsymbol{\partial}\cdot\mathbf{A})=-J_v\,. \tag{5.10}$$

Here we defined

$$\mathbf{J}\equiv-ie\Phi^*\overleftrightarrow{\boldsymbol{\partial}}\Phi-2e^2(\mathbf{A}+\mathbf{A}^e)\Phi^*\Phi \tag{5.11}$$

and

$$J_v\equiv-ie\Phi^*\overleftrightarrow{\partial}_v\Phi\,. \tag{5.12}$$

We observe that there are only three Maxwell equations, the one corresponding to (5.5) with $\mu=u$ is absent. Correspondingly, there are only three components to the current, J_1, J_2, and J_v. There is no component J_u in Maxwell's equations. This is consistent with (5.6); *all* field equations are of that form with L independent of ∂_u.

Now Eq. (5.10) contains no ∂_u at all (our "time derivative") so that this is a *constraint equation* imposed on the fields on the initial surface. Consequently only two of the three component A_1, A_2, and A_u are dynamically independent. We shall choose A_1 and A_2 as in Ref. 18, but shall leave this matter to the next section.

We now turn to the momentum four-vector as determined in null coordinates from the canonical energy tensor which in turn is derived from the Lagrangian (5.1) in the null-plane gauge. We find first

$$T_{\mu\nu}=-(D_\mu\Phi)^*\partial_\nu\Phi-\partial_\nu\Phi^*D_\mu\Phi-F_{\mu\alpha}\partial_\nu A^\alpha-g_{\mu\nu}\mathfrak{L}. \quad (5.13)$$

This tensor is not symmetric. While symmetrization is clearly possible by well-known techniques, it is this nonsymmetric canonical tensor which yields the correct field equations.[23]

The momentum P_μ can now be given as an integral of $T_{\mu\nu}$ over the null plane $u=$const,

$$P_\mu=\int d^3\bar{x}\, n^\nu T_{\nu\mu}. \quad (5.14)$$

The momenta P_j are then seen to be

$$P_j=\int d^3\bar{x}[\partial_v\Phi^*\partial_j\Phi+\partial_j\Phi^*\partial_v\Phi+\partial_v\mathbf{A}\cdot\partial_j\mathbf{A}] \quad (j=1,\,2) \quad (5.15)$$

and the third momentum in the $u=$const hyperplane is

$$P_v=-\int d^3\bar{x}\, n^\nu T_{\nu\mu}n^\mu =\int d^3\bar{x}[2\partial_v\Phi^*\partial_v\Phi+\partial_v\mathbf{A}\cdot\partial_v\mathbf{A}]. \quad (5.16)$$

But the most interesting momentum is the one which is in the u direction and which corresponds to the Hamiltonian in the spacelike-plane formulation

$$P_u=-\int d^3\bar{x}\, n^\nu T_{\nu\mu}m^\mu =\int d^3\bar{x}[\boldsymbol{\partial}\Phi^*\cdot\boldsymbol{\partial}\Phi+m^2\Phi^*\Phi+ie(\mathbf{A}+\mathbf{A}^e)\cdot(\Phi^*\overleftrightarrow{\boldsymbol{\partial}}\Phi) -ieA_u(\Phi^*\overleftrightarrow{\partial_v}\Phi)+e^2(\mathbf{A}+\mathbf{A}^e)^2\Phi^*\Phi +\tfrac{1}{2}(F_{12})^2-\tfrac{1}{2}(\partial_vA_u)^2+\partial_v\mathbf{A}\cdot\boldsymbol{\partial}A_u]. \quad (5.17)$$

It is conveniently split into a radiation-free part $P_u^{(0)}$ and a radiation-interaction part $P_u^{(1)}$. The latter can be written in terms of the current components (5.12):

$$P_u^{(1)}=\int d^3\bar{x}\Big\{ie\mathbf{A}\cdot(\Phi^*\overleftrightarrow{\boldsymbol{\partial}}\Phi)+e^2\mathbf{A}\cdot(\mathbf{A}+2\mathbf{A}^e)\Phi^*\Phi +A_uJ_v-\frac{1}{8}\Big[\int\varepsilon(v-v')J_v'dv'\Big]^2\Big\}. \quad (5.18)$$

The last term arises from $-\frac{1}{2}(\partial_vA_u)^2$ upon substitution of the constraint equation (5.10) as integrated in Appendix B.

Before we proceed to the quantized version of the classical theory developed here, an important mathematical comment must be made. The Lagrangian density, the current density, the momentum density, and other densities of the theory which are to be integrated over three-dimensional hyperplanes will yield divergent integrals, in general. This is physically obvious when one considers the emitted radiation field which does not vanish asymptotically along a null cone so that integration over the variable v cannot converge. It is necessary to consider these classical expressions as distribution densities which are mathematically meaningful when integrated (formally) over suitable test functions. Thus, the integrand of P_u in (5.17), for example, should contain a factor $\sigma(\bar{x})$ which is 1 over a very large but finite region of the $u=$const hyperplane and then goes to zero smoothly such that there exists a bound M for which $\sigma(\bar{x})=0$ for $(x_1^2+x_2^2+v^2)^{1/2}>M$. This "cutoff" can later be taken to the limit $\sigma=1$ for all $\bar{x}$, and such a limit will have to be proven to exist. These considerations are in fact necessary in what follows so that ambiguities are eliminated. In the quantized theory these ambiguities are related to vacuum divergences and Haag's theorem.

Quantum electrodynamics is now obtained from the above considerations in the time-honored way of regarding the fields A_μ and Φ as operators in a Hilbert space and by requiring the field equations (5.8)–(5.10) to hold for them. Of course, this does not give a mathematically well-defined result, even when the fields are treated as operator-valued distributions, because products such as $A_j\Phi^\dagger\Phi$ are not defined (Wick ordering does not exist for interacting fields). We shall return to this point below.

Let us at first ignore this problem and treat A_μ and Φ as operators, so that the P_μ will also be operators. One must then add to the field equations suitable commutation relations. Since the dynamics consists in the development of the system off a given null plane $u=$const, these commutation relations must be specified on such a plane. As long as these do not involve derivations with respect to the "time" variable u, they are formally those of the free fields.

The free-field commutation relations for Φ on a null plane were derived three years ago[1] by one of us and were extended by us more recently to other fields.[17]

[23] G. Wentzel, *Quantum Theory of Fields* (Interscience, New York, 1949).

They are

$$[\Phi(x),\Phi(x')]_{u=u'}=0\,, \tag{5.19}$$

$$[\Phi(x),\Phi^\dagger(x')]_{u=u'}=-\tfrac{1}{4}i\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\,. \tag{5.20}$$

Similarly, one has for the components A_1 and A_2 of the electromagnetic potential

$$[A_j(x),A_k(x')]_{u=u'} =-\tfrac{1}{4}i\delta_{jk}\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\quad (j,k=1,2)\,. \tag{5.21}$$

As we shall see below, commutation relations involving A_u cannot be specified but must be derived. Thus, only the commutation relation between A_j and Φ must be given, and since we are dealing with free-field analogs, we must have

$$[A_j(x),\,\Phi(x')]_{u=u'}=0\,. \tag{5.22}$$

If A_j ($j=1,2$), Φ, and their adjoints are the only independent dynamical variables, the above list of commutation relations is complete. The operator is then obtained from (5.10) as we shall see.

The quantized form of the previously developed classical theory (apart from the mathematical questions to which we shall return later) now takes on the following structure.

On a fixed initial null plane the independent fields A_1, A_2, Φ and their adjoints are related by the commutation relations (5.19)–(5.22) which have been given. The operator A_u is derived with (5.10). The unitary generator of space-time translations (in null coordinates) $U(x)$ satisfies

$$i\partial_\mu U=P_\mu U \tag{5.23}$$

with P_μ given by the operators (5.15)–(5.17). In particular, the null translation generator P_u determines the dynamics as it develops off the null plane $u=$const. We have the relation

$$\Phi(x)=U(x)\Phi(0)U^{-1}(x) \tag{5.24}$$

and the first derivative

$$\partial_\mu\Phi(x)=i[\Phi(x),P_\mu]\,. \tag{5.25}$$

Analogous relations hold for A_μ.

By means of these equations and the given commutation relations, one must be able to derive the field equations (5.8)–(5.10). That this is indeed the case is the outcome of a somewhat lengthy and tedious calculation which we have performed in detail. We shall not present it here, but instead we shall establish the relations between the above Heisenberg picture and the Furry picture in the next section. This will lead to equivalent results.

VI. FURRY PICTURE

The Furry picture for a system involving both quantized and external fields is characterized by the field equations satisfied by the quantized fields. These are not the free-field equations as in the interaction picture; rather, they are those which take full account of the external field but ignore the interaction between the quantized fields. In our case they are Eq. (2.1), which was solved exactly in closed form in Secs. II and III, and the free-field equations for the two dynamically independent components a_1 and a_2:

$$2\partial_u\partial_v\phi=[\boldsymbol{\partial}^2-m^2-2ie\mathbf{A}_e\cdot\boldsymbol{\partial}-e^2\mathbf{A}_e{}^2]\phi\,, \tag{6.1}$$

$$(2\partial_u\partial_v-\boldsymbol{\partial}^2)a_j=0\quad(j=1,2)\,. \tag{6.2}$$

Equations (6.2) are formally the Maxwell equations in the Lorentz gauge which is, however, modified in the Furry picture to (6.8) below. (See also Appendix A.)

To these equations are added the free-field commutations (5.19)–(5.22) on a $u=$const plane but with the Heisenberg fields $\mathbf{A}$, Φ replaced by the Furry fields $\mathbf{a}$, ϕ. The remaining u dependence is in the state vectors and is characterized by the unitary operators $V(u)$ which transform between the two pictures:

$$\phi(x)=V(u)\Phi(x)V^{-1}(u)\,, \tag{6.3}$$

$$\mathbf{a}(x)=V(u)\mathbf{A}(x)V^{-1}(u)\,. \tag{6.4}$$

We note that $V(u)$ is indeed unitary since it contains the cutoff function $\sigma(\bar{x})$ which we discussed in Sec. V, and we shall exhibit it explicitly. We can simply specify $V(u)$ and then verify explicitly that it leads to the correct equations in the Heisenberg picture. But it is fairly obvious that it is just $P_u{}^{(1)}$ of (5.18) transformed to the Furry picture. The essential point here is that *this operator contains no u derivatives*. Otherwise that transformation would lead to an integral equation for $P_{uF}{}^{(1)}$, a matter that was discussed in our previous paper.[17] Thus we have

$$P_{uF}{}^{(1)}(u)\equiv V(u)P_u{}^{(1)}(u)V^{-1}(u) =\int\sigma(\bar{x})d^3\bar{x}\colon\Big\{ie\mathbf{a}\cdot(\phi^\dagger\overleftrightarrow{\boldsymbol{\partial}}\phi)+e^2\mathbf{a}\cdot(\mathbf{a}+2\mathbf{A}^e)\phi^\dagger\phi +\frac{e^2}{8}\Big[\int\varepsilon(v-v')(\phi'^\dagger\overleftrightarrow{\partial}_v{}'\phi)dv'\Big]^2 -iea_u(\phi^\dagger\overleftrightarrow{\partial}_v\phi)\Big\}\colon. \tag{6.5}$$

The ill-defined expression (5.18) has now been made precise by the explicit appearance of $\sigma(\bar{x})$ and by the well-defined normal ordering of the fields. The latter shows that, taking (6.5) as the starting point, the Heisenberg operator (5.18) is derived in a mathematically much more meaningful way than could be obtained within the Heisenberg picture. Of course, this is an implicit definition, since $V(u)$ itself is given by

$$-i\partial_uV(u)=P_{uF}{}^{(1)}(u)V(u)\,. \tag{6.6}$$

An initial condition such as $V(0)=1$ or $V(-\infty)=1$ makes $V(u)$ unique. We choose the latter.

By means of $P_{uF}^{(1)}$ the u derivatives of the fields are determined. For example,

$$\partial_u\phi = V\partial_u\Phi V^{-1} - i[\phi, P_{uF}^{(1)}]. \tag{6.7}$$

We next turn to the derived operator a_u. It is specified by a defining equation in the initial null plane. Since this defining equation in the Heisenberg picture, Eq. (5.10), is independent of ∂_u, the corresponding equation in the Furry picture must be

$$\partial_v(\boldsymbol{\partial}\cdot\mathbf{a} - \partial_v a_u) = j_v \equiv -ie{:}\phi^\dagger \overleftrightarrow{\partial}_v \phi{:}\,. \tag{6.8}$$

This seems rather surprising in view of the impression that in this picture the quantized electromagnetic field is free. But only the dynamically independent components, A_1 and A_2 in our choice, can be expected to be free. Since the Maxwell equation (5.10) is independent of ∂_u and is therefore an "initial constraint," a transformation to a different picture cannot change this constraint, i.e., cannot uncouple the *dependent* component A_u. Thus, in null-plane quantum electrodynamics not all components of A_μ can be decoupled from the matter field by a change of the picture. Since clearly only relative values of a_u can be of physical significance, only the first integral of (6.8) must be determined. From Appendix B we find

$$\partial_v a_u = \boldsymbol{\partial}\cdot\mathbf{a} + \tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v-v')\, j_v(u,\mathbf{x},v')\,dv'. \tag{6.9}$$

The next integration can, however, also be given and is conveniently expressed as

$$a_u(x) = -\tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v-v')dv'\sigma(v')\bigg[\boldsymbol{\partial}\cdot\mathbf{a}(u,\mathbf{x},v') + \tfrac{1}{2}\int_{-\infty}^{\infty}\varepsilon(v'-v'')\,j_v(u,\mathbf{x},v'')dv''\bigg]. \tag{6.10}$$

The additive integration "constants" must be chosen consistently as is discussed in Appendix B. The relevant commutation relations involving a_u then follow as

$$[\partial_v a_u(x), \mathbf{a}(x')]_{u=u'} = -\tfrac{1}{4}i\varepsilon(v-v')\,\boldsymbol{\partial}\delta_2(\mathbf{x}-\mathbf{x}')\,, \tag{6.11}$$

$$[\partial_v a_u, \partial_v'\phi']_{u=u'} = -\tfrac{1}{2}e\delta_3(\bar{x}-\bar{x}')\phi(x') + \tfrac{1}{2}e\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\partial_v'\phi(x)\,. \tag{6.12}$$

It is now a matter of straightforward computation to verify that the field equations (6.1)–(6.3), by means of the transformations (6.3)–(6.6) and the indicated commutation relations, yield the Heisenberg field equations (5.8)–(5.10). The commutation relations, of course, remain form invariant. This computation is carried out in Appendix C.

One can now proceed to the evaluation of the scattering matrix in the spirit of Feynman-Dyson quantum electrodynamics. Leaving aside an improved treatment of ultraviolet or infrared divergences, one has the formal expression

$$S = V(\infty) = U_+ \exp\bigg[-i\int P_{uF}^{(1)}(u)du\bigg] \tag{6.13}$$

with $P_{uF}^{(1)}$ given by (6.5). The symbol U_+ stands for (positive) u-ordering in complete analogy to the usual time-ordering. This analogy continues through the Wick theorems to a perturbation expansion in the quantized-field coupling in terms of Feynman diagrams in the Furry picture. The corresponding propagators were given in Sec. IV.

For practical calculations the S matrix looks very much like the usual one, but it is cast in null coordinates, because the integrand of $P_{uF}^{(1)}$, after all, can be written

$$:iea^\mu(\phi^\dagger\overleftrightarrow{\partial}_u\phi) + e^2a_\mu(a^\mu + 2A_e^{\ \mu})\phi^\dagger\phi - \tfrac{1}{2}(\partial_\mu a^\mu)^2: \tag{6.14}$$

and is only restricted by $a_v = A_v^{\ e} = A_u^{\ e} = 0$. In the Lorentz gauge the last term in (6.14) would vanish, but in the null-plane gauge this term is of order e^2 by (6.9). Since it is not clear how a *consistent* calculation can be done in the Lorentz gauge when null coordinates are used, the assertion[4] that one can carry out the Feynman-Dyson theory by simply changing the S matrix to null coordinates seems without foundation.

As far as u-ordering is concerned, we do note that for timelike $x-x'$, one has $\theta(u-u') = \theta(t-t')$. This is, of course, the reason for the similarity in the Green functions between the present formulation and the usual one as discussed in Sec. IV.

For the evaluation of S-matrix elements, we shall need the quantized version of the wave-packet decomposition (3.4). We shall distinguish the annihilation part $\phi^{(-)}(x)$ and the creation part $\phi^{(+)}(x)$ of the Furry-picture field operator $\phi(x)$. They correspond, respectively, to the positive-energy part $\phi_+(x)$ and the negative-energy part $\phi_-(x)$ of the classical plane-wave solution (2.13) or $\phi_{\alpha+}(x)$ and $\phi_{\alpha-}(x)$ of the classical wave-packet solution (3.4). Thus we have

$$\phi(x) = \phi^{(-)}(x) + \phi^{(+)}(x)\,, \tag{6.15}$$

$$\phi^{(-)}(x) = \sum_\alpha a_\alpha \phi_{\alpha+}(x)\,, \tag{6.16}$$

$$\phi^{(+)}(x) = \sum_\alpha b_\alpha^\dagger \phi_{\alpha-}(x)\,, \tag{6.17}$$

where the creation and annihilation operators satisfy the usual commutation relations

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}\,,\quad [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}\,, \tag{6.18}$$

$$[a_\alpha, a_\beta] = [a_\alpha, b_\beta] = [a_\alpha, b_\beta^\dagger] = [b_\alpha, b_\beta] = 0\,. \tag{6.19}$$

It then follows from the completeness relations (3.8a) and from (3.8b) that

$$\begin{aligned}[\phi(x), \phi^\dagger(x')] &= \sum_\alpha\sum_\beta \{[a_\alpha\phi_{\alpha+}(x), a_\beta^\dagger\phi_{\beta+}^*(x')] + [b_\alpha^\dagger\phi_{\alpha-}(x), b_\beta\phi_{\beta-}^*(x')]\} \\ &= \sum_\alpha \phi_{\alpha+}(x)\phi_{\alpha+}^*(x') - \sum_\alpha \phi_{\alpha-}(x)\phi_{\alpha-}^*(x') \\ &= -i\Delta(x,x';A^e)\,. \end{aligned} \tag{6.20}$$

When we restrict the commutation relation to the null plane $u=u'$, we find

$$[\phi(x),\phi^\dagger(x')]_{u=u'}=-\tfrac{1}{4}i\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}') \quad (6.21)$$

in agreement with (5.20) and (6.3). This follows from (4.12a).

Completely analogous relations can be written down for the free electromagnetic fields a_j ($j=1, 2$). These are, of course, well known and are therefore given here only very briefly for the sake of completeness and notation:

$$a_j(x)=\sum_\alpha [c_{\alpha j}a_{\alpha+}{}^j(x)+c_{\alpha j}{}^\dagger a_{\alpha-}{}^j(x)], \quad (6.22)$$

$$[c_{\alpha j},c_{\beta k}{}^\dagger]=\delta_{jk}\delta_{\alpha\beta}, \quad [c_{\alpha j},c_{\beta k}]=0. \quad (6.23)$$

The operators $c_{\alpha j}$ commute with all a_α, b_α and their adjoints. Finally,

$$[a_j(x),a_k(x')]=-i\delta_{jk}D(x-x') \quad (6.24)$$

is again a consequence of the completeness relations of the classical wave-packet solutions of the free Maxwell equations,

$$\sum_\alpha a_{\alpha\epsilon}(x)a_{\alpha\epsilon}(x')=-i\epsilon D_\epsilon(x-x') \quad (6.25)$$

and

$$D(x-x')=D_+(x-x')+D_-(x-x'). \quad (6.26)$$

The restriction of (6.24) to the null plane $u=u'$ yields

$$[a_j(x),a_k(x')]_{u=u'}=-\tfrac{1}{4}i\delta_{jk}\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}') \quad (6.27)$$

in agreement with (5.21) and (6.4).

In practice we use the plane-wave expansion of the field operators $a_j(x)$ in terms of null coordinates

$$a_j(x)=(4\pi)^{-3/2}\sum_{\lambda=1}^{2}\int [a(\bar{k},\lambda)e_j(\bar{k},\lambda)$$
$$\times\exp(i\bar{k}\cdot\bar{x}-i|k_u|u)+a^\dagger(\bar{k},\lambda)e_j{}^*(\bar{k},\lambda)$$
$$\times\exp(-i\bar{k}\cdot\bar{x}+i|k_u|u)]d^3\bar{k}/|k_v| \quad (6.28)$$

with the commutation relations for the creation and annihilation operators $a^\dagger(\bar{k},\lambda)$ and $a(\bar{k},\lambda)$ as

$$[a(\bar{k},\lambda),a^\dagger(\bar{k}',\lambda')]=|k_v|\delta_{\lambda\lambda'}\delta_3(\bar{k}-\bar{k}'),$$
$$[a(\bar{k},\lambda),a(\bar{k}',\lambda')]=0, \quad (6.29)$$

and the completeness relation for the unit polarization vectors $e_j(\bar{k},\lambda)$,

$$\sum_{\lambda=1}^{2} e_j(\bar{k},\lambda)e_{j'}{}^*(\bar{k},\lambda)=\delta_{jj'}. \quad (6.30)$$

This expansion for $a_j(x)$ yielded (6.24).

VII. NULL-PLANE GAUGE

In the usual formulation of quantum electrodynamics (with respect to spacelike planes), the gauge problem plays a large and annoying role. The potentials and the associated gauge freedom, which were originally introduced into classical electrodynamics in order to simplify the mathematics of the Maxwell equations, become a major part of the complications of the theory. The consistency between the commutation relations among the potentials, the Lorentz or Coulomb gauge condition, and the requirement of a vacuum state, is usually achieved only by means of rather far-fetched methods such as the Bleuler-Gupta indefinite-metric technique, the finite-mass photon, or the separate treatment of the Coulomb interaction as a direct interaction following Fermi's original work.

In the null-plane formulation of quantum electrodynamics the situation is radically different. One is led to a particular gauge, the null-plane gauge, characterized by $A_v=0$. In this gauge, one of the Maxwell equations, (5.10), permits us to express one of the remaining components in terms of the other two on the initial null surface, as discussed in Sec. VI. Consequently there are only two dynamically independent components of A_μ. If these are chosen to be A_1 and A_2, one obtains A_u in the Heisenberg picture by means of the boundary condition (6.9) in terms of A_1, A_2, and Φ exactly as in the Furry picture (6.10). Since only the commutation relations involving A_1 and A_2 are specified on the initial null surface, those involving A_u are derived and cannot lead to any conflict between the gauge choice and these relations.

A comparison of the null-plane gauge with the Coulomb gauge used in the usual formulation is instructive. In the latter, one of the Maxwell equations $\nabla^2A^0=-j^0$ also becomes a constraint permitting the elimination of A^0 in terms of j^0; the gauge condition $\nabla\cdot\mathbf{A}=0$ then permits the reduction to only A_1 and A_2 as independent fields as in the null-plane gauge. Thus there is considerable similarity between these gauges.

This similarity fails as far as covariance is concerned. Since we treat m^μ and n^μ as four-vectors [see Eqs. (2.2) ff], A_v is an invariant. One must therefore compare the null-plane gauge to the *covariant* Coulomb gauge (see, e.g., Ref. 22). We also note that (three-dimensional) rotation invariance alone suffices to establish that the theory is independent of the choice of the null vectors m^μ, n^μ which, in particular, makes the S matrix invariant. Rotation invariance can be made manifest by defining the null directions as follows: Let $\hat{u}$ be an arbitrary unit vector in 3-space and define $n^\mu=(1,\hat{u})/\sqrt{2}$ and $m^\mu=(1,-\hat{u})/\sqrt{2}$. Then A_v and A_u are manifestly rotation invariant. Our choice $\hat{u}=\hat{k}$, where $\hat{k}$ is the unit vector along the z axis, was made for convenience.

To be sure, manifest Poincaré invariance is not a necessary requirement on a relativistic theory. It suffices that the Poincaré algebra is satisfied, which is indeed the case.[18]

The theory is, however, manifestly gauge invariant only under the restricted class of gauge transformations $A_\mu\to A_\mu+\partial_\mu\Lambda$, where Λ is independent of v.

The definition of the vacuum in the Furry (or interaction) picture offers no difficulties: The conditions

$$\phi^{(-)}|0\rangle=0\,,\quad a_j{}^{(-)}|0\rangle=0\quad(j=1,2)\,,\qquad(7.1)$$

where the superscript $(-)$ indicates the annihilation part of the operator, imply via (6.10) that the vacuum will also satisfy

$$a_u{}^{(-)}|0\rangle=0\,.\qquad(7.2)$$

For this purpose, we define

$$j_v{}^{(\pm)}(x)\equiv-\tfrac{1}{2}ie\!:\![\phi^{\dagger}(x)\overleftrightarrow{\partial}_v\phi^{(\pm)}(x)+\phi^{\dagger(\pm)}(x)\overleftrightarrow{\partial}_v\phi(x)]\!:\,,\qquad(7.3)$$

so that the creation and annihilation parts of a_u become

$$a_u{}^{(\pm)}(x)=-\tfrac{1}{2}\int_{-\infty}^{\infty}\varepsilon(v-v')dv'\sigma(v')\Big[\boldsymbol{\partial}\cdot\mathbf{a}^{(\pm)}(u,\mathbf{x},v')+\tfrac{1}{2}\int_{-\infty}^{\infty}\varepsilon(v'-v'')j_v{}^{(\pm)}(u,\mathbf{x},v'')\Big]dv''\,.\qquad(7.4)$$

Equation (7.2) is then an obvious consequence.

Furthermore, the commutation relations are consistent with the definition of the vacuum. For example,

$$\langle[a_j(x),a_j(x')]\rangle_0=2i\,\mathrm{Im}\langle a_j{}^{(-)}(x)a_j{}^{(+)}(x')\rangle_0$$

is clearly consistent with

$$[a_j(x),a_j(x')]_{u=u'}=-\tfrac{1}{4}i\varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\,.$$

Thus we see that the null-plane gauge is very simple and is not beset by the difficulties encountered in various gauges of the spacelike-plane formulation of quantum electrodynamics.

Finally, one may wonder whether the use of a fixed gauge did not cost us something. Did we not lose gauge invariance, and did we not lose one of the four inhomogeneous Maxwell equations in the set (5.9), (5.10)? The answer to this question is as follows.

The Lagrangian (5.1) in the null-plane gauge is still invariant under gauge transformations of the first kind, i.e., $\Phi\to e^{i\alpha}\Phi$ with α a constant. As a result there still exists a *conserved* current J_μ with components $\mathbf{J}$ and J_v given by (5.11) and (5.12), and J_u is now known explicitly as

$$J_u=-ie\Phi^*\overleftrightarrow{\partial}_u\Phi-2e^2A_u\Phi^*\Phi\,.\qquad(7.5)$$

Now in the usual Maxwell theory, current conservation is a consequence of the field equations (5.5); the left-hand side becomes an identity upon taking the divergence. The "missing" Maxwell equation, the one with $\mu=u$, is now restored since current conservation is already ensured for our matter field:

$$\begin{aligned}\partial_v(\partial_jF_{ju}-\partial_uF_{vu})&=-\partial_u(\partial_jF_{jv}-\partial_vF_{uv})+\partial_k(\partial_jF_{jk}-\partial_uF_{vk}-\partial_vF_{uk})\\&=\partial_uJ_v-\partial_kJ_k\\&=-\partial_vJ_u\,.\end{aligned}\qquad(7.6)$$

The first equality is a consequence of $\partial_\mu\partial_\nu F^{\mu\nu}\equiv0$, the second one follows from the three Maxwell equations (5.9) and (5.10), and the last one is $\partial_\mu J^\mu=0$. Equation (7.6) is the v derivative of the "missing" equation. Thus, the invariance of the Lagrangian under gauge transformations of the first kind ensures that the missing equation be satisfied if and only if there is no integration function from (7.6),

$$(2\partial_u\partial_v-\boldsymbol{\partial}^2)A_u-\partial_u(\partial_vA_u-\boldsymbol{\partial}\cdot\mathbf{A})-J_u=C(u,\mathbf{x})\qquad(7.7)$$

with

$$C(u,\mathbf{x})=0\,.\qquad(7.8)$$

The use of the null-plane gauge thus requires (7.8) as a condition which ensures the full Maxwell theory.

Now in the distant past, A_μ and Φ become free fields. Therefore the current, being associated with particles of finite mass, will correspond to timelike world lines and will be bounded on spacelike planes. Thus it will have no support in the null direction $v\to-\infty$. Similarly, we can easily avoid choosing our physical system so that incident photons come from $v=-\infty$ in a u=const plane. Thus, it is physically evident that the matrix elements of the left-hand side of (7.7) vanish in the $v=-\infty$ limit. Hence, (7.8) must hold.

VIII. COMPTON SCATTERING

The lowest-order matrix element to contribute to Compton scattering is

$$R^{(1)}\equiv\langle\alpha+,\,k'\lambda';\,\mathrm{in}\,|S^{(1)}|\beta+;\,\mathrm{in}\rangle\,,\qquad(8.1)$$

where

$$S^{(1)}=-i\int P_{uF}{}^{(1)}(u)du\,.\qquad(8.2)$$

This matrix element represents the absorption of one or more photons out of the pulse with the resultant emission of one photon into the state $|k'\lambda';\mathrm{in}\rangle$. Note that in the absence of the external field this matrix element vanishes.

Since only terms linear in $a^\mu(x)$ contribute to $R^{(1)}$, we write out explicitly the effective part of $S^{(1)}$ using (6.5),

$$S_{\mathrm{eff}}{}^{(1)}=e\int d^4x\!:\!\{\mathbf{a}(x)\cdot[\phi^{\dagger}(x)\overleftrightarrow{\boldsymbol{\partial}}\phi(x)-2ie\mathbf{A}^e(u)\phi^{\dagger}(x)\phi(x)]-a_u(x)\phi^{\dagger}(x)\overleftrightarrow{\partial}_v\phi(x)\}\!:\,.\qquad(8.3)$$

The component a_u is given by (6.10) in terms of a_j and j_v. To first order, we replace a_u by

$$a_u{}^{(1)}=-\tfrac{1}{2}\int_{-\infty}^{\infty}\varepsilon(v-v')\,\boldsymbol{\partial}\cdot\mathbf{a}(u,\mathbf{x},v')dv'\,.\qquad(8.4)$$

Using expansions (6.15)–(6.17) for the field operator $\phi(x)$ and expressions (3.4) for the wave packets $\phi_{\alpha+}(x)$, combined with the plane-wave expansion (6.28) for the electromagnetic field, we find

$$R^{(1)}=\frac{e}{(4\pi)^{3/2}}\int\frac{d^3\bar{p}'}{2|p_v'|}\int\frac{d^3\bar{p}}{2|p_v|}\times\Big(h_{\alpha+}{}^*(\bar{p}')h_{\beta+}(\bar{p})\delta_3(\bar{p}-\bar{p}'-\bar{k}')\times\int du\{i[\mathbf{q}-2e\mathbf{A}^e(u)]\cdot\mathbf{e}^*(\bar{k}'\lambda')\}\times\exp i\Big\{(|k_u'|+|p_u'^f|-|p_u^f|)u-\int_{ui}^{u}2e\mathbf{A}^e(\eta)[\boldsymbol{\xi}-\tfrac{1}{2}e\mathbf{A}^e(\eta)\zeta]d\eta\Big\}\Big),\quad(8.5)$$

where

$$\zeta=\tfrac{1}{2}(|p_v'|^{-1}-|p_v|^{-1}),\quad(8.6a)$$

$$\boldsymbol{\xi}=\tfrac{1}{2}(\mathbf{p}'/|p_v'|-\mathbf{p}/|p_v|),\quad(8.6b)$$

$$\mathbf{q}=(\mathbf{p}+\mathbf{p}'+\mathbf{k}'(|p_v|+|p_v'|)/|k_v'|),\quad(8.6c)$$

$$p_u{}^f=(\mathbf{p}^2+m^2)/(2p_v),\quad k_u'=\mathbf{k}'^2/(2k_v').\quad(8.6d)$$

The δ function $\delta_3(\bar{p}-\bar{p}'-\bar{k}')$ appearing in (8.5) is a statement of the conservation of the three momentum components p_1, p_2, and p_v. If $A_\mu{}^e=0$ in (8.5), the integration over u yields a fourth δ function $\delta(|k_u'|+|p_u'^f|-|p_u^f|)$, which ensures that $R^{(1)}$ vanishes whenever a photon is scattered, since $|k_u'|+|p_u'^f|-|p_u^f|>0$ whenever $|k_v'|>0$ and $|\mathbf{k}'|>0$.

If $a_u{}^{(1)}$ included the additional term $\alpha(u,\mathbf{x})$ as in (A14) and (B6), which does not depend on v, it would yield in expression (8.5) a term multiplied by the δ function $\delta(|p_v|-|p_v'|)$. Physically this is a process in which either no photon is scattered, or it is scattered in the n^μ direction, $k'^\mu=\omega' n^\mu$. As these photons are scattered into the direction of the external pulse, we shall not consider them to be distinguished from the pulse in this approximation.

In order to perform the u integration in (8.5), we make the following explicit choice for $A_\mu{}^e(u)$:

$$\mathbf{A}^e(u)=\begin{cases}\mathrm{Re}[\mathbf{A}^e e^{-i\omega u}], & |u|<u_0\\ 0, & |u|>u_0\end{cases}\quad(8.7a)$$

with

$$\mathbf{A}^e=(1,i)A^e\quad(8.7b)$$

and $\omega=|k_u|$. We thus have a circularly polarized monochromatic plane wave with a square-pulse shape. Inserting expression (8.7) into (8.5) and performing the integration, we obtain

$$R^{(1)}=\frac{2ie}{(4\pi)^{3/2}}\int\frac{d^3\bar{p}d^3\bar{p}'}{4|p_v||p_v'|}h_{\alpha+}{}^*(\bar{p}')h_{\beta+}(\bar{p})\delta_3(\bar{p}-\bar{p}'-\bar{k}')e^{i\varphi_0}\times\left[I_0+e^{i\varphi_1}\sum_{r=-\infty}^{\infty}M_r\frac{\sin[(|k_u'|+|p_u'^f|-|p_u^f|+e^2\zeta A_e{}^2-r\omega)u_0]}{|k_u'|+|p_u'^f|-|p_u^f|+e^2\zeta A_e{}^2-r\omega}\right],\quad(8.8)$$

where

$$M_r=e^{-ir\varphi_2}\left[\mathbf{q}\cdot\mathbf{e}^*(\bar{k}',\lambda')-r\omega\xi^{-1}\cos\varphi_++i\omega^2\xi^{-1}\sin\varphi_+\frac{d}{d\omega}\right]J_r\left(\frac{2eA^e}{\omega}\xi\right),\quad(8.9)$$

$$I_0=\mathbf{q}\cdot\mathbf{e}^*(\bar{k}',\lambda')\int_{u_0}^{\infty}\cos[(|k_u'|+|p_u'^f|-|p_u^f|)u+\varphi_0]du,\quad(8.10)$$

$$\varphi_0=e^2A_e{}^2u_0\zeta-2eA_e\omega^{-1}\xi_1\sin\omega u_0,\quad\varphi_1=-2eA_e\omega^{-1}\xi_2\cos\omega u_0,\quad(8.11a)$$

$$\xi e^{i\varphi_2}=-\xi_1+i\xi_2,\quad\xi=|\boldsymbol{\xi}|,\quad(8.11b)$$

$$\xi\cos\varphi_+=-\boldsymbol{\xi}\cdot\mathbf{e}^*(\bar{k}',\lambda'),\quad\xi\sin\varphi_+=-\xi_1e_2{}^*(\bar{k}',\lambda')+\xi_2e_1{}^*(\bar{k}',\lambda'),\quad(8.11c)$$

and J_r is the Bessel function of the first kind of order r. The I_0 term arises from integration over the regions outside the pulse and is unimportant for our considerations. (Note that $I_0\to 0$ for $u_0\to\infty$ and $\lim I_0$ for $u_0\to 0$ does not contribute for noninfinite values of A^e, since $|k_u'|+|p_u'^f|-|p_u^f|>0$.) The term $(\sin\chi u_0)/\chi$ in (8.8) with

$$\chi\equiv|k_u'|+|p_u'^f|-|p_u^f|-r\omega+e^2A_e{}^2\zeta\quad(8.12)$$

has a maximum value u_0 at

$$\chi=0\quad(8.13)$$

and as $|\chi|$ increases, $(\sin\chi u_0)/\chi$ oscillates with amplitude decreasing as $1/|\chi|$. Since $|p_v|-|p_v'|=|k_v'|>0$, we have $\zeta>0$, and with $|k_u'|+|p_u'^f|-|p_u^f|>0$, therefore, only values of $r\geqslant 1$ will give a maximum of $(\sin\chi u_0)/\chi$ and consequently a maximum of $R^{(1)}$.

In the limit $u_0 \to \infty$, i.e., for an infinite plane-wave train,

$$\lim_{u_0\to\infty} \frac{\sin\chi u_0}{\chi} = \pi\delta(\chi) \tag{8.14}$$

and

$$\lim_{u_0\to\infty} R^{(1)} = \frac{ie}{4\sqrt{\pi}} \int \frac{d^3\bar{p}d^3\bar{p}'}{4|p_v||p_v'|} \sum_{r=1}^{\infty} M_r \delta(\chi) \times [h_{\alpha+}^*(\bar{p}')h_{\beta+}(\bar{p})\delta_3(\bar{p}+\bar{p}'-\bar{k}') \lim_{u_0\to\infty} e^{i(\varphi_0+\varphi_1)}]. \tag{8.15}$$

The phase factor $\lim_{u_0\to\infty} e^{i\varphi_0+i\varphi_1}$ is inessential for the calculation of the cross section. The important factor in (8.15) is the second Dirac δ function. Interpreting the rth term in the summation to represent the absorption of r photons from the pulse with the resultant emission of one photon (k_μ',λ'), it must be concluded that the frequency of this scattered photon is shifted by an amount depending on the intensity of the pulse. To make this clear, we write out explicitly the values of k_μ' required by the presence of the four momentum-space δ functions in (8.15):

$$\mathbf{k}' = \mathbf{p} - \mathbf{p}', \tag{8.16a}$$

$$k_3' = p_3^f - p_3'^f + rk_3 - \tfrac{1}{4}e^2 A_e^2 k^0 |k\cdot k'|/|k\cdot pk\cdot p'|, \tag{8.16b}$$

$$k^{0\prime} = p_f^0 - p_f^{0\prime} + rk^0 - \tfrac{1}{4}e^2 A_e^2 k^0 |k\cdot k'|/|k\cdot pk\cdot p'| \tag{8.16c}$$

with

$$p_f^0 = (p_u^f + p_v)/\sqrt{2}, \quad p_3^f = (p_u^f - p_v)/\sqrt{2}. \tag{8.16d}$$

At the same time it is clear that even before taking the limit $u_0 \to \infty$, the same values for k_μ' as given in (8.16) are required in order to give $R^{(1)}$ its maximum value. Therefore, the intensity-dependent frequency shift is present also in the case of a finite-length pulse, and this frequency shift is the same as that predicted by Brown and Kibble[8] for an infinitely long plane-wave train.

APPENDIX A: FREE ELECTROMAGNETIC FIELD

The Lagrangian of the free electromagnetic field in null coordinates and in the null-plane gauge is

$$\mathcal{L} = -\tfrac{1}{2}(F_{12})^2 + \partial_u \mathbf{a}\cdot\partial_v \mathbf{a} - \partial_v \mathbf{a}\cdot\boldsymbol{\partial} a_u + \tfrac{1}{2}(\partial_v a_u)^2 \tag{A1}$$

from which follow the classical Maxwell equations. They take on the form

$$(2\partial_u\partial_v - \boldsymbol{\partial}^2)a_j = \partial_j(\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}) \tag{A2}$$

and

$$\partial_v(\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}) = 0. \tag{A3}$$

The last equation is a constraint from which we can determine a_u in terms of the a_j $(j=1, 2)$, which we take to be the primary dynamically independent fields. This classical theory also has a canonical energy tensor given by the electromagnetic part of (5.13) from which the momenta follow:

$$P_j = \int \partial_v \mathbf{a}\cdot\partial_j \mathbf{a}\sigma(\bar{x})d^3\bar{x}, \tag{A4}$$

$$P_v = \int \partial_v \mathbf{a}\cdot\partial_v \mathbf{a}\sigma(\bar{x})d^3\bar{x}, \tag{A5}$$

$$P_u = \int [\tfrac{1}{2}(f_{12})^2 - \tfrac{1}{2}(\partial_v a_u)^2 + \partial_v \mathbf{a}\cdot\boldsymbol{\partial} a_u]\sigma(\bar{x})d^3\bar{x}. \tag{A6}$$

These were used in Eqs. (5.15)–(5.17). The function $\sigma(\bar{x})$ following (5.18) is written explicitly. So much for the classical theory.

The quantized theory starts with (A4) to (A6) as normal-ordered operators whose commutation relations follow from (6.27) for a given null plane $u=u'$. In order to derive the Maxwell equations for the quantized fields a_j, one must start with

$$\partial_u a_j = i[a_j, P_u], \tag{A7}$$

and we see that commutation relations with the derived field a_u are needed. These are derived from the constraint (A3) which is given as part of the initial data [while (A2) is not given]; we find

$$[\partial_v^2 a_u, a_j']_{u=u'} = \partial_v \boldsymbol{\partial}\cdot[\mathbf{a}, a_j']_{u=u'} = -\tfrac{1}{4}i\partial_v \varepsilon(v-v')\partial_j\delta_2(\mathbf{x}-\mathbf{x}'), \tag{A8}$$

where the prime on the field indicates primed arguments. Since $\partial_v = -\partial/\partial_v$, this yields

$$[\partial_v a_u, a_j']_{u=u'} = -\tfrac{1}{4}i\varepsilon(v-v')\partial_j\delta_2(\mathbf{x}-\mathbf{x}') + f(u, \mathbf{x}-\mathbf{x}'). \tag{A9}$$

The additional function f which could be operator valued must be independent of v, and if it is a c-number must depend on $\mathbf{x}-\mathbf{x}'$ only, because of translation invariance. Translation invariance in v then implies

$$[a_u, \partial_v' a_j']_{u=u'} = \tfrac{1}{4}i\varepsilon(v-v')\partial_j\delta_2(\mathbf{x}-\mathbf{x}') - f(u, \mathbf{x}-\mathbf{x}'). \tag{A10}$$

In order to obtain further information on f we compute (A2) explicitly from (A7) and the commutation relations

$$2\partial_u\partial_v a_j = 2i\int \sigma(\bar{x}')d^3\bar{x}' \times :\{f_{12}'[\partial_v a_j, f_{12}'] - \partial_v' a_u'[\partial_v a_j, \partial_v' a_u'] + [\partial_v a_j, \partial_v' \mathbf{a}']\cdot\boldsymbol{\partial}' a_u' + \partial_v' \mathbf{a}'\cdot[\partial_v a_j, \boldsymbol{\partial}' a_u']\}:.$$

In the last term, an integration by parts can bring ∂_v' into the commutator, while the integrated part vanishes due to $\sigma(\bar{x})$. Thus, all commutators involving a_u contain two v derivatives so that f in (A9) or (A10) does not contribute. The second and third terms cancel, and after taking the limit $\sigma \to 1$, we are left with

$$2\partial_u \partial_v a_j = \partial_k f_{kj} + \partial_j \boldsymbol{\partial}\cdot\mathbf{a} = \boldsymbol{\partial}^2 a_j . \qquad \text{(A11)}$$

Consistency with Maxwell's equations (A2) and (A3) thus requires that $\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}$ is independent of both $\mathbf{x}$ and v. Its commutator with $a_j(x')$ must then be independent of $\mathbf{x}-\mathbf{x}'$ and $v-v$. Thus f can depend only on u.

An immediate consequence of this result is that one cannot choose arbitrarily $a_u=0$ for some v such as $v \to -\infty$. $f(u)$ in (A10) cannot be chosen to satisfy such a requirement. The most convenient choice for f is $f=0$. This implies that $\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}$ commutes with the a_j on the null plane. But since the a_j are complete (irreducible),[24] this means that $\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}$ is a multiple of the identity operator. One is then free to choose $\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a}=0$. This means that the null-plane gauge $a_v=0$ and the Lorentz gauge $\partial_\mu a^\mu=0$ coexist, a matter which is very convenient. We shall therefore make this choice.

A further integration of (A10) with $f=0$ yields

$$[a_u, a_j']_{u=u'} = \tfrac{1}{4} i \,|v-v'|\, \partial_j \delta_2(\mathbf{x}-\mathbf{x}') . \qquad \text{(A12)}$$

An additive term has again been chosen to vanish.

The potential a_u can now be computed explicitly relative to some reference value at $v=v_0$ (which is not completely arbitrary, from what was said above); one has the formal integral

$$a_u(x) = a_u(u,\mathbf{x},v_0) = -\int_{v0}^{v} \boldsymbol{\partial}\cdot\mathbf{a}(u,\mathbf{x},v')dv' . \qquad \text{(A13)}$$

Of course, only relative values of a_u have physical significance. An alternative to (A13) which does not involve reference to a point v_0 is, formally,

$$a_u(x) = -\tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v-v')\,\boldsymbol{\partial}\cdot\mathbf{a}(u,\mathbf{x},v')\sigma(v')dv' + \alpha(u,\mathbf{x}) . \qquad \text{(A14)}$$

The additive term $\alpha(u,\mathbf{x})$ is arbitrary. These integrals are made mathematically meaningful with a test function $\sigma(v)$ which is 1 over a large but finite domain.

We conclude this appendix with the observation that the expression for P_u, (A6), can be simplified in view of the Lorentz gauge $\partial_v a_u = \boldsymbol{\partial}\cdot\mathbf{a}$ and the factor σ, which permits integration by parts and eliminates all contributions from surface terms. One finds

$$P_u = \int \sigma(\bar{x}) d^3\bar{x} [\tfrac{1}{2}(f_{12})^2 + \tfrac{1}{2}(\boldsymbol{\partial}\cdot\mathbf{a})^2] . \qquad \text{(A15)}$$

APPENDIX B: SOLUTION OF THE CONSTRAINT EQUATION

In this appendix we want to prove that (6.10) is a consistent solution of the constraint equation (6.8). Thereby we want to choose the integration constants such that for $j_v=0$ one obtains the results of Appendix A.

An immediate integral of (6.8) is formally

$$\partial_v a_u - \boldsymbol{\partial}\cdot\mathbf{a} - \tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v-v') j_v(u,\mathbf{x},v')dv' = f_1(u,\mathbf{x}) . \qquad \text{(B1)}$$

Since f_1 could be an operator and could depend on j_v, we cannot at this point choose $f_1=0$. The commutators of $\partial_v a_u$ with a_j and with $\partial_q \phi^\dagger$ are on the null plane

$$[\partial_v a_u, a_j']_{u=u'} = -\tfrac{1}{4} i \varepsilon(v-v')\partial_j \delta_2(\mathbf{x}-\mathbf{x}') + ig(u,\mathbf{x}-\mathbf{x}') \qquad \text{(B2)}$$

from (A9), and

$$[\partial_v a_u, \partial_v{}'\phi^\dagger(x')]_{u=u'} = \tfrac{1}{2} e \delta_3(\bar{x}-\bar{x}')\phi^\dagger + \tfrac{1}{2} e \varepsilon(v-v')\delta_2(\mathbf{x}-\mathbf{x}')\partial_v{}'\phi^\dagger(x') + ih(u,\mathbf{x},\mathbf{x}') \qquad \text{(B3)}$$

from (6.8) and (6.21).

When these commutation relations are transformed to the Heisenberg picture (fields in capital letters), they enable us to derive the Maxwell equations (5.9) from the Heisenberg equation of motion

$$\partial_u A_j = i[A_j, P_u] , \qquad \text{(B4)}$$

using P_u in (5.17). This calculation, which is quite analogous to that of the free case, yields

$$(2\partial_u\partial_v - \boldsymbol{\partial}^2)A_j = J_j + \tfrac{1}{2}\partial_j \int \varepsilon(v-v')J_v(u\mathbf{x}v')dv' + C(u,\mathbf{x}) ,$$

where C contains the commutator of F_1 with A_j. Consistency with (5.9) then requires

$$\partial_j(\partial_v A_u - \boldsymbol{\partial}\cdot\mathbf{A}) = \tfrac{1}{2}\partial_j \int \varepsilon(v-v')J_v(u,\mathbf{x},v')dv' + C(u,\mathbf{x}) \qquad \text{(B5)}$$

or

$$\partial_j F_1(u,\mathbf{x}) = C(u,\mathbf{x}) .$$

Thus, if F_1 is a multiple of the identity operator so that $G=H=0$ and $C=0$, we must have F_1 independent of $\mathbf{x}$. The simplest consistent choice is clearly $F_1=0$ which makes (B5) consistent in view of (B1). This choice requires that the left-hand side of (B1) commute with the complete set A_1, A_2, and Φ on the null plane, which is indeed satisfied.

Returning to the Furry picture, we can integrate (B1) formally with $f_1=0$:

$$a_u(x) = -\tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v-v')\sigma(v')dv' \left[\boldsymbol{\partial}\cdot\mathbf{a}(u,\mathbf{x},v') + \tfrac{1}{2}\int_{-\infty}^{\infty} \varepsilon(v'-v'') j_v(u,\mathbf{x},v'')dv'' \right] + \alpha(u,\mathbf{x}) . \qquad \text{(B6)}$$

This reduces to (A14) for $j_v=0$. If α is a c number, (B2)

[24] J. R. Klauder, H. Leutwyler, and L. Streit, Nuovo Cimento **66A**, 536 (1970).

and (B3) integrate to (A12) and to

$$[a_u,\partial_v'\phi^{\dagger\prime}]_{u=u'}=-\tfrac{1}{4}e\delta_2(\mathbf{x}-\mathbf{x}')\Big[\varepsilon(v-v')\phi^\dagger(x') + \int\varepsilon(v-v'')\varepsilon(v''-v')\sigma(v'')dv''\partial_v'\phi^\dagger(x')\Big]. \quad \text{(B7)}$$

APPENDIX C: TRANSFORMATION TO HEISENBERG PICTURE

We want to prove that the transformation (6.3), (6.4), and (6.6) with $P_{uF}^{(1)}$ given by (6.5) will lead from the Furry-picture field equations (6.1), (6.2), and (6.8) to the Heisenberg field equations (5.8)–(5.10). The commutation relations and the constraint (6.8) are given on the initial null surface. They remain invariant.

Now,

$$\begin{aligned}
2i[\partial_v\phi,P_{uF}^{(1)}]
&=2i\int\sigma(\bar{x}')d^3\bar{x}':\Big\{ie\mathbf{a}\cdot([\partial_v\phi,\phi'^\dagger]\overleftrightarrow{\boldsymbol{\partial}}'\phi')\\
&\quad -ie[\partial_v\phi,a_u'](\phi'^\dagger\overleftrightarrow{\partial}_v\phi')-iea_u'([\partial_v\phi,\phi'^\dagger]\overleftrightarrow{\partial}_v'\phi')\\
&\quad +e^2\mathbf{a}'\cdot(\mathbf{a}'+2\mathbf{A}_e')[\partial_v\phi,\phi'^\dagger]\phi'\\
&\quad -\tfrac{1}{4}\int\varepsilon(v'-v''')j_v'''dv'''\int\varepsilon(v'-v'')dv''\\
&\qquad\times[\partial_v\phi,j_v(u',\mathbf{x}',v'')]\Big\}:\\
&=2i\times\tfrac{1}{2}i:\{ie\mathbf{a}\cdot\boldsymbol{\partial}\phi+ie\boldsymbol{\partial}\cdot(\mathbf{a}\phi)-iea_u\partial_v\phi\\
&\quad -ie\partial_v(a_u\phi)+e^2\mathbf{a}\cdot(\mathbf{a}+2\mathbf{A}_e)\phi\}:. \quad \text{(C1)}
\end{aligned}$$

In the first equation, the second and the last term inside the curly brackets cancel because by (6.9) and (5.22) the last term is

$$\begin{aligned}
\int\sigma dv'(-\tfrac{1}{2})\int\varepsilon(v'-v''')dv''':j_v'''[\partial_v\phi,\partial_v'a_u']:\\
=\int\sigma dv':[\partial_v\phi,a_u']\tfrac{1}{2}\partial_v'\int\varepsilon(v'-v''')j_v'''dv'''\\
=-\int\sigma dv':[\partial_v\phi,a_u']j_v'':.
\end{aligned}$$

Various terms in (C1) now combine in view of (6.9), and with (6.7) and (6.1) one finds

$$\begin{aligned}
V2\partial_u\partial_v\Phi V^{-1}&=2\partial_u\partial_v\phi+2i[\partial_v\phi,P_{uF}^{(1)}]\\
&=(\boldsymbol{\partial}^2-m^2-2ie\mathbf{A}_e\cdot\boldsymbol{\partial}-e^2\mathbf{A}_e^2)\phi\\
&\quad -2ie:\mathbf{a}\cdot\boldsymbol{\partial}\phi:+2ie:a_u\partial_v\phi:\\
&\quad -e^2:\mathbf{a}\cdot(\mathbf{a}+2\mathbf{A}_e)\phi:+ie:(\partial_va_u-\boldsymbol{\partial}\cdot\mathbf{a})\phi:\\
&=:\{[\boldsymbol{\partial}-ie(\mathbf{a}+\mathbf{A}_e)]^2-m^2\}\phi:\\
&\qquad +ie:a_u\partial_v\phi:+ie:\partial_v(a_u\phi)::. \quad \text{(C2)}
\end{aligned}$$

When the $V\cdots V^{-1}$ operation is brought to the right-hand side, one has

$$(\mathbf{D}^2-m^2)\Phi-D_u\partial_v\Phi-\partial_v(D_u\Phi)=0 \quad \text{(C3)}$$

in agreement with (5.8). This is the Heisenberg-picture field equation for Φ. The normal ordering in (C2) gives a more precise meaning to (C3).

The derivation of the Maxwell equations in the Heisenberg picture proceeds along similar lines. We have first

$$\begin{aligned}
2i[\partial_va_j,P_{uF}^{(1)}]
&=2i\int d^3\bar{x}':\{ie[\partial_va_j,\mathbf{a}'](\phi'^\dagger\overleftrightarrow{\boldsymbol{\partial}}'\phi'-2ie(\mathbf{a}'+\mathbf{A}_e)\phi'^\dagger\phi')\\
&\qquad +[\partial_va_j,a_u']j_v'\}:\\
&=2i\Big\{-\tfrac{1}{2}ij_j-\tfrac{1}{4}i\partial_j\int\varepsilon(v-v')j_v(u,\mathbf{x},v')dv'\Big\}.
\end{aligned}$$

Again, the constraint equation (6.9) is necessary, and we obtain

$$V2\partial_u\partial_vA_jV^{-1}=2\partial_u\partial_va_j+j_j+\partial_j(\partial_va_u-\boldsymbol{\partial}\cdot\mathbf{a}).$$

The Maxwell equations in the Furry picture (6.1) make the right-hand side independent of ∂_u:

$$V2\partial_u\partial_vA_jV^{-1}=\boldsymbol{\partial}^2a_j+j_j+\partial_j(\partial_va_u-\boldsymbol{\partial}\cdot\mathbf{a}).$$

When the V operators are brought to the right-hand side, we find the desired equations (5.9). Again these are more precisely given as transformed Furry-picture equations than can be specified in the context of the Heisenberg picture.