gravitational theories. Three of these gauge choices have been used in obtaining isotropic spatial coordinates. Consider the fourth gauge transformation of the time coordinate:

$$x^{0\prime}=x^0+\xi\frac{\partial}{\partial x^0}\sum_i m_i|\mathbf{r}-\mathbf{r}_i|\,,$$

where ξ is the arbitrary gauge parameter. Under this gauge transformation the Greek-letter parameters of the metric given in Eqs. (1a)–(1c) transform as

$$\Delta\to\Delta+\tfrac{1}{4}\xi,\quad \Delta'\to\Delta'-\tfrac{1}{4}\xi,\quad \chi\to\chi+2\xi,$$
$$\alpha''\to\alpha''+\tfrac{1}{2}\xi,\quad \alpha'''\to\alpha'''+2\xi,$$

with all other parameters unaltered.

A variety of observable physical effects have been discussed in this paper. The coefficients of the various effects are listed below:

Eq. (2) $4\Delta+(8/3)\Delta'-\tfrac{1}{6}\chi-\tfrac{3}{2}\gamma-\tfrac{1}{3}\alpha'-\tfrac{2}{3}$,

Eq. (7) $\chi+4\beta+3\gamma-8\Delta$,

Eq. (7) $8\Delta'+\chi-\alpha'+2\beta-2$,

Eq. (20) $2\beta+\alpha'-\gamma-2$,

Eq. (20) $4\Delta+4\Delta'-2\gamma-2$,

Eq. (20) $2\alpha'-\chi-\gamma-8\Delta'$.

All these collections of parameters are invariant under the above gauge transformation.

PHYSICAL REVIEW D VOLUME 3, NUMBER 8 15 APRIL 1971

Scale Invariance of the Second Kind and the Brans-Dicke Scalar-Tensor Theory

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(Received 22 June 1970)

Invariance of the Brans-Dicke scalar-tensor theory of gravity under scale transformations of the second kind is discussed. It is shown that the requirement of scale invariance of the second kind leads to the unique value $-\frac{3}{2}$ of the adjustable parameter ω in the Brans-Dicke theory. It is further shown that, in order to obtain a consistent set of equations, matter must be coupled to the scalar-tensor field in such a way that the whole theory can be transformed into the usual Einstein tensor theory. This is accomplished by performing a space-time dependent scale change such that the transformed scalar field is everywhere a constant.

INTRODUCTION

The invariance of physical laws under a change of units was first discussed by Weyl.[1] In addition to the "metric" tensor $g_{\mu\nu}$, Weyl introduced a further geometrical object into the theory, a vector field ϕ_μ. This latter field served to characterize the change in length of a vector as it is transported from the point x^μ to the point $x^\mu+dx^\mu$. Weyl associated the field ϕ_μ with the vector potential of the electromagnetic field, hoping thereby to obtain a "unified" theory of gravitation and electromagnetism. The resultant theory was required to be invariant both with respect to arbitrary coordinate transformations and gauge or scale transformations. Under the latter group of transformations $g_{\mu\nu}$ was multiplied by an arbitrary space-time function while the gradient of this function was added to ϕ_μ. Since $g_{\mu\nu}$ and ϕ_μ transformed independently of each other, they did not together constitute an irreducible object under the combined coordinate-plus-gauge group. Furthermore, the total invariance group was just the direct product of the coordinate group and the gauge group. Consequently, Weyl's theory failed on two counts from being a unified theory. The chief draw back to Weyl's theory, however, was that it lead to fourth-order differential equations for the $g_{\mu\nu}$.

Although Weyl's theory has by and large been discarded, the idea of scale invariance has continued to play a role in physical thinking. Quite recently there has been renewed interest in scale invariance as an asymptotic symmetry in high-energy physics.[2] We will not be directly concerned here with this aspect of the theory but rather with its implications for the Brans-Dicke scalar-tensor theory of gravity.[3] Our main conclusion will be that the requirement of scale invariance of the second kind, i.e., invariance under arbitrary space-time dependent scale changes, leads to a form of the Brans-Dicke theory that is completely equivalent to the Einstein theory. Conversely, we show that the Einstein theory can be put in a form that is invariant under scale changes of the second kind.

BRANS-DICKE THEORY AND SCALE INVARIANCE

The equations of motion of matter and the gravitational field of the Brans-Dicke theory can be derived

[1] H. Weyl, Sitzber. Kgl. Preuss. Akad. Wiss., **495** (1918); Math. Z. **2**, 384 (1918); Ann. Phys. (Leipzig) **59**, 101 (1919); Phys. Z. **21**, 649 (1920).

[2] See, for example, C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) (to be published).

[3] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

from a variational principle with an action given by[4]

$$S=\int(\sqrt{(-g)})\left(\phi^2R-4\omega g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+\frac{16\pi}{c^4}L\phi\right)d^4x\,, \quad (1)$$

where R is the curvature scalar formed from $g_{\mu\nu}$ and ω is an adjustable dimensionless constant. For convenience we use here a scalar function which is the square root of Dicke's scalar function. Dicke takes L appearing in Eq. (1) to be the usual covariant Lagrangian density for whatever matter is present. Thus, for a system of massive particles moving along trajectories $x^\mu=z_i^\mu(\tau_i)$, L would be given by

$$L=\frac{1}{\sqrt{(-g)}}\sum_i\int m_ic^2[g_{\mu\nu}(x)\dot{z}_i^\mu\dot{z}_i^\nu]^{1/2}\delta^4(x-z_i)d\tau_i\,, \quad (2)$$

where $\dot{z}^\mu\equiv dz_i^\mu/d\tau_i$, and τ_i is an arbitrary monotonically increasing path parameter along the ith particle trajectory (not necessarily the proper time, as assumed by Dicke.) It is unnecessary to fix the path parameters further since in the form given L is invariant under an arbitrary change in path parameter.

Let us now consider the Brans-Dicke theory in the absence of matter. Since there are no dimensional parameters appearing in the action the resulting equations of motion should be invariant under a scale change. Indeed, if we make the substitutions

$$g_{\mu\nu}\rightarrow g'_{\mu\nu}=\lambda g_{\mu\nu}\,, \quad (3a)$$

$$\phi\rightarrow\phi'=\lambda^{-1/2}\phi\,, \quad (3b)$$

with λ a constant, we find that the action given by Eq. (1) is unchanged in form. This form invariance follows from the fact that under the scale change (3)

$$g'^{\mu\nu}=\lambda^{-1}g^{\mu\nu}\,, \quad (4a)$$

$$\sqrt{(-g')}=\lambda^2\sqrt{(-g)}\,, \quad (4b)$$

and

$$R'=\lambda^{-1}R\,. \quad (4c)$$

While the matter-free Brans-Dicke action is invariant under a constant scale transformation, it is not in general invariant under a position-dependent scale change. In a later paper Dicke[5] argues that "the equations of motion of matter must be invariant under a general coordinate-dependent transformation of units." Such an invariance we call scale invariance of the second kind. We do not intend here to enter into a detailed justification for such an invariance since probably no such justification can be given. Aside from the arguments presented by Dicke, we merely note that two observers separated by a spacelike interval would have no way of informing each other of the particular value of λ which they were going to use in carrying out the transformation (3).

Under a space-time dependent scale change Eq. (4c) is no longer valid. Rather we have

$$R=\lambda(R'+3\Box'\ln\lambda-\tfrac{3}{2}\lambda^{-2}\lambda_{,\mu}\lambda_{,\nu}g'^{\mu\nu})\,, \quad (5)$$

where

$$\Box'\ln\lambda=\frac{1}{\sqrt{(-g')}}[(\sqrt{(-g')})g'^{\mu\nu}\lambda^{-1}\lambda_{,\mu}]_{,\nu}\,. \quad (6)$$

Also,

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}=\lambda^2g'^{\mu\nu}\phi'_{,\mu}\phi'_{,\nu}+\lambda g'^{\mu\nu}\lambda_{,\mu}\phi'_{,\nu}\phi'+\tfrac{1}{4}g'^{\mu\nu}\lambda_{,\mu}\lambda_{,\nu}\phi'^2\,. \quad (7)$$

Therefore,

$$\begin{aligned}(\sqrt{(-g)})(R\phi^2-4\omega g^{\mu\nu}\phi_{,\mu}\phi_{,\nu})&=(\sqrt{(-g')})(R'\phi'^2-4\omega g'^{\mu\nu}\phi'_{,\mu}\phi'_{,\nu})-(\omega+\tfrac{3}{2})\\&\quad\times(\sqrt{(-g')})g'^{\mu\nu}\lambda^{-2}\lambda_{,\mu}\lambda_{,\nu}\phi'^2-(4\omega+6)(\sqrt{(-g')})\\&\quad\times g'^{\mu\nu}\lambda^{-1}\lambda_{,\mu}\phi'_{,\nu}\phi'+3[(\sqrt{(-g')})g'^{\mu\nu}\lambda^{-1}\lambda_{,\mu}\phi'^2]_{,\nu}\,. \quad (8)\end{aligned}$$

We see from this result that the action given by Eq. (1) (with $L=0$) is invariant under a scale change of the second kind only if

$$\omega=-\tfrac{3}{2}\,, \quad (9)$$

since the last term on the right-hand side of Eq. (8) is a complete divergence, and so does not contribute when S is varied to obtain the equations of motion.

COUPLING TO MATTER

If we attempt to couple the scalar-tensor field to matter in the way suggested by Dicke, a difficulty arises when $\omega=-\frac{3}{2}$. Deser[6] has pointed out that in this case the trace of the matter stress-energy tensor must vanish if the equations of motion are to be self-consistent. The equation of motion gotten by varying ϕ in the action S is

$$(\Box-\tfrac{1}{6}R)\phi=0\,, \quad (10)$$

while varying $g_{\mu\nu}$ yields

$$G_{\mu\nu}=6\phi^{-2}[T_{\mu\nu}(\phi)+T_{\mu\nu}(q)]\,, \quad (11)$$

where $T_{\mu\nu}(q)$ is the stress-energy tensor associated with the matter variables q. It is gotten in the usual way by varying the matter part of the action with respect to $g_{\mu\nu}$:

$$T^{\mu\nu}(q)\equiv\frac{2}{\sqrt{(-g)}}\frac{\delta S_M}{\delta g_{\mu\nu}}\,, \quad (12)$$

where

$$S_M=\frac{16\pi}{c^4}\int(\sqrt{(-g)})Ld^4x\,. \quad (13)$$

Normally the stress-energy tensor associated with the ϕ field would be gotten in the same way. However, since

[4] In our equations a subscript comma followed by a subscript denotes ordinary differentiation, a subscript semicolon followed by a subscript denotes covariant differentiation, the signature of the metric is $(-,+,+,+)$, and the Einstein summation convention is used throughout.

[5] R. H. Dicke, Phys. Rev. **125**, 2163 (1962).

[6] S. Deser, Ann. Phys. (N. Y.) **59**, 248 (1970).

there is no "free" gravitational action we take $T_{\mu\nu}(\phi)$ as given by Eq. (11), namely,

$$T_{\mu\nu}(\phi)=\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}+\tfrac{1}{6}(g_{\mu\nu}\Box-D_{\mu\nu})\phi^2, \quad (14)$$

where

$$D_{\mu\nu}\phi^2\equiv(\phi^2)_{;\mu\nu}. \quad (15)$$

It is perhaps interesting to note that in the flat-space limit $g_{\mu\nu}\to\eta_{\mu\nu}=\text{diag}(-1,+1,+1,+1)$, $T_{\mu\nu}(\phi)$ reduces to

$$T_{\mu\nu}(\phi)=\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}+\tfrac{1}{6}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\phi^2, \quad (16)$$

which is just the stress-energy tensor adopted for a scalar field by Callen, Coleman, and Jackiw.[2]

If we now take the trace of Eq. (11), we obtain

$$\phi(\Box-\tfrac{1}{6}R)\phi=T^\mu{}_\mu(q). \quad (17)$$

Use of Eq. (10) then yields the result that

$$T^\mu{}_\mu(q)=0. \quad (18)$$

While Eq. (18) holds for the electromagnetic field, it does not hold in general. Thus, for L given by Eq. (2) we have

$$T^{\mu\nu}(q)=\sum_i\int m_ic^2\frac{\dot{z}_i{}^\mu\dot{z}_i{}^\nu}{(g_{\rho\sigma}\dot{z}_i{}^\rho\dot{z}_i{}^\sigma)^{1/2}}\delta^4(x-z_i)d\tau_i, \quad (19)$$

and its trace is clearly not zero.

To get around this difficulty, Deser suggests that one add an invariance breaking term $\frac{1}{2}\mu^2\int(\sqrt{(-g)})\phi^2d^4x$ to the action, in which case Eq. (18) gets replaced by

$$\mu^2\phi^2=-T^\mu{}_\mu(q). \quad (20)$$

If we wish to preserve scale invariance, we must proceed differently. We can obtain a consistent set of equations if we replace $g_{\mu\nu}$ in the matter action everywhere by $g_{\mu\nu}\phi^2$ since this product is scale invariant. In this case Eq. (10) is replaced by

$$\phi(\Box-\tfrac{1}{6}R)\phi=T^\mu{}_\mu(q) \quad (21)$$

and is thus just the trace of Eq. (11).

It is not surprising that there is an algebraic identity between the equations of motion for ϕ and $g_{\mu\nu}$ in the present case since replacing $g_{\mu\nu}$ by $g_{\mu\nu}\phi^2$ in the matter action lead to a total action that is still scale invariant. Thus, in expression (2) for the matter Lagrangian, $z_i{}^\mu$ and τ_i are scale invariant while m_i is a dimensionless numerical constant. It is a general rule that whenever one has an invariance with respect to a group whose elements are specified by one or more space-time functions, the equations of motion will satisfy a number of "Bianchi"-type identities equal to the number of functions that appear in the group. Thus in the present case, in addition to the four Bianchi identities associated with the general coordinate invariance we have an additional identity associated with the scale invariance.

RELATION TO GENERAL RELATIVITY

With $g_{\mu\nu}\phi^2$ replacing $g_{\mu\nu}$ in the matter action and with $\omega=-\frac{3}{2}$, we have a scale-invariant theory. We are, therefore, free to impose a scale condition analogous to the coordinate conditions permitted by a coordinate-invariant theory. In particular we will impose the condition

$$\phi=\text{const}. \quad (22)$$

If then we take this constant equal to $G^{-1/2}$, where G is the gravitational constant, the action (1) reduces to that for the Einstein equations. Furthermore, the dimensionless constant m_i appearing in the matter action (2) gets replaced by the constant

$$\bar{m}_i=m_iG^{-1/2} \quad (23)$$

which (in units where the velocity of light and Planck's constant are dimensionless) has the dimensions of a reciprocal length. Thus the gravitational constant acts as a scale for all dimensional equalities. We note finally that we can include a scale-invariant term $\alpha(\sqrt{(-g)})\phi^4$ in the Brans-Dicke action which reduces to a cosmological term $(\alpha G^{-2})(\sqrt{(-g)})$ with the above choice of scale.

CONCLUSIONS

We see that the requirement of scale invariance of the second kind leads to a version of the Brans-Dicke theory that is equivalent to the usual Einstein theory. Of course, one could always drop the requirement of scale invariance of the second kind. But then it would no longer be possible to transform the Brans-Dicke theory to a form in which the usual Einstein equations hold and the scalar field appears as a "matter field," since to do so requires a position-dependent scale transformation. Thus the Brans-Dicke theory with $\omega\neq-\frac{3}{2}$ must be considered as an inequivalent alternate description of the gravitational interaction of matter.