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Dispersion Calculation of Isoscalar 3π and $K\bar{K}$ Electromagnetic Form Factors*

GORDON J. AUBRECHT, II†
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
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Using dispersion theory and the assumption that the three-pion cut is well represented by the $\rho\pi$ cut, the $I=0$, $J=1$ $\rho\pi$ and $K\bar{K}$ scattering amplitudes are calculated from a matrix integral equation, using only right-hand cuts, in an effective-range–type approximation. Subtraction constants are determined by the experimental masses and widths of the isoscalar vector resonances. There are no ghosts (spurious first-sheet poles) in the model. The $\rho\pi$ and $K\bar{K}$ isoscalar form factors are calculated and compared to the available e^+e^- colliding-beam data near the ω and ϕ masses. The isoscalar charge radii are also computed. The agreement of the model with the experimental values is satisfactory. An estimate $\Gamma(\rho\to\pi^0\gamma)\simeq 40$ keV is obtained (the decay has not yet been observed).

I. INTRODUCTION

THE burgeoning number of electron-positron colliding-beam experiments has lately spurred intense experimental and theoretical interest in form factors of low-mass particles (low compared to the nucleon). Historically, the nucleon form factors have generated the most interest[1] because they were easiest to measure. The investigation of lower-mass form factors is of interest from the standpoint of nucleon form factors as well, for those investigated so far are the ones of most importance in a dispersion calculation of nucleon form factors (i.e., the lowest-mass intermediate states).

It is the purpose of this work to investigate the form factors of the $\rho\pi$ transition and the K from the dispersion viewpoint. These form factors are important for the calculation of the isoscalar nucleon form factor. In the approximation considered here, the two form factors are coupled. The work presented may be useful in itself; however, it should perhaps be regarded as the first stage in a program to study isoscalar form factors completely. Various possible improvements might be incorporated in the future.

In Sec. II, the relevant amplitudes and form factors are obtained. Section III describes the determination of subtraction constants. The problem of possible ghosts is discussed in Sec. IV. Results are obtained and discussed in Sec. V, and the work summarized in Sec. VI.

II. DETERMINATION OF SCATTERING AMPLITUDES AND FORM FACTORS

In order to utilize the considerable intuition developed over the last decades on the two-body problem, and because of the difficulties inherent in three-body calculations, it is essential to approximate the three-pion states by ones in which two particles "stand in" for the three particles which are physically there. It seems a reasonable first approximation to insert for n final-state pions

$$[(n-1)\text{-pion resonance}]+(\text{pion}).$$

This final state must be symmetrized or antisymmetrized as the particular case requires.

The three-pion state will be represented by a ρ resonance plus a pion. As a consequence, the analytic structure of the amplitude is changed. Originally, a three-pion intermediate state implied that there was a cut in the complex plane, running from $9m_\pi^2$ to infinity[2] in the variable t, the mass-squared of the virtual photon. When the physics is changed to make the photon–three-pion vertex into a photon–ρ-meson–pion vertex, the new form factor has a cut which begins at $t=(m_\rho+m_\pi)^2$. This requires the ω particle, which appears as a resonance in the physical process, to instead become a bound-state pole.[3] This may not be such a violent approximation scheme, even though some information of the three-pion system is undoubtedly lost. Much simplicity is gained by this approach—and hopefully insight into the more complicated version of the problem as well. There are, moreover, several reasons to believe that this is in some ways not such a drastic step.

(i) One expects the complex $\rho\pi$ cut to dominate the discontinuity across the three-pion cut for small t.

(ii) The Gell-Mann–Sharp–Wagner (GMSW) model of ω decay,[4] seems to describe the physics of the decay successfully.

(iii) Schwarz's[5] derivation of the GMSW result confers upon the model a greater validity than one would have at first suspected.

(iv) The amplitude near threshold is suppressed by a factor q (unless there is a resonant state).

(v) Higher angular momentum states will be unim-

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† Present address: High Energy Physics Laboratory, Department of Physics, The Ohio State University, Columbus, Ohio 43210.

[1] One compendium of the older data is R. Hofstadter, *Nuclear and Nucleon Structure* (Benjamin, New York, 1963). More recent data appear, for example, in Richard Wilson, Phys. Today **22**, No. 1, 47 (1969).

[2] All masses will henceforth be given in terms of pion masses (unless explicitly stated otherwise); $9m_\pi^2=9$.

[3] The pole presents the usual difficulties in the definition of resonance height, for example.

[4] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[5] J. H. Schwarz, Phys. Rev. **175**, 1852 (1968).

portant for small t because of the threshold suppression factor. States of higher angular momentum are possible because the three particles are free to combine in any way so that the vector sum of their angular momenta is 1 (the spin of the photon is 1).

Another approximation made is the total neglect of left-hand cuts in the partial-wave amplitudes. These cuts are generated by "driving forces" in the crossed channels, and are an integral part of the solution of such a problem when calculated via the N/D method,[6] for example. Instead of doing a dynamical calculation along such lines, we shall use a coupled-channel effective-range approximation. The theory to be discussed in this paper is an effective-range theory. The two subtraction constants (subtraction constant matrices here) correspond to matrix generalizations of scattering length and effective range.[7]

For the case of spinless particles, the threshold behavior follows easily from the asymptotic properties of the Q_l's and the use of the Froissart-Gribov formula. The t-channel kinematic singularities of the $d_{nn'}{}^J$ are explicitly removed in the form of factors[8,9] ($n \geq |n'|$):

$$[\tfrac{1}{2}(1+z)]^{(n+n')/2}.$$

In a manner similar to the spinless case the partial-wave equation is written, and the analog of the Froissart-Gribov procedure executed. It is interesting to investigate the behavior as t approaches the threshold. Two parts contribute to this behavior: the explicit kinematic singularities in s, and the asymptotic behavior of the function $e_{nn'}{}^J$ (the analog to the Q_l) where[10] the notation of Collins and Squires[9] is used (also see Huang[11]).

$$\left(\frac{1+z}{2}\right)^{(n+n')/2}\left(\frac{1-z}{2}\right)^{(n-n')/2} e_{nn'}{}^J(z) \underset{z\to\infty}{\longrightarrow} z^{-J-1}z^n,$$

so that

$$a_J(t)=\text{const } f(q^2)(q^2)^{J+1}(q^2)^{-n}(q^2)^{-1} = \text{const } (q^2)^{J-n}f(q^2),$$

where $f(q^2)$ is finite at $q^2=0$. In the case of the $J=1$ $\pi\rho$ scattering amplitude, both J and n are one. Hence the factor containing $J-n$ does not contribute.

The last consideration concerns knowledge of the parities of all the objects. Namely, in this problem a negative-parity object must be formed from two objects of the same parity. These objects are therefore in a state of odd angular momentum ($l \geq 1$). Thus $f(q^2)$ is finally given by

$$f(q^2)=q^{2l}=q^2$$

(using the simplest assumption for l). This recovers the same answer that would have been obtained for spinless particles with $J=1$.

Consider the various $J=1$ partial-wave amplitudes defined

$$\begin{aligned}
A_{\rho\pi\to\rho\pi}(M'\to M)&\equiv f_{11}(M',0;\,M,0),\\
A_{\rho\pi\to K\bar{K}}(M\to 0)&\equiv f_{12}(M,0;\,0,0),\\
A_{K\bar{K}\to\rho\pi}(0\to M)&\equiv f_{21}(0,0;\,M,0),\\
A_{K\bar{K}\to K\bar{K}}&\equiv f_{22}.
\end{aligned}$$

With these definitions, analytically continued elastic unitarity can be written

$$\begin{aligned}
\text{disc}f_{11}(M,0;\,M',0)/2i&=\rho_{\rho\pi}(t)\theta(t-m_{\rho\pi}{}^2)\sum_{M''} f_{11}{}^*(M,0;\,M'',0)f_{11}(M'',0;\,M',0)\\
&\qquad+\rho_K(t)\theta(t-4m_K{}^2)f_{12}{}^*(M,0;\,0,0)f_{21}(0,0;\,M',0),\\
\text{disc}f_{12}(M,0;\,0,0)/2i&=\rho_{\rho\pi}(t)\theta(t-m_{\rho\pi}{}^2)\sum_{M''} f_{11}{}^*(M,0;\,M'',0)f_{12}(M'',0;\,0,0)\\
&\qquad+\rho_K(t)\theta(t-4m_K{}^2)f_{12}{}^*(M,0;\,0,0)f_{22},\\
\text{disc}f_{21}(0,0;\,M,0)/2i&=\rho_{\rho\pi}(t)\theta(t-4m_{\rho\pi}{}^2)\sum_{M''} f_{21}{}^*(0,0;\,M'',0)f_{11}(M'',0;\,M,0)\\
&\qquad+\rho_K(t)\theta(t-4m_K{}^2)f_{22}{}^*f_{21}(0,0;\,M,0),\\
\text{disc}f_{22}/2i&=\rho_{\rho\pi}(t)\theta(t-m_{\rho\pi}{}^2)\sum_{M''} f_{21}{}^*(0,0;\,M'',0)f_{12}(M'',0;\,0,0)+\rho_K(t)\theta(t-4m_K{}^2)f_{22}{}^*f_{22}.
\end{aligned} \tag{2.1}$$

Here $m_{\rho\pi}=m_\rho+m_\pi$, $\rho_{\rho\pi}(t)$ and $\rho_K(t)$ are kinematic factors to be defined below, and higher-mass intermediate states are neglected. The equations are coupled because, in the approximation of the 3π cut by a $\rho\pi$ cut, the threshold is at 42.8 (in units of $m_\pi{}^2$) whereas the two-K threshold is at 51.6. The thresholds are too close to allow the $K\bar{K}$ threshold to be called "far away." On the basis of subsequent calculations, we estimate that the contribution from higher states is less than 1.5%.

[6] This approximation eliminates several problems. The existence of the left-hand cut complicates the analysis of the asymptotic behavior at the pseudothreshold. Also missing are problems brought on by overlapping right and left cuts which arise when a crossed particle can have low mass (here the pion) and the particle is considered in the narrow-resonance approximation.

[7] The familiar meanings of scattering length and effective range do not necessarily correspond to those which would be expected in the one-channel problem. For example, in the multichannel case with several different thresholds, it is not clear about which threshold the expansion is made.

[8] Ling-Lie Wang, Phys. Rev. **142**, 1187 (1966); T. L. Trueman, *ibid.* **173**, 1684 (1968); J. Franklin, *ibid.* **170**, 1606 (1968).

[9] P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer, New York, 1968).

[10] E. J. Squires, *Complex Angular Momentum and Particle Physics* (Benjamin, New York, 1964).

[11] K. Huang, in Theories of Strong Interaction at High Energies, Brookhaven Summer School in Elementary Particle Physics, 1969 (unpublished).

A similar equation may be written for the form factor. The form factor is here a four-entry coulmn vector.

It can be shown (see, e.g., Goldberger and Watson,[12] Appendix E) that for the $\rho\pi$ scattering amplitude the partial-wave amplitude $|J,n_\rho,n_\pi\rangle$ can be written in terms of $|JLS\rangle$ amplitudes:

$$|J,\,n_\rho,\,n_\pi=0\rangle=\sum_{L,S}\left(\frac{2L+1}{2J+1}\right)^{1/2}\langle 1,0;\,M,0|Sn_\rho\rangle \times\langle L,S;\,0,n_\rho|Jn_\rho\rangle|JLS\rangle.$$

Using the standard definitions, and defining eigenstates of parity

$$|Jn_an_b\rangle\pm=(1/\sqrt{2})(|Jn_an_b\rangle\pm|J-n_a-n_b\rangle),$$

it is found that the only object with $J^P=1^-$ is the amplitude

$$|J=1,\,L=1,\,S=1\rangle_-.$$

Using the restrictions of parity and time-reversal invariance on the partial-wave amplitudes, this particular linear combination decouples in the unitarity equation. The $\rho\pi$ state is therefore described by only one amplitude (three *a priori*), which automatically is seen to have the correct threshold behavior.

The coupled matrix equations have thus been reduced from 4×4 to 2×2. Define $\mathfrak{M}(t)$ to be the matrix of amplitudes, and $F(t)$ to be the two-entry form-factor column vector. Then (2.1) and its analog form factors may be written

$$\mathrm{Im}\mathfrak{M}(t)=\mathfrak{M}^*(t)R(t)\mathfrak{M}(t)\,, \tag{2.2}$$

$$\mathrm{Im}F(t)=\mathfrak{M}^*(t)R(t)F(t)\,, \tag{2.3}$$

where $R(t)$ is the diagonal matrix:

$$R(t)=\begin{pmatrix}\rho_{\rho\pi}(t)\theta[t-(m_\rho+1)^2] & 0\\ 0 & \rho_K(t)\theta(t-4m_K^2)\end{pmatrix},$$

with

$$\rho_{\rho\pi}(t)=\{[t-(m_\rho+1)^2][t-(m_\rho-1)^2]\}^{3/2}/8t^2,$$

$$\rho_K(t)=\frac{[t-4m_K^2]^{3/2}}{8\sqrt{t}}=m_K^2\rho_\pi\left(\frac{t}{m_K^2}\right),$$

and m_ρ is taken to be real (in line with the preceding approximations).

If it is supposed that $\mathfrak{M}(t)$ has no left-hand cuts, a solution of

$$F(t)=\frac{1}{\pi}\int_{(m_\rho+1)^2}^{\infty}dt'\frac{\mathfrak{M}^*(t')R(t')F(t')}{t'-t}$$

(making the appropriate number of subtractions) is given by

$$F(t)=\mathfrak{M}(t)\begin{pmatrix}c_1\\ c_2\end{pmatrix}, \tag{2.4}$$

where c_1 and c_2 are arbitrary at present, but will be fixed later. Since $F(t)$ has no left-hand cuts, Eq. (2.4) clearly assumes that $\mathfrak{M}(t)$ has none either. Otherwise, the integral equation for $F(t)$ is no longer trivial.

Equation (2.2) may be rewritten

$$\mathrm{Im}\mathfrak{M}^{-1}(t)=-R(t)\,, \tag{2.5}$$

where $R(t)$ is defined above and $\mathfrak{M}^{-1}(t)$ is the matrix inverse of $\mathfrak{M}(t)$. This is the matrix analog of the well-known one-channel result. The solution to Eq. (2.5) follows from standard dispersion theory (making the appropriate number of subtractions)

$$\mathfrak{M}^{-1}(t)=\frac{1}{\pi}\int_{(m_\rho+1)^2}^{\infty}dt'\frac{\mathrm{Im}\mathfrak{M}^{-1}(t')}{t'-t}, \tag{2.6}$$

where it is assumed that $\mathfrak{M}$ has no left-hand cuts. At this point, there is some difficulty with subtractions (to be dealt with later) and ghosts. By ghosts is meant: The dispersion integral for $\mathfrak{M}^{-1}(t)$ is not precisely the same as that for $\mathfrak{M}(t)$. Possible zeros of $\mathfrak{M}^{-1}(t)$ become possible poles in $\mathfrak{M}(t)$. Thus, a zero on the first sheet of the complex t plane not ruled out by Eqs. (2.5) and (2.6) is (possibly) translated into an impermissible pole in $\mathfrak{M}(t)$. In the one-channel calculation, the justification for ignoring ghosts is the Omnès function.[13,14] Detailed arguments must be deferred until the solution of Eq. (2.6) (assuming meanwhile that there are no problems with ghosts) is established. $\rho(t)$ can be written $\rho(t)=q^{2l+1}/\sqrt{t}$. It is clear that two subtractions are necessary in Eq. (2.6) in order that the integrals converge. This is reasonable—the theory is better for low energy than high because more and more cuts come in to contribute to the phase shifts. Equation (2.6) has thus become

$$\mathfrak{M}^{-1}(t)=a+bt-\frac{t^2}{\pi}\int_{(m_\rho+1)^2}^{\infty}dt'\frac{R(t')}{t'^2(t'-t)}, \tag{2.7}$$

where a and b are 2×2 matrices. It must be emphasized again that Eq. (2.7) assumes that $\mathfrak{M}(t)$ has no left-hand cuts. a and b contain *a priori* eight independent parameters which must be determined from experiment: the eight matrix elements $a_{11},\ldots,b_{22}$. Time-reversal invariance requires the scattering matrix to be symmetric and reduces this number to six unknown constants. This was proved in the multichannel case by Bjorken and Nauenberg.[15] (If the N/D equations are used, these considerations do not of course imply that either $\mathfrak{N}$ or $\mathfrak{D}$ is symmetric.) More detail is given in Pilkuhn.[16]

In the notation adopted here,

$$\mathfrak{M}(t)=[\mathfrak{M}(t)]_{\mathrm{transpose}} \tag{2.8}$$

[12] M. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1965).

[13] R. Omnès, Nuovo Cimento **8**, 316 (1958).

[14] M. Jacob and G. F. Chew, *Strong Interaction Physics* (Benjamin, New York, 1964).

[15] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

[16] H. Pilkuhn, *The Interactions of Hadrons* (North-Holland, Amsterdam, 1967), Chap. 3.

or, in terms of the unknown matrices a and b, $a_{12}=a_{21}$, $b_{12}=b_{21}$. Consequently, there are but six unknowns: a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}.

The integrals involved in Eq. (2.7) are relatively straightforward (although tedious) to do. The solutions are

$$\mathfrak{M}^{-1}{}_{11}(t)=a_{11}+b_{11}t+A(t) \\ -i\rho_{\rho\pi}(t)\theta[t-(m_\rho+1)^2], \quad (2.9a)$$

$$\mathfrak{M}^{-1}{}_{22}(t)=a_{22}+b_{22}t+k(t)-i\rho_K(t)\theta(t-4m_K^2), \quad (2.9b)$$

with

$$A(t)=K_{\rho\pi}(t)+\frac{(m_\rho^2-1)^2}{16\pi t^2}\ln m_\rho^2 - \frac{(m_\rho^2-1)[m_\rho^2-1+\frac{3}{2}(m_\rho^2+1)\ln m_\rho^2]}{8\pi t},$$

$$K_{\rho\pi}(t)=\frac{\rho_{\rho\pi}(t)}{\pi}\ln\frac{[t-(m_\rho-1)^2]^{1/2}+[t-(m_\rho+1)^2]^{1/2}}{[t-(m_\rho-1)^2]^{1/2}-[t-(m_\rho+1)^2]^{1/2}},$$

and

$$k(t)=\frac{\rho_K(t)}{\pi}\ln\frac{\sqrt{t}+[t-4m_K^2]^{1/2}}{\sqrt{t}-[t-4m_K^2]^{1/2}}.$$

Although $A(t)$ may appear to diverge as $t\to 0$, it can be shown that

$$A(t)\xrightarrow[t\to 0]{} -\frac{5[(m_\rho^2)^2-1]+3[(m_\rho^2)^2+1]\ln m_\rho^2}{16\pi(m_\rho^2-1)},$$

$$k(t)\xrightarrow[t\to 0]{} -\frac{m_K^2}{\pi}.$$

III. DETERMINATION OF SUBTRACTION CONSTANTS

The fact that the physical three-pion cut begins at 9 (units of m_π^2), but that in the approximation considered here, where three pions are represented by the $\pi\rho$ combination, the cut begins at 42, means that the physical resonance at the ω mass-squared becomes a bound-state pole in this formalism. The present experimental data (world-averaged) found in Rosenfeld *et al.*[17] give the following branching ratios of ω and ϕ decays:

$\omega\to K\bar{K}$,	kinematically forbidden
$\omega\to$ three pions,	$87\pm4\%$
$\phi\to$ three pions (and $\pi\rho$),	$18.1\pm4.9\%$
$\phi\to K\bar{K}$,	$81.9\pm4.0\%$.

[17] N. Barash-Schmidt, A. Barbaro-Galtieri, C. Bricman, S. B. Derenzo, L. R. Price, A. Rittenberg, M. Roos, A. H. Rosenfeld, P. Söding, and C. G. Wohl, Rev. Mod. Phys. **42**, 87 (1970).

This makes possible a major hypothesis: That, so far as is possible, ϕ occurs only in the $K\bar{K}$ channel. This is reasonable in view of the data at hand. For, suppose all three-pion events detected are two-body $\pi\rho$ decays followed by ρ decaying to two pions. Letting P denote Lorentz-invariant phase space,

$$dP_{\rho\pi}/d\Omega=q_{\rho\pi}(t)/16\pi^2\sqrt{t}.$$

Similarly, in the case of the $K\bar{K}$ channel

$$dP_{K\bar{K}}/d\Omega=2q_{K\bar{K}}(t)/16\pi^2\sqrt{t},$$

where $q_{K\bar{K}}$ is computed using the average kaon mass. Evaluated at the ϕ mass, the ratio of available phase-space volumes is

$$P_{K\bar{K}}/P_{\rho\pi}=0.025 \quad (3.1)$$

whereas, from above, a computation implies

$$\text{rate}(\phi\to K\bar{K})/\text{rate}(\phi\to\rho\pi)=4.53. \quad (3.2)$$

If it can seem reasonable that there are no dynamical effects, then Eqs. (3.1) and (3.2) should more or less agree. In fact,

$$(\text{actual result})/(\text{naive-phase-space result})>150.$$

This makes it highly unlikely that effects of three-body phase space can adequately account for this ratio; there is probably a dynamical reason for the suppression. It is most significant that the decay of ϕ into $\rho\pi$ does not occur, considering the adequate phase space available.

In the quark model, the $SU(3)$ singlet and octet isosinglet are mixed. If $\tan^2\theta_V=\frac{1}{2}$, then ϕ is completely constituted of strange quarks and ω is made up only of nonstrange quarks. Expressing $\rho\pi$ and $K\bar{K}$ in quark terms, $\rho\pi$ is completely made of nonstrange quarks and $K\bar{K}$ has both strange and nonstrange parts. This forbids the decay $\phi\to\rho\pi$ but allows both $\omega\to\bar{K}K$ (forbidden kinematically) and $\phi\to K\bar{K}$. The small experimental rate for $\phi\to\rho\pi$ is taken to indicate a small deviation of θ_V from $\tan^{-1}1/\sqrt{2}$.

Other data determining the parameters are the masses and residues of the poles or resonances which are in the equations. To this end, the matrix of amplitudes must be more closely studied.

The solution $\mathfrak{M}^{-1}(t)$ was found. We are, however, interested in the matrix of amplitudes

$$\mathfrak{M}(t)=\text{adjoint}\,\mathfrak{M}^{-1}(t)/\det\mathfrak{M}^{-1}(t). \quad (3.3)$$

A pole occurs when $\det\mathfrak{M}^{-1}(t)=0$; alternatively, a resonance occurs for $\text{Re}\det\mathfrak{M}^{-1}(t)=0$. If there is a pole or resonance at $t=m_R^2$, then

$$\text{Re}\det\mathfrak{M}^{-1}(t)|_{m_R^2}=0.$$

The residues are determined by expansion about the resonance position. One finds

$$\mathfrak{M}(t)=\text{adjoint}\mathfrak{M}^{-1}(t)/\det\mathfrak{M}^{-1}(t)\,|_{m_R^2}=\{\text{adjoint}\mathfrak{M}^{-1}(t)/\text{Re}\det\mathfrak{M}^{-1}(t)+i\,\text{Im}\det\mathfrak{M}^{-1}(t)\}\,|_{m_R^2}$$
$$=\frac{\text{adjoint}\mathfrak{M}^{-1}(t)}{-(m_R^2-t)[d\,\text{Re}\det\mathfrak{M}^{-1}(t)/dt]\,|_{m_R^2}+\cdots+i\,\text{Im}\det\mathfrak{M}^{-1}(t)}\Bigg|_{t=m_R^2}$$
$$=\frac{\{\text{adjoint}\mathfrak{M}^{-1}(t)/[-d\,\text{Re}\det\mathfrak{M}^{-1}(t)/dt]\}\,|_{m_R^2}}{m_R^2-t+\cdots+i\{\text{Im}\det\mathfrak{M}^{-1}(t)/[-d\,\text{Re}\det\mathfrak{M}^{-1}(t)/dt]\}\,|_{m_R^2}}.$$

We expect that, to order Γ_R/m_R, the matrix of residues is given by

$$G=\frac{\text{Re adj}\mathfrak{M}^{-1}(t)}{[-d\,\text{Re}\det\mathfrak{M}^{-1}(t)/dt]}\Bigg|_{m_R^2}. \tag{3.4}$$

The equations for the various residues calculated from Eq. (3.4) can also be identified with experimental quantities.

The matrix G is defined to be real. Small imaginary parts have been neglected. G is expected to be real to order Γ/m for two reasons. First, the off-diagonal elements are real because of time-reversal invariance, and second, these residues correspond to inverse lifetimes (which are real[18]). We shall compute the size of the imaginary part presently.

The third condition arises from the "normalization" of the form factor. It is well known that, for small t, the form factor can be thought of as the Fourier transform of the charge distribution.[19,20] Thus an experiment in which there is no momentum transfer just measures the zeroth moment, i.e., the charge.

The form factor is usually thought of in terms of scattering from a virtual photon. The charge by which the form factor is normalized is the charge carried through by the external particle for a one-particle form factor. For form factors of dissimilar particles, $F(0)$ is not really a charge, but can be determined in principle by observing a decay such as $\rho\to\pi\gamma$. Such channels can be reached by analytic continuation in the four-momenta of the particles involved.

Spin-zero form factors are written in terms of the two available linearly independent vectors

$$(4k_1^0k_2^0)^{1/2}\langle k_1|J_\lambda|k_2\rangle=(k_1+k_2)_\lambda f+(k_1-k_2)_\lambda g.$$

For equal masses, g is zero by current conservation (the K form-factor case). Since there is a single parity-conserving antisymmetric matrix element for the decay of a photon into three pseudoscalar mesons, there is but a single 3π form factor. Similarly (neglecting charge labels) there is only one amplitude for $\gamma\to\rho\pi$, and thus there is only one $\rho\pi$ form factor.

The form factors are normalized by their values at $t=0$. It is trivial to evaluate in the $K\bar{K}$ case

$$F_K(t)=F_K{}^S(t)+\tau_3F_K{}^V(t),$$
$$F_{K^+}(t)=F_K{}^S(t)+F_K{}^V(t),$$
$$F_{K^0}(t)=F_K{}^S(t)-F_K{}^V(t),$$

so that

$$F_{K^+}(0)=1,\quad F_{K^0}(0)=0$$

implies

$$F_K{}^S(0)=F_K{}^V(0)=\tfrac{1}{2}.$$

The $\rho\pi$ case is slightly different. The isovector form factor is not considered because the three-pion isovector form factor is zero by G parity. The $\rho\pi$ form factor can be calculated from the width of the decay $\rho\to\pi$ +photon if it is known. Experimentally, a limit

$$|F_{\rho\pi}(0)|<0.98\pm0.12$$

can be put on the form factor from the upper limit on the branching ratio of that process (see the Rosenfeld tables[17]). In the Appendix the value of $\rho\pi$ transition form factor at zero momentum transfer is given. For the present, its value will be unspecified. Therefore,

$$F(0)=\begin{pmatrix}F_{\rho\pi}(0)\\ \tfrac{1}{2}\end{pmatrix}. \tag{3.5}$$

Recalling the discussion below (2.4), this means

$$F(t)=\mathfrak{M}(t)\mathfrak{M}^{-1}(0)\begin{pmatrix}F_{\rho\pi}(0)\\ \tfrac{1}{2}\end{pmatrix} \tag{3.6}$$

for the present case in which there are no left-hand cuts. Had $F(t)$ been defined as a row vector, Eq. (3.6) would hold for the transpose.

It is not quite obvious (for constant N matrices) that Eq. (3.6) involves only $\mathfrak{D}$. This must, however, be the case because F can have no left-hand cuts. Notice that by Eq. (2.8) $\mathfrak{M}(t)\mathfrak{M}^{-1}(0)$ can be written (where the subscript tr stands for the transpose)

$$\mathfrak{M}(t)\mathfrak{M}^{-1}(0)=[[\mathfrak{M}^{-1}(0)]_{\text{tr}}[\mathfrak{M}(t)]_{\text{tr}}]_{\text{tr}}$$
$$=[\mathfrak{M}^{-1}(0)\mathfrak{M}(t)]_{\text{tr}}$$
$$=[\mathfrak{D}(0)\mathfrak{N}^{-1}\mathfrak{N}\mathfrak{D}^{-1}(t)]_{\text{tr}}$$
$$=[\mathfrak{D}(0)\mathfrak{D}^{-1}(t)]_{\text{tr}}$$
$$=[\mathfrak{D}^{-1}(t)]_{\text{tr}}[\mathfrak{D}(0)]_{\text{tr}}.$$

[18] Goldberger and Watson, Ref. 12, Sec. 8.5; E. P. Wigner, Phys. Rev. **98**, 145 (1955); F. T. Smith, *ibid.* **118**, 349 (1960).
[19] G. Barton, *Dispersion Techniques in Field Theory* (Benjamin, New York, 1965).
[20] S. D. Drell and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford U. P., London, 1965).

This involves only the $\mathfrak{D}$ function. Moreover, it verifies the expectation (nurtured by the one-channel case) that $F(t)$ is essentially $\mathfrak{D}^{-1}(t)$.

The conditions on the solution are summarized in Table I. Conditions I and II can be directly compared to the results obtained in the one-dimensional case by Gell-Mann and Zachariasen[21] using the N/D approach in a scalar-resonance-model field theory. It will easily be seen that, in the one-channel problem, II reduces to

$$\gamma_R^{-1}=-d\,\mathrm{Re}\mathfrak{M}^{-1}(t)/dt|_{m_R^2}$$

and I to

$$\mathrm{Re}\mathfrak{M}^{-1}(t)|_{m_R^2}=0,$$

the same results as those of Ref. 21. The same remarks made above concerning the neglect of small imaginary parts and so on, of course, continue to hold.

The solution to Eq. (3.6) now reads

$$\mathfrak{M}^{-1}(t)=\begin{pmatrix} a_{11}+b_{11}t+A(t)-i\rho_{\rho\pi}(t)\theta[t-(m_\rho+1)^2] & a_{12}+b_{12}t \\ a_{12}+b_{12}t & a_{22}+b_{22}t+k(t)-i\rho_K(t)\theta(t-4m_K^2) \end{pmatrix}, \tag{3.7}$$

where $k(t)$ and $A(t)$ are the respective real parts of the several amplitudes and the θ functions are to remind us that there are no imaginary parts until the indicated energies squared (so that the symbols k, A may be reserved to the real parts of the amplitudes). The values of $A(t)$ and $k(t)$ in the various t regions are displayed in Table II and the $B(t)$ of the table is given by

$$B(t)=\frac{(m_\rho^2-1)^3\ln m_\rho^2}{16\pi t^2}-\frac{(m_\rho^2-1)[m_\rho^2-1+\frac{3}{2}(m_\rho^2+1)\ln m_\rho^2]}{8\pi t}.$$

The form (3.1) for $\mathfrak{M}^{-1}(t)$ implies

$$\mathfrak{M}(t)=\frac{\begin{pmatrix}\mathfrak{M}^{-1}{}_{22}(t) & -\mathfrak{M}^{-1}{}_{12}(t)\\ -\mathfrak{M}^{-1}{}_{12}(t) & \mathfrak{M}^{-1}{}_{11}(t)\end{pmatrix}}{\mathfrak{M}^{-1}{}_{11}(t)\mathfrak{M}^{-1}{}_{22}(t)-[\mathfrak{M}^{-1}{}_{12}(t)]^2}.$$

Consider $\mathfrak{M}(t)$ first at $t=m_\phi^2$. In view of the strong hypothesis made above—that the ϕ occurs only in the $K\bar{K}$ channel (condition IV)—it follows that

$$a_{12}+b_{12}m_\phi^2=0. \tag{3.8}$$

Since $m_\phi^2>(m_\rho+1)^2$ and $m_\phi^2>4m_K^2$, both $\mathfrak{M}^{-1}{}_{11}$ and $\mathfrak{M}^{-1}{}_{22}$ are complex; condition I states [taking Eq. (3.5) into account]

$$[a_{11}+b_{11}m_\phi^2+A(m_\phi^2)][a_{22}+b_{22}m_\phi^2+k(m_\phi^2)]=\rho_K(m_\phi^2)\rho_{\rho\pi}(m_\phi^2). \tag{3.9}$$

The right-hand side is very small, so the left-hand side is as well. This is good, since, if the resonance is to occur as far as possible in the $K\bar{K}$ channel, then $a_{22}+b_{22}m_\phi^2+k(m_\phi^2)$ should be small. At $t=m_\omega^2$ condition I is simply

$$[a_{11}+b_{11}m_\omega^2+A(m_\omega^2)][a_{22}+b_{22}m_\omega^2+k(m_\omega^2)]-(a_{12}+b_{12}m_\omega^2)^2=0. \tag{3.10}$$

Two independent ω coupling constants are determined:

$$-\gamma_{\omega\rho\pi}=[a_{22}+b_{22}m_\omega^2+k(m_\omega^2)]/d_1(m_\omega^2), \tag{3.11}$$

$$-\gamma_{\omega K\bar{K}}=[a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]/d_1(m_\omega^2), \tag{3.12}$$

with

$$d_1(m_\omega^2)=[b_{22}+k'(m_\omega^2)][a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]-2b_{12}(a_{12}+b_{12}m_\omega^2)+[b_{11}+A'(m_\omega^2)]\times[a_{22}+b_{22}m_\omega^2+k(m_\omega^2)].$$

TABLE I. Four conditions determining the six subtraction constants.

Condition I:	$\mathrm{Re}\,\det\mathfrak{M}^{-1}(t)\|_{m_R^2}=0.$
Condition II:	$G=\left.\dfrac{\mathrm{Re\ adjoint}\mathfrak{M}^{-1}(t)}{-d\,\mathrm{Re}\,\det\mathfrak{M}^{-1}(t)/dt}\right\|_{m_R^2}.$
Condition III:	$F(0)$ normalized correctly.
Condition IV:	ϕ is decoupled from $\rho\pi$.

TABLE II. Solutions $k(t)$ and $A(t)$ in various t regions.

$t\leq(m_\rho-1)^2$,

$$A(t)=B(t)-\{[(m_\rho+1)^2-t]^{3/2}[(m_\rho-1)^2-t]^{3/2}/8\pi t^2\}\times\ln[\{[(m_\rho+1)^2-t]^{1/2}+[(m_\rho-1)^2-t]^{1/2}\}/\{[(m_\rho+1)^2-t]^{1/2}-[(m_\rho-1)^2-t]^{1/2}\}].$$

$(m_\rho-1)^2\leqslant t\leqslant(m_\rho+1)^2$,

$$A(t)=B(t)+\{[(m_\rho+1)^2-t]^{3/2}[t-(m_\rho-1)^2]^{3/2}/8\pi t^2\}\times[2\tan^{-1}\{[(m_\rho+1)^2-t]/[t-(m_\rho-1)^2]\}^{1/2}-\pi].$$

$t\geqslant(m_\rho+1)^2$,

$$A(t)=B(t)+\{[t-(m_\rho+1)^2]^{3/2}[t-(m_\rho-1)^2]^{3/2}/8\pi t^2\}\times\ln[\{[t-(m_\rho-1)^2]^{1/2}+[t-(m_\rho+1)^2]^{1/2}\}/\{[t-(m_\rho-1)^2]^{1/2}-[t-(m_\rho+1)^2]^{1/2}\}].$$

$0\leqslant t\leqslant 4m_K^2$,

$$k(t)=[(4m_K^2-t)^{3/2}/8\pi\sqrt{t}]\{2\tan^{-1}[(4m_K^2-t)/t]^{1/2}-\pi\}.$$

$t\geqslant 4m_K^2$,

$$k(t)=[(t-4m_K^2)^{3/2}/8\pi\sqrt{t}]\ln[\{t^{1/2}+[t-4m_K^2]^{1/2}\}/\{t^{1/2}-[t-4m_K^2]^{1/2}\}]$$

with

$$B(t)=\frac{(m_\rho^2-1)^3\ln m_\rho^2}{16\pi t^2}-\frac{(m_\rho^2-1)[m_\rho^2-1+\frac{3}{2}(m_\rho^2+1)\ln m_\rho^2]}{8\pi t}.$$

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[21] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

TABLE III. Five equations in five unknowns to be solved.

$$f_1(\mathbf{e})=[a_{11}+m_\phi^2b_{11}+A(m_\phi^2)][a_{22}+m_\phi^2b_{22}+k(m_\phi^2)]-\rho_K(m_\phi^2)\rho_{\rho\pi}(m_\phi^2),$$
$$f_2(\mathbf{e})=[a_{11}+m_\omega^2b_{11}+A(m_\omega^2)][a_{22}+m_\omega^2b_{22}+k(m_\omega^2)]-a_{12}^2(1-m_\omega^2/m_\phi^2)^2,$$
$$f_3(\mathbf{e})=[a_{11}+m_\phi^2b_{11}+A(m_\phi^2)][b_{22}+k'(m_\phi^2)+\gamma_{\phi K\bar{K}}^{-1}]-d[\rho_K(t)\rho_{\rho\pi}(t)]/dt|m_\phi^2+[b_{11}+A'(m_\phi^2)]\times[a_{22}+m_\phi^2b_{22}+k(m_\phi^2)],$$
$$f_4(\mathbf{e})=[a_{22}+m_\omega^2b_{22}+k(m_\omega^2)][b_{11}+A'(m_\omega^2)+\gamma_{\omega\rho\pi}^{-1}]+\frac{2a_{12}(1-m_\omega^2/m_\phi^2)}{m_\phi^2}+[b_{22}+k'(m_\omega^2)]\times[a_{11}+m_\omega^2b_{11}+A(m_\omega^2)],$$
$$f_5(\mathbf{e})=[a_{11}+m_\omega^2b_{11}+A(m_\omega^2)][b_{22}+k'(m_\omega^2)+\gamma_{\omega K\bar{K}}^{-1}]+\frac{2a_{12}^2(1-m_\omega^2/m_\phi^2)}{m_\phi^2}+[b_{11}+A'(m_\omega^2)]\times[a_{22}+m_\omega^2b_{22}+k(m_\omega^2)].$$

Furthermore, from condition II, the decoupling of ϕ and $\rho\pi$, and the smallness of $a_{22}+b_{22}m_\phi^2+k(m_\phi^2)$, one obtains the result

$$-\gamma_{\phi K\bar{K}}=[a_{11}+b_{11}m_\phi^2+A(m_\phi^2)]/d_2(m_\phi^2), \quad (3.13)$$

where Eq. (3.3) was used and $d_2(m_\phi^2)$ is given by

$$d_2(m_\phi^2)=[b_{11}+A'(m_\phi^2)][a_{22}+b_{22}m_\phi^2+k(m_\phi^2)]-d[\rho_K(t)\rho_{\rho\pi}(t)]/dt|_{m_\phi^2}+[b_{22}+k'(m_\phi^2)]\times[a_{11}+b_{11}m_\phi^2+A(m_\phi^2)].$$

(The fact that $\omega\to K\bar{K}$ is kinematically forbidden does not imply that the coupling constant has any specific value.) The value of $\gamma_{\omega\rho\pi}$ is calculated by following the Gell-Mann, Sharp, and Wagner[4] calculation of $\Gamma(\omega\to 3\pi)$:

$$\Gamma(\omega\to 3\pi)=(3m_\omega/4m_\rho^2)\gamma_\rho\gamma_{\omega\rho\pi}I(m_\omega^2),$$

where $\gamma_\rho=m_\rho^2\Gamma(\rho\to\pi\pi)/q_\rho^3$ and $I(m_\omega^2)=0.105$ in units $m_\pi=1$. Thus

$$\gamma_{\omega\rho\pi}=\frac{4m_\rho^2\Gamma(\omega\to 3\pi)}{3m_\omega\gamma_\rho I(m_\omega^2)}=11.84.$$

[As an indication of the sensitivity of I to three-body phase space, we note that if a pion mass $m_\pi=140$ MeV is used instead of $m_\pi=138$ MeV, $\gamma_{\omega\rho\pi}=12.44$. See Sec. V for the definition of $I(m_\omega^2)$.]

The $\phi K\bar{K}$ coupling constant is computed from $\Gamma(\phi\to K\bar{K})$:

$$\gamma_{\phi K\bar{K}}=m_\phi^2\Gamma(\phi\to K\bar{K})/q_{K\bar{K}}^3=\tfrac{1}{3}g_{\phi K\bar{K}}^2/4\pi.$$

It might be noted that

$$\det G|_{m_\omega^2}\equiv 0$$

by definition [it is just Eq. (3.8)]. However, it will be seen that

$$\det G|_{m_\phi^2}\neq 0, \quad \operatorname{Re}\det G|_{m_\phi^2}=0,$$

because in the definition used here the very small imaginary parts of G were neglected in the transition from Eq. (3.4) to Eq. (3.9).

In order to determine the $\omega K\bar{K}$ coupling, additional information is required ($\omega\to K\bar{K}$ is kinematically forbidden). It is natural to select another dynamical model to obtain a prediction for this ratio—the most flexible model should be preferred. In $SU(3)$ with ω-ϕ mixing, one finds that

$$\gamma_\omega/\gamma_\phi=\tan^2\theta_V.$$

In the quark model discussed above, $\tan^2\theta_V=\frac{1}{2}$ (see, e.g., Kokkedee[22]). In any case the value of this ratio is uncertain up to factors $m_\phi^2/m_\omega^2\cong 2$; we use

$$\gamma_{\omega K\bar{K}}=\tfrac{1}{2}\gamma_{\phi K\bar{K}}.$$

At this juncture, there are six equations in the six unknowns $a_{11}, \ldots, b_{22}$. Equation (2.8) is trivially used to leave five equations in five unknowns. These are presented in Table III in terms of vectors

$$\mathbf{e}=\text{column}(a_{11},b_{11},a_{22},b_{22},a_{12}),$$
$$\mathbf{f}=\text{column}(f_1,f_2,f_3,f_4,f_5),$$

where f_i is defined in Table III. The solutions of these equations should be real and may be found by finding the solution to $\phi(\mathbf{e})=\mathbf{f}\cdot\mathbf{f}=0$. Alternatively, the equations may be used to eliminate certain variables, which reduces it to the problem of two equations in two unknowns (which is soluble graphically).

If the set of equations is to be solved simultaneously in five variables, it is useful to "have a handle" on the size of a_{11}, b_{11}, a_{22}, and b_{22}. These are easily found in analogy with the $\rho\to\pi\pi$ case[23,24] which has already been calculated, by considering the coupling $\gamma_{\omega K\bar{K}}$ to be turned off adiabatically. The ϕ meson is already shut off from the $\rho\pi$ channel, so in this limit the problem is simply that of two uncoupled one-channel equations. Solving

$$a_{11}=m_\omega^2[A'(m_\omega^2)+\gamma_{\omega\rho\pi}^{-1}]-A(m_\omega^2),$$
$$b_{11}=-[A'(m_\omega^2)+\gamma_{\omega\rho\pi}^{-1}],$$
$$a_{22}=m_\phi^2[k'(m_\phi^2)+\gamma_{\phi K\bar{K}}^{-1}]-k(m_\phi^2),$$
$$b_{22}=-[k'(m_\phi^2)+\gamma_{\phi K\bar{K}}^{-1}].$$

TABLE IV. One-channel estimates and solutions to $\phi(\mathbf{e})\cong 0$.

	One-channel	Physical solution: $\phi(\mathbf{e})=0$	$\phi(\mathbf{e})=0$	$\phi(\mathbf{e})=0.007$
a_{11}	11.71	13.37	16.98	8.30
b_{11}	−0.195	−0.220	−0.248	−0.089
a_{22}	23.27	23.77	118.29	2.62
b_{22}	−0.427	−0.436	−2.543	−0.047
a_{12}	···	±6.67	±27.32	0.0

[22] J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969).
[23] J. H. Schwarz, Phys. Rev. **179**, 1486 (1969).
[24] G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters **21**, 244 (1968).

These values can be used as the starting point, from which a certain solution results when the five equations in five unknowns are solved. With general starting places, one other solution and an almost-solution are found. The solutions are indicated in Table IV.

The one-channel calculation leads to the values listed immediately thereafter—the adiabaticity seems well justified *a posteriori*, as the changes in $a_{11}, \ldots, b_{22}$ are $<11\%$. There is an additional (physical) criterion for choosing this as the physical solution: The quantity

$$a_{22}+m_\phi^2+b_{22}+k(m_\phi^2)$$

should be very small if the $\rho\pi$ channel is to be decoupled from the ϕ. For the leftmost column of solutions, this quantity is -0.0133, while for the other solution it is -20.18 and the determination is thus doubly unambiguous. The sign ambiguity of a_{12} (see Table III) is unfortunately unavoidable at this stage (but will be resolved later).

TABLE V. The two $\phi(\mathrm{e})=0$ solutions of Table IV when $m_\pi=140$ MeV.

	Physical solution II	Nonphysical solution II
a_{11}	12.52	16.73
b_{11}	-0.208	-0.253
a_{22}	17.92	93.72
b_{22}	-0.339	-2.043
a_{12}	5.36	24.67

As a consistency check, the equations were also reduced to two equations in two unknowns and solved graphically. Then either

$$\begin{aligned} a_{11}+m_\omega^2 b_{11}+A(m_\omega^2)&=0\,,\\ a_{22}+m_\omega^2 b_{22}+k(m_\omega^2)&=0\,,\\ a_{12}&=0\,, \end{aligned} \tag{3.14}$$

or

$$\begin{aligned} &[a_{11}+m_\phi^2 b_{11}+A(m_\phi^2)](a_{11}+m_\phi^2 b_{11}+A(m_\omega^2)+(m_\phi^2-m_\omega^2)A'(m_\omega^2)\\ &\quad+(\gamma_{\omega K\bar K}/\gamma_{\phi K\bar K})\{[k(m_\omega^2)-k(m_\phi^2)]+(m_\phi^2-m_\omega^2)[k'(m_\omega^2)+\gamma_{\omega K\bar K}^{-1}]\})+(\gamma_{\omega K\bar K}/\gamma_{\omega\rho\pi})\rho_K(m_\phi^2)\rho_{\rho\pi}(m_\phi^2)=0\,,\\ &[a_{11}+m_\phi^2 b_{11}+A(m_\phi^2)]\{a_{11}+\tfrac12(m_\phi^2+m_\omega^2)b_{11}+A(m_\omega^2)+\tfrac12(m_\phi^2-m_\omega^2)A'(m_\omega^2)+\tfrac12(m_\phi^2-m_\omega^2)(\gamma_{\omega K\bar K}/\gamma_{\omega\rho\pi})\\ &\quad\times[k'(m_\omega^2)+\gamma_{\omega K\bar K}^{-1}-k'(m_\phi^2)-\gamma_{\phi K\bar K}^{-1}]\}-\tfrac12(m_\phi^2-m_\omega^2)[b_{11}+A'(m_\phi^2)]\{a_{11}+m_\phi^2 b_{11}+A(m_\phi^2)\\ &\quad+(m_\phi^2-m_\omega^2)A'(m_\omega^2)+(\gamma_{\omega K\bar K}/\gamma_{\omega\rho\pi})[k(m_\omega^2)-k(m_\phi^2)+(m_\phi^2-m_\omega^2)(k'(m_\omega^2)+\gamma_{\omega K\bar K}^{-1})]\}\\ &\qquad+\tfrac12(m_\phi^2-m_\omega^2)(\gamma_{\omega K\bar K}/\gamma_{\omega\rho\pi})(d/dt)[\rho_{\rho\pi}(t)\rho_K(t)]|_{m_\phi^2}=0\,. \end{aligned} \tag{3.15}$$

The solutions are shown in Fig. 1. One of the points at which Eq. (3.14) crosses the ellipse [given by the first of Eqs. (3.15)] is the relative minimum of Table IV. Physically, this is not a true solution because it requires that $a_{12}=0$ (i.e., that the equations uncouple[25]). Inspection verifies that the solutions are those indicated in Table IV.

Table V indicates the strong effect of electromagnetic particle splittings. The numbers of Table IV were calculated using average pion and kaon masses. The masses used in Table V are the charged-particle masses. This effect was noted by others; especially important in the context of this work is the remark of Schwarz[5] that it is more reasonable to use an average mass because in the transition all three charge states are emitted. At least it is one more indication that the numbers presented here are only qualitative, for the solutions are rather sensitive to small perturbations.

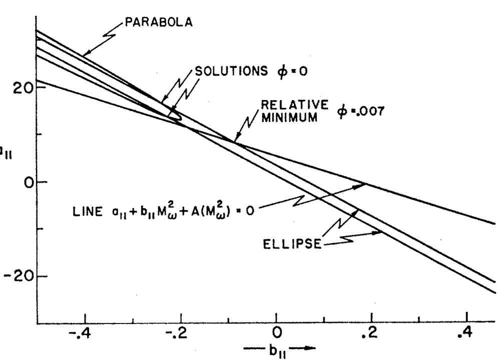

FIG. 1. Graphical solution to Eqs. (3.14) and (3.15).

[25] The whole point of the calculation is that the channels are coupled.

As a further check, the values of the preferred solution were recalculated using ratios $\gamma_\omega/\gamma_\phi$ modified by a factor of 2. The results of the calculation are presented in Table VI.

In regard to the assertions made just below Eq. (2.4) with the physical solution chosen, the size of the $\phi\rho\pi$ coupling is computed:

$$\gamma_{\phi\rho\pi}/\gamma_{\phi K\bar K}=0.006\,,$$

and one finds for the $\phi K\bar K$ coupling that

$$\mathrm{Im}G_{22}/\mathrm{Re}G_{22}\simeq 0.16=\tfrac16\,,$$

while for the $\phi\rho\pi$ coupling (which is not exactly zero because of finite-width effects),

$$\mathrm{Im}G_{11}/\mathrm{Re}G_{11}=6.3\,.$$

TABLE VI. Effect of changes in the ratio of $\gamma_\omega/\gamma_\phi$ on the physical solution to $\phi(\mathrm{e})=0$.

	$(\gamma_\omega/\gamma_\phi)/(\gamma_\omega/\gamma_\phi)_{SU(3)}=1$	2	$\frac12$
a_{11}	13.37	14.94	12.60
b_{11}	-0.220	-0.241	-0.209
a_{22}	23.77	23.87	23.73
b_{22}	-0.436	-0.438	-0.436
a_{12}	6.67	9.48	4.71

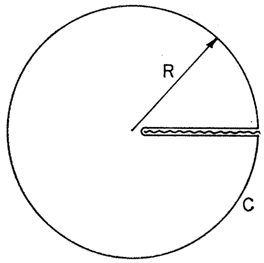

FIG. 2. Contour of integration for the determination of whether or not there are ghosts.

This, however, is to be expected because $\mathrm{Re}G_{11}\simeq 0$. It appears that the assumptions made are *a posteriori* reasonable.

IV. PROBLEM OF GHOSTS

The problem of ghosts has been ignored since the cut contribution to the integral depends on the values $a_{11}, \ldots, a_{12}$. Ghosts are sometimes found in one-channel problems,[26] especially in the Frye-Warnock procedure[27] in which ghosts arise in the attempt to replace coupled-channel equations by inelasticity functions as the coupling strength increases.

As far as we know, no such work has been done in the coupled-channel problem. In the present instance a well-known formula from complex-variable theory[28] giving the difference of the number of zeros and poles contained within a contour is of considerable use:

$$N_C = Z_C - P_C = \frac{1}{2\pi i}\oint_C dz \frac{\phi'(z)}{\phi(z)}.$$

The general contour of integration C is shown in Fig. 2, and

$$\begin{aligned} N_C &= \frac{1}{2\pi i}\oint_C dt \frac{[d\,\det\mathfrak{M}^{-1}(t)/dt]}{\det\mathfrak{M}^{-1}(t)}, \\ &= Z_C - P_C, \end{aligned}$$

where Z_C is the number of zeros in C and P_C is the number of poles in C. In the present situation,

$$\begin{aligned} N_C &= \frac{1}{2\pi i}\oint_C dt \frac{d\,\det\mathfrak{M}^{-1}(t)/dt}{\det\mathfrak{M}^{-1}(t)} \\ &= \frac{1}{2\pi i}\oint dt \frac{d}{dt}\ln\det\mathfrak{M}^{-1}(t) \\ &= \frac{1}{2\pi i}\oint_{C_R} dt \frac{d\,\det\mathfrak{M}^{-1}(t)/dt}{\det\mathfrak{M}^{-1}(t)} \\ &\quad + \frac{1}{2\pi i}\oint_{\mathrm{cut}} dt \frac{d}{dt}\ln\det\mathfrak{M}^{-1}(t), \end{aligned} \tag{4.1}$$

with C_R the contribution of the circle of radius R. The decomposition of Eq. (4.1) is the natural one to make in view of Fig. 2. It is easily shown that N_C is an integer:

$$N_C = \frac{1}{2\pi i}\oint_{\mathrm{cut}} dt \frac{d}{dt}\ln\det\mathfrak{M}^{-1}(t).$$

Let $\det\mathfrak{M}^{-1}(t) = V(t)e^{i\delta(t)}$ on the top of the cut. Then[29]

$$\begin{aligned} &\frac{1}{2\pi i}\left[\int_\infty^{t_0} + \int_{t_0}^\infty\right] dt \frac{d}{dt}\ln\det\mathfrak{M}^{-1}(t) \\ &= \frac{1}{2\pi i}\int_{t_0}^\infty dt \frac{d}{dt}\ln V(t)e^{i\delta(t)} \\ &\quad - \frac{1}{2\pi i}\int_{t_0}^\infty dt \frac{d}{dt}\ln V(t)e^{-i\delta(t)} \\ &= \frac{1}{2\pi i}\int_{t_0}^\infty dt \frac{d}{dt}\ln e^{2i\delta(t)} = \frac{1}{\pi}[\delta(\infty) - \delta(t_0)] \\ &= m, \end{aligned} \tag{4.2}$$

where m must be a function of the parameters which will enter later into the problem. One finds that m is an integer by examining $\tan\delta(t_0)$ and $\tan\delta(\infty)$:

$$\tan\delta(\infty) = \lim_{t\to\infty} \frac{\mathrm{Im}\,\det\mathfrak{M}^{-1}(t)}{\mathrm{Re}\,\det\mathfrak{M}^{-1}(t)}. \tag{4.3}$$

It will subsequently be shown that this limit is zero. Similarly

$$\tan\delta(t_0) = \left.\frac{\mathrm{Im}\,\det\mathfrak{M}^{-1}(t)}{\mathrm{Re}\,\det\mathfrak{M}^{-1}(t)}\right|_{t_0} = 0 \tag{4.4}$$

and Eqs. (4.3) and (4.4) together imply that

$$\delta(\infty) - \delta(t_0) = \pi(\text{integer}). \tag{4.5}$$

Define

$$W(t) \equiv \det\mathfrak{M}^{-1}(t).$$

The asymptotic behavior in t may be found by straightforward (although tedious) expansion. Then for the present case in which there are no left-hand cuts,

$$\begin{aligned} \mathrm{Re}W(t) &= (t/8\pi)^2[(\ln t)^2 + b_1(\ln t) + c_1] \\ &\qquad \times [1 + O(t^{-1})], \\ -\mathrm{Im}W(t) &= (t/8\pi)^2[2(\ln t) + b_1][1 + O(t^{-1})], \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} b_1 &= 8\pi(b_{11} + b_{22}) - \ln m_\rho - \ln m_K^2 = 20.84, \\ c_1 &= (8\pi)^2[b_{11}b_{22} - \tfrac{1}{64} - (a_{12}/m_\phi^2)^2] \\ &\quad - 8\pi[b_{11}\ln m_K^2 + b_{22}\ln m_\rho] + \ln m_\rho \ln m_K^2 = 79.33, \end{aligned}$$

and the numbers given use $m_\pi = 138$ MeV.

[26] M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 270 (1965); J. Finkelstein, Phys. Rev. **140**, B175 (1965); W. D. Heiss, Nuovo Cimento **52A**, 1201 (1967).

[27] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

[28] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U. P., New York, 1963), 4th ed., Chap. 6.

[29] $d\delta(t)/dt$ is continuous but not differentiable at the $K\bar{K}$ and $\rho\pi$ thresholds.

The integral over the contour is broken up as in Eq. (4.1). The integral over the cut is further divided:

$$N_{\rm cut}=(2\pi i)^{-1}\int_{C_R}dt\frac{W'(t)}{W(t)}+\pi^{-1}\int_{(m_\rho+1)^2}^{4m_K^2}dt\,I_1(t)+\pi^{-1}\int_{4m_K^2}^{\infty}dt\,I_2(t). \quad (4.7)$$

The cut integral is so divided because, for the region $(m_\rho+1)^2\leq t\leq 4m_K^2$,

$$W(t)=[a_{11}+b_{11}t+A(t)][a_{22}+b_{22}t+k(t)]-a_{12}^2(1-t/m_\phi^2)^2-i\rho_{\rho\pi}(t)[a_{22}+b_{22}t+k(t)] \quad (4.8)$$

and, in the region $t\geq 4m_K^2$,

$$W(t)=[a_{11}+b_{11}t+A(t)][a_{22}+b_{22}t+k(t)]-a_{12}^2(1-t/m_\phi^2)^2-i\{\rho_{\rho\pi}(t)[a_{22}+b_{22}t+k(t)]+\rho_K(t)[a_{11}+b_{11}t+A(t)]\}. \quad (4.9)$$

Furthermore, $I_1(t)$ is defined by

$$I_1(t)=-g(t)/h(t),$$

with

$$\begin{aligned}g(t)=&[a_{22}+b_{22}t+k(t)]\{\rho_{\rho\pi}(t)[b_{11}+A'(t)]\\&-\rho_{\rho\pi}'(t)[a_{11}+b_{11}t+A(t)]\}+a_{12}(1-t/m_\phi^2)\\&\times\{(2/m_\phi^2)\rho_{\rho\pi}(t)[a_{22}+b_{22}t+k(t)]\\&+(1-t/m_\phi^2)\rho_{\rho\pi}(t)[b_{22}+k'(t)]+\rho_{\rho\pi}'(t)(1-t/m_\phi^2)\\&\times[a_{22}+b_{22}t+k(t)]\},\\h(t)=&[a_{11}+b_{11}t+A(t)][a_{22}+b_{22}t+k(t)]\\&-a_{12}^2(1-t/m_\phi^2)^2+\rho_{\rho\pi}(t)[a_{22}+b_{22}t+k(t)].\end{aligned}$$

The integral for $I_2(t)$ is analogously defined in the region $t>4m_K^2$ using (4.9).

Equation (4.6) defines $W'(t)/W(t)$ for large t. First, the assertion below (4.3) is verified:

$$\begin{aligned}\tan\delta(\infty)&=\lim_{t\to\infty}\frac{\operatorname{Im}\det\mathfrak{M}^{-1}(t)}{\operatorname{Re}\det\mathfrak{M}^{-1}(t)}\\&=\lim_{t\to\infty}-\frac{\pi[2(\ln t)+b_1][1+O(t^{-1})]}{[(\ln t)^2+b_1(\ln t)+c_1][1+O(t^{-1})]}\\&=-2\pi\lim_{t\to\infty}\frac{1}{\ln t}=0.\end{aligned} \quad (4.10)$$

In order to compute $W'(t)/W(t)$, a Phragmèn-Lindelöf theorem (see, e.g., Titchmarsh[30] or Kanazawa and Sugawara[31]) is necessary: If some function of t is bounded by a power of $|t|$ as $t\to\infty$, then it has the same behavior in any direction in the complex plane. In consequence, (4.6) implies

$$W'(t)/W(t)\to 2/t$$

and, therefore,

$$\frac{1}{2\pi i}\int_{C_R}dt\frac{W'(t)}{W(t)}\to\lim_{R\to\infty}\frac{1}{2\pi i}\int_0^{2\pi}Re^{i\theta}id\theta\frac{2}{Re^{i\theta}}=2. \quad (4.11)$$

The integral of $I_2(t)$ presents a difficult problem. One can show that for large t (defining $x=\ln t$), $I_2(t)$ is given by

$$I_2(t)=-(2\pi/t)\zeta(x)+R(x)O(t^{-2}), \quad (4.12)$$

with

$$\zeta(x)=\frac{x^2+b_1x-c_1+\frac{1}{2}b_1^2}{x^4+2b_1x^3+(b_1^2+2c_1+4\pi^2)x^2+2b_1(c_1+2\pi^2)x+\pi^2b_1^2+c_1^2},$$

and $R(x)$ is a ratio of fourth-order polynomials in x. The cancellation of third-order polynomials in $\zeta(x)$ is a requisite for the convergence of the integral $I_2(t)$ [i.e., it must hold if (4.10) is to be true]. The x behavior of $\zeta(x)$ for extremely large x is the slowest integer power convergence possible. It is thus interesting to do the integral analytically in the asymptotic region described by Eq. (4.12). The cutoff chosen is $T=2^{20}$, for which

$$|I_2(t)+2\pi\zeta(x)/t|<0.03\%.$$

It is then possible to write

$$-\int_T^\infty dt\,2\pi\frac{\zeta(x)}{t}=-2\pi\int_{\ln T}^\infty dx\,\zeta(x)=-2\pi\int_{\ln T}^\infty dx\,|n(x)|^2,$$

where

$$n(x)=\frac{x+\frac{1}{2}b_1+i[(\frac{1}{2}b_1)^2-c_1]^{1/2}}{[x+\frac{1}{2}b_1+J(b_1,c_1)+i\pi][x+\frac{1}{2}b_1-J(b_1,c_1)+i\pi]},$$

where

$$J(b_1,c_1)=[(\tfrac{1}{2}b_1)^2-\pi^2-c_1]^{1/2}.$$

With this result, the integral of $I_2(t)$ from T to infinity is found to give

$$-2\pi\int_T^\infty dt\,I_2(t)=\tan^{-1}\frac{\pi}{\ln T+\frac{1}{2}b_1-J(b_1,c_1)}+\tan^{-1}\frac{\pi}{\ln T+\frac{1}{2}b_1+J(b_1,c_1)}.$$

Table VII lists the numerical results for the integration along the cut for both the true and spurious solutions to the simultaneous equations solved in the Sec. III.

[30] E. C. Titchmarsh, *The Theory of Functions* (Oxford U. P., Oxford, 1958).

[31] A. Kanazawa and M. Sugawara, Phys. Rev. **123**, 1895 (1961).

TABLE VII. Contributions from the cut integral to the number of zeros minus the number of poles in the physical amplitude $\mathfrak{M}^{-1}(t)$.

Region	Solution	Nonsolution
$(m_\rho+1)^2 \leqslant t \leqslant 4m_K^2$	$-0.038\,729\,6$	$+0.120\,656\,8$
$4m_K^2 \leqslant t \leqslant T$	$-0.250\,188\,2$	$-0.032\,804\,1$
$t \geqslant T$	$-0.711\,083\,0$	$-1.087\,920\,8$
Therefore, the entire cut contribution to the integral is given by		
	$-1.000\,001$	$-1.000\,068$

For $m_\pi=140$ MeV, the numbers are the same up to the final digits. The function $d\delta/dt$ is shown in the resonance region in Fig. 3.

The final result, then, is that the number of zeros minus the number of poles is given by

$$N_C=2-1=1=Z_C-P_C.$$

Since one zero is the ω bound state and there are obviously no poles in $\mathfrak{M}^{-1}$, the result implies that there are no ghost problems in the model.

V. RESULTS AND DISCUSSION

The three-pion cross section determines the $\rho\pi$ form factor through the Orsay[32] and Novosibirsk[33] data:

$$\sigma(e^+e^-\to\omega\to3\pi)=1.76\pm0.32\ \mu\text{b},$$
$$\sigma(e^+e^-\to\phi\to3\pi)=1.01\pm0.21\ \mu\text{b}.$$

In this paper $\gamma\to3\pi$ is taken actually to be the process $\gamma\to\rho\pi$, $\rho\to\pi\pi$. (This is just the model of GMSW[4] extended to all t for which an effective-range approximation is useful. The GMSW model is reasonable because it takes into account the variation over wide ranges of the effective masses of the several pion pairs.) Neglecting the electron mass compared to others in the problem, one can obtain

$$|F_{\rho\pi}(t)|^2=\frac{2\pi m_\rho^2 t\sigma(e^+e^-\to3\pi;\,t)}{\alpha\gamma_\rho I(t)},$$

where $I(t)$, the integral over the Dalitz-plot region, can be put into the form

$$I(t)=\left[\frac{(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)}{2\sqrt{t}}\right]^4 \times\int_0^1 d\alpha\int_0^1 d\beta(1-\beta^2)h^3(\alpha)U(\alpha,\beta)$$

with

$$h^2(\alpha)\equiv\alpha(1-\alpha)\times\frac{(\sqrt{t}-m_\pi)(\sqrt{t}+3m_\pi)-(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)\alpha}{4m_\pi^2+(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)\alpha}$$

and

$$\begin{aligned}U(\alpha,\beta)\equiv{}&[4m_\pi^2+(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)\alpha]\\&\times|\{2m_\rho^2-im_\rho\Gamma_\rho-t+m_\pi^2+(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)\\&\times[\alpha+\beta h(\alpha)]\}^{-1}+\{2m_\rho^2-im_\rho\Gamma_\rho-t+m_\pi^2\\&+(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)[\alpha-\beta h(\alpha)]\}^{-1}+[2m_\rho^2-im_\rho\Gamma_\rho\\&-8m_\pi^2-2(\sqrt{t}+m_\pi)(\sqrt{t}-3m_\pi)\alpha]^{-1}|^2.\end{aligned}$$

$I(t)$ is sensitive to the pion mass chosen. One finds, for $m_\pi=138$ (140) MeV, that $I(m_\omega^2)=0.105$ (0.097), and $I(m_\phi^2)=4.514$ (4.306). The value of $I(m_\omega^2)$ was used in determining $\gamma_{\omega\rho\pi}$ in Sec. III.

The K form factors are determined in much the same

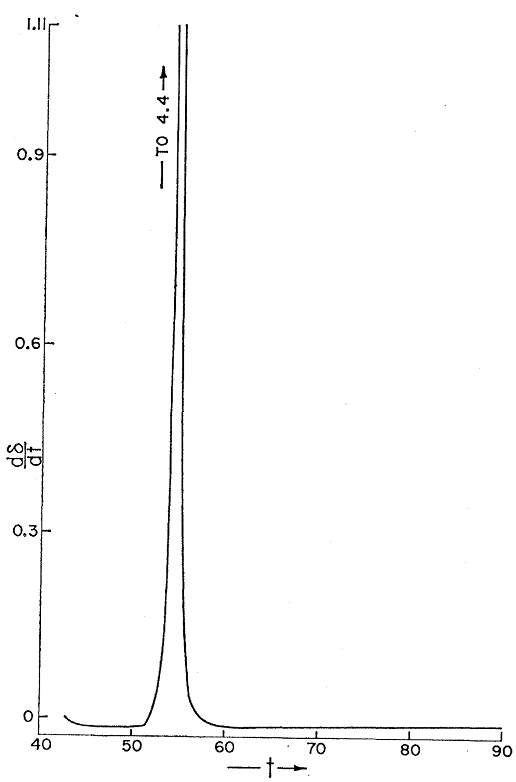

FIG. 3. $d\delta/dt$ in the resonance region. In Figs. 3–7, t is given in units in which $m_\pi^2=1$.

[32] J. B. Augustin, D. Benaksas, J. Buon, V. Gracco, J. Haissinski, D. Lalanne, F. Laplanche, J. Le Francois, P. Lehmann, P. Martin, F. Rumpf, and E. Silva, Phys. Letters **28B**, 513 (1969); J. E. Augustin, J. C. Bizot, J. Buon, B. Delcourt, J. Haissinski, J. Jeanjean, D. Lalanne, P. C. Marin, H. Nguyen Ngoc, J. Perez-y-Jorba, F. Richard, F. Rumpf, and D. Treille, *ibid.* **28B**, 517 (1969); J. Haissinski, in *Proceedings of the Conference on ππ and Kπ Interactions* (Argonne National Laboratory, 1969), p. 373; F. Laplanche, thesis, Production du meson vectoriel ω par annihilations electrons-positrons (detection du mode $\pi^+\pi^-\pi^0$). Indication d'une interference ρ-ω dans le mode $\pi^+\pi^-$, University of Paris, 1970 (unpublished); J. Perez-y-Jorba, in Proceedings of the Daresbury Conference on High Energy Photon and Electron Interactions, Daresbury, England, 1970, p. 213 (unpublished); J. C. Bizot, J. Buon, Y. Chatelus, J. Jeanjean, D. Lalanne, H. Nguyen Ngoc, J. Perez-y-Jorba, P. Petroff, F. Richard, F. Rumpf, and D. Treille, Phys. Letters **32B**, 416 (1970).

[33] V. A. Sidorov, in Proceedings of the Daresbury Conference on High Energy Photon and Electron Interactions, Daresbury, England, 1970, p. 227 (unpublished).

fashion as the calculation above. One finds

$$|F_K(t)|^2=\frac{3t^{5/2}\sigma(e^+e^-\to K\bar{K};\,t)}{\alpha p_K{}^3(t)},\tag{5.1}$$

where

$$p_K(t)=\tfrac{1}{2}(t-4m_K{}^2)^{1/2},$$

α is the fine-structure constant, and the Orsay data[32] give

$$\sigma(e^+e^-\to K\bar{K};\,m_\phi{}^2)=3.88\pm0.59\ \mu\text{b}.$$

Evaluating (5.1) at the ϕ mass determines $|F_K(m_\phi{}^2)|^2$. The numbers resulting from the calculations described are displayed in Table VIII along with the computed theoretical values.

There is some inconvenience in the calculation of the form factors at the ω mass because there is a pole there.

TABLE VIII. Calculated form factors compared with values extracted from cross-section data (Refs. 32 and 33).

Form factor	Calculated $F_{\rho\pi}(0)=0.42\pm0.06$	Extracted from experiment
$\lvert F_{\rho\pi}(m_\phi{}^2)\rvert^2$	0.63 ± 0.16	$0.89\pm\ 0.20$
$\lvert F_{\rho\pi}(m_\omega{}^2)\rvert^2$	490 ± 230	$462\pm\ 117$
$\lvert F_K(m_\phi{}^2)\rvert^2$	6930 ± 790	$10\,770\pm1640$

One must have recourse to the usual method of dealing with δ functions: The area under the curve remains constant as the peak gets higher and narrower. This means

$$|F(m^2)|^2=(2/m_\omega\Gamma_\omega)^2|[\text{residue of }F(t)]|_{m_\omega{}^2}|^2.\tag{5.2}$$

Using Eq. (3.6), the form factors are given by

$$F_{\rho\pi}(t)=\frac{p_1[a_{22}+b_{22}t+k(t)-i\rho_K(t)\theta(t-4m^2)]-p_2a_{12}(1-t/m_\phi{}^2)}{\det\mathfrak{M}^{-1}(t)}\tag{5.3}$$

and

$$F_K(t)=\frac{p_2\{a_{11}+b_{11}t+A(t)-i\rho_{\rho\pi}(t)\theta[t-(m_\rho+1)^2]\}-p_1a_{12}(1-t/m_\phi{}^2)}{\det\mathfrak{M}^{-1}(t)},\tag{5.4}$$

where

$$p_1=a_{12}F_K(0)+[a_{11}+A(0)]F_{\rho\pi}(0),\quad p_2=a_{12}F_{\rho\pi}(0)+[a_{22}+k(0)]F_K(0).$$

The value of $F_{\rho\pi}(0)$ is obtained from the Appendix. It is taken to be 0.42. The form factors computed from Eqs. (5.3) and (5.4) are displayed in Figs. 4–7. The form-factor values calculated are all reasonable. Even the kaon form factor squared at the ϕ is all right (it is calculated in the approximation of no interchannel coupling—the same calculation as elsewhere[23,24,34] performed for the pion—the result is about 13 000).

The charge radius is defined generally as

$$\langle r^2\rangle=6\left.\frac{dF(t)}{dt}\right|_{t=0}.\tag{5.5}$$

Putting Eqs. (5.3) and (5.4) into (5.5), we obtain

$$\langle r_{\rho\pi}{}^2\rangle=6\frac{[p_1-a_{11}-A(0)][b_{22}+k'(0)]-[a_{22}+k(0)][b_{11}+A'(0)]+(p_2-2a_{12})a_{12}/m_\phi{}^2}{[a_{11}+A(0)][a_{22}+k(0)]-a_{12}{}^2},\tag{5.6}$$

$$\langle r_K{}^2\rangle=6\frac{[p_2-a_{22}-k(0)][b_{11}+A'(0)]-[a_{11}+A(0)][b_{22}+k'(0)]+(p_1-2a_{12})a_{12}/m_\phi{}^2}{[a_{11}+A(0)][a_{22}+k(0)]-a_{12}{}^2}.\tag{5.7}$$

Then

$$\langle r_{\rho\pi}{}^2\rangle=0.251\ \text{F}^2.$$

Similarly, the K isoscalar charge radius is given by

$$\langle r_K{}^2\rangle=0.260\ \text{F}^2.$$

The numerical result is relatively insensitive to the value of $F_{\rho\pi}(0)$ chosen for a wide range of $F_{\rho\pi}(0)$'s. The coupling of the channels thus seems to lead to roughly equal charge radii in the two channels. These values may be compared to an isovector radius of approximately 0.4. Here, the minus sign is chosen for a_{12}, in agreement with the Appendix. We did not really expect the $\rho\pi$ form factor at the ϕ mass to be accurate. The important thing is that it be small (for the ϕ is to be decoupled from $\rho\pi$).

Conditions are somewhat sensitive to the size of the coupling in the off-diagonal channel at the ϕ mass. Several quantities are small there and respond to slight variations in almost everything. The approximately 10% variations in the a's and b's were demonstrated above.

The agreement of the experimental numbers with the numbers calculated from this model is not spectacular. Nevertheless, the numbers are of approximately the correct size. This suggests that it may be worthwhile to extend the approach used here to include more infor-

[34] G. J. Aubrecht, II, Phys. Rev. D 1, 284 (1970).

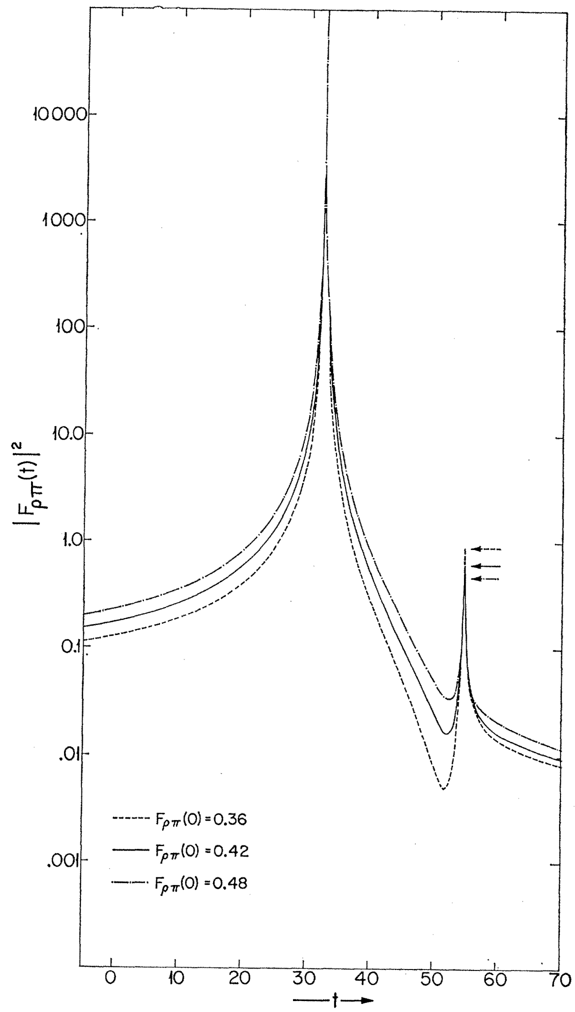

FIG. 4. $|F_{\rho\pi}(t)|^2$ computed from (5.3).

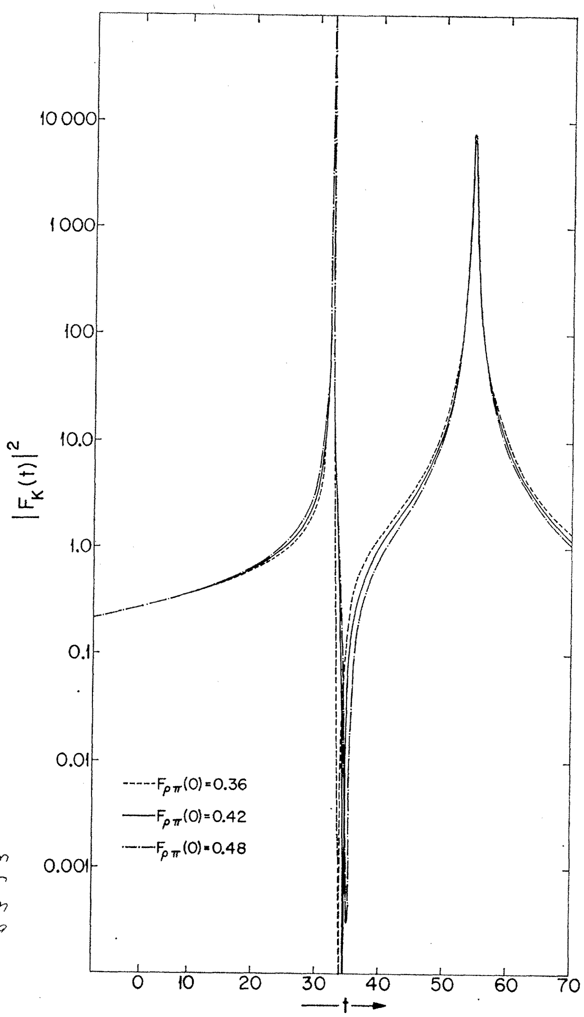

FIG. 5. $|F_K(t)|^2$ computed from (5.4).

mation which is available (for example, left-hand cuts may be considered as indicated below). Further work on the problem is obviously required so that the experimental results can eventually be calculated. We feel that this paper is a first step in a program to completely describe these form factors. As more (and more accurate) data become available, this model and its descendants can be more rigidly constrained. In particular, it will be extremely interesting to have a good experimental determination of $F_{\rho\pi}(0)$. When this is available, it can be compared with the value calculated in the Appendix. If it is allowed to vary over the range $0.36\leq F_{\rho\pi}(0)\leq 0.48$, the variation induced in the calculated form factors is

$$6160\leq |F_K(m_\phi^2)|^2\leq 7735,$$

$$0.48\leq |F_{\rho\pi}(m_\phi^2)|^2\leq 0.80,$$

$$720\geq |F_{\rho\pi}(m_\omega^2)|^2\geq 300.$$

The inclusion of various modifications will alter the results given. In particular, this model with the left-hand cut corrections put in via the N/D method might sufficiently alter the coupling so that the form factors at the ϕ mass could emerge closer to experiment. In the formulation of the coupled-channel problem with left-hand cuts, the N/D formalism may be used (cursive letters refer to matrices):

$$\mathcal{M}(t)=\mathcal{N}(t)\mathcal{D}^{-1}(t), \tag{5.8}$$

so that $\mathcal{N}(t)$ has only the left-hand and $\mathcal{D}(t)$ only the right-hand cut. This formal decomposition is well established.[35] The N/D method is justified by Wiener-Hopf techniques.[36,37] On the right-hand cut, (5.8) is

[35] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[36] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. I.
[37] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Benjamin, New York, 1964).

replaced as follows:

$$\mathfrak{D}(t)=\mathfrak{M}^{-1}(t)\mathfrak{N}(t)\,,$$

$$\begin{aligned}\mathrm{Im}\mathfrak{D}(t)&=\mathrm{Im}[\mathfrak{M}^{-1}(t)\mathfrak{N}(t)]=[\mathrm{Im}\mathfrak{M}^{-1}(t)]\mathfrak{N}(t)\\&=R(t)\mathfrak{N}(t)\,,\end{aligned}\tag{5.9}$$

$$\mathfrak{D}(t)=\alpha+\beta t-\frac{t^2}{\pi}\int_{t_0}^{\infty}dt'\frac{R(t')\mathfrak{N}(t')}{t'^2(t'-t)},$$

while on the left-hand cut the discontinuity $\mathcal{L}(t)$ is calculated. The driving forces are the crossed-channel singularities in the amplitude with appropriate spin and isospin—for example, π, K, ϕ, and so on in the narrow-resonance approximation.[38] Then

$$\mathrm{Im}\mathfrak{M}(t)=\mathrm{Im}[\mathfrak{N}(t)\mathfrak{D}^{-1}(t)]=[\mathrm{Im}\mathfrak{N}(t)]\mathfrak{D}^{-1}(t)=\mathcal{L}(t)\,,$$

or, $\mathrm{Im}\mathfrak{N}(t)=\mathcal{L}(t)\mathfrak{D}(t)$, so that with the appropriate number of subtractions,

$$\begin{aligned}\mathfrak{N}(t)&=\frac{1}{\pi}\int_L dt'\frac{\mathrm{Im}\mathfrak{N}(t')}{t'-t}\\&=\frac{1}{\pi}\int_L dt'\frac{\mathcal{L}(t')\mathfrak{D}(t')}{t'-t}.\end{aligned}\tag{5.10}$$

The $\mathfrak{D}$ corresponding to the $\mathfrak{M}$ we computed earlier in this paper could be called $\mathfrak{D}^{(0)}(t)$; this could be used to generate $\mathfrak{N}^{(1)}(t)$ to give $\mathfrak{D}^{(1)}(t)$, using Eqs. (5.9) and (5.10) in tandem. We call our value $\mathfrak{D}^{(0)}$ because we start with $\mathfrak{N}^{(0)}$ taken as a constant. The chain is constructed as far as one has the patience to compute. If no new information is introduced, it is unlikely that further iterations after the first will improve the calculation.

The connection between $\mathfrak{N}$ and $\mathfrak{D}$ matrices has been exploited particularly in bootstrap theory (for example,

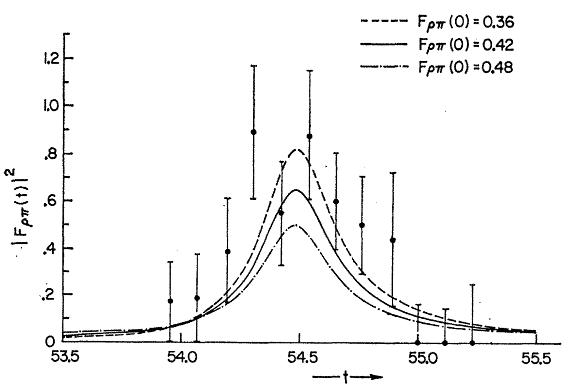

FIG. 6. $|F_{\rho\pi}(t)|^2$ near $t=m_\phi^2$ compared to the Orsay colliding-beam results.

[38] G. Chew, *S-Matrix Theory of Strong Interactions* (Benjamin, New York, 1963).

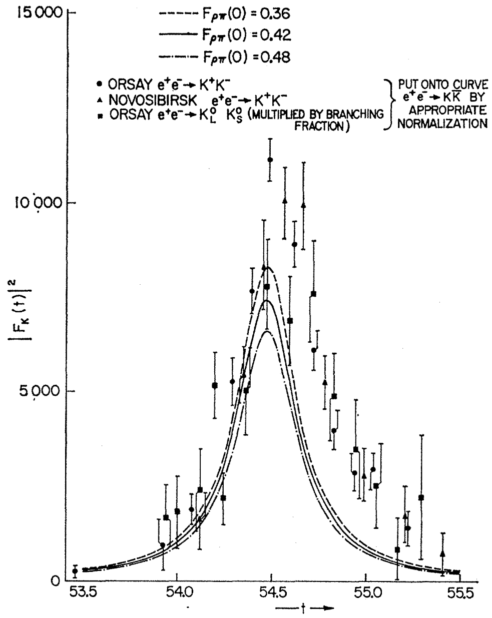

FIG. 7. $|F_K(t)|^2$ near $t=m_\phi^2$ compared to the Orsay and Novosibirsk colliding-beam results.

Zachariasen and Zemach[39] consider a coupled-amplitude problem which requires the matrix formalism). It has also been used in form-factor calculations, though not in matrix problems.

Rewriting $\mathfrak{M}(t)$,

$$\mathfrak{M}(t)=\mathfrak{N}(t)\mathfrak{D}^{-1}(t)=\mathfrak{N}(t)[\mathrm{cofactor}\mathfrak{D}(t)]/\det\mathfrak{D}(t)\,.$$

In the N/D approximation, $\mathfrak{D}(t)$ is real analytic in the right half-plane, and $\mathfrak{D}(t)$ is real analytic in the left half-plane [$\mathfrak{N}(t)$ contains all left-hand cuts and $\mathfrak{D}(t)$ contains all right-hand cuts]. The condition for a resonance is given by

$$\mathrm{Re}\,\det\mathfrak{D}(t)|_{m_R^2}=0\,.$$

When $\mathfrak{N}$ is a constant matrix and there are no left-hand cuts, it can easily be shown that $\mathfrak{M}^{-1}(t)$ can be used in place of $\mathfrak{D}(t)$. This is the situation obtaining in the calculation presented here. There is thus a well-defined way to extend these calculations.

VI. SUMMARY

In the approximation in which the 3π state is taken to be represented by the $\rho\pi$ state (the ρ subsequently decaying), the $\rho\pi$ and $K\bar{K}$ partial-wave scattering amplitudes and form factors are coupled. Higher-mass intermediate states are, as usual, disregarded. The resulting integral equation for the inverse matrix of amplitudes is solved; because two subtractions are necessary, there result six undetermined parameters in the solution for $\mathfrak{M}^{-1}$. These six numbers are determined

[39] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 894 (1962).

by inserting the experimental masses and widths of the ω and ϕ vector meson, and by requiring (in accordance with experimental evidence) that the ϕ be, as far as possible, uncoupled from $\rho\pi$. Electromagnetic effects and possible errors in $SU(3)$ values are demonstrated to engender effects of $\sim 10\%$ on the values of these six parameters.

From a consistency condition on the value of the residue of (the experimentally inaccessible) $F_K(m_\omega^2)$, the value of the $\rho\pi$ form factor at zero-momentum transfer is calculated (in the Appendix). This predicts a value for the width $\Gamma(\rho\rightarrow\pi^0\gamma)\sim 40$ keV. The experimental upper limit is presently

$$\Gamma(\rho\rightarrow\pi^0\gamma)<0.24 \text{ MeV}.$$

The form factors are then calculated (assuming that there are no left-hand cuts so that the form factors are proportional to the matrix of amplitudes). The values compared with the Orsay data are reasonably close for a first approximation. It is indicated how the method may be generalized to include left-hand cuts in the matrix $\mathfrak{M}(t)$.

Squared charge radii are computed for the K and for the $\rho\pi$ state and found to be approximately the same size, ~ 0.25 F^2, as compared to the pion isovector charge radius ~ 0.4 F^2.

No ghosts are found in the model by explicit computation.

Since the values quoted for the form factors depend sensitively on the value of $F_{\rho\pi}(0)$, it will be interesting to obtain a firm experimental value for it in order that the model be more stringently tested against experiment.

ACKNOWLEDGMENTS

I am grateful for the invaluable assistance of Dr. John Schwarz, first for suggesting that this would be an interesting project, and later for continued helpful criticism in the course of the work. I would like to thank the Princeton University Physics Department for supplying the computer time required for this work.

APPENDIX: CALCULATION OF THE $\pi\rho$ TRANSITION FORM FACTOR AT ZERO-MOMENTUM TRANSFER

In this appendix, the value of $F_{\rho\pi}(0)$ is calculated by two methods. First, the calculation of $F_{\omega\pi}(0)$ is reviewed. One writes the pole-dominance diagram for $\gamma\rightarrow\omega\pi$ [40] as in Fig. 8(a), which suggests the estimate for the form factor

$$F_{\omega\pi}(t)=g_{\omega\rho\pi}\lambda/(m_\rho^2-t).$$

The constant λ describes the ρ-photon coupling. It may be determined from the corresponding approxi-

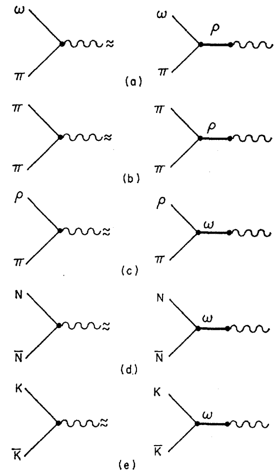

FIG. 8. Pictorial representation of the vector dominance approximation for the form factors considered in the text: (a) $\omega\pi$ form factor at $t=m_\rho^2$; (b) pion form factor at $t=m_\rho^2$; (c) $\rho\pi$ form factor at $t=m_\omega^2$; (d) nucleon form factor at $t=m_\omega^2$; (e) kaon isoscalar form factor at $t=m_\omega^2$.

mation to the pion form factor as in Fig. 8(b), or

$$F_\pi(t)=g_{\rho\pi\pi}\lambda/(m_\rho^2-t) \quad\text{and}\quad F_\pi(0)=1.$$

Then we obtain

$$F_{\omega\pi}(0)=g_{\omega\rho\pi}/g_{\rho\pi\pi}=(\gamma_{\omega\rho\pi}/\gamma_\rho)^{1/2}\cong 2.4.$$

This number gives approximately the correct decay rate for $\omega\rightarrow\pi^0\gamma$; it can further be used to make an $SU(3)$ estimate of $\rho\rightarrow\pi^0\gamma$. One uses the quark-model prediction that the $\phi\rho\pi$ coupling is small (for other justifications, see Harari[41]). Then, using the "ideal" $SU(3)$ mixing angle,

$$\Gamma(\rho\rightarrow\pi^0\gamma)=\tfrac{1}{9}\Gamma(\omega\rightarrow\pi^0\gamma)$$

or

$$F_{\rho\pi}(0)=\tfrac{1}{3}F_{\omega\pi}(0)\cong 0.8.$$

Recent data[42] suggest

$$F_{\rho\pi}(0)\sim 0.48\text{–}1.8$$

if one believes one-pion exchange (the authors of Ref. 42 feel that one-pion exchange is inadequate to describe their data). The upper limit on the form factor by direct measurement was given in the Rosenfeld tables[17] as 0.98 ± 0.12.

The next estimate is necessarily crude. It utilizes a calculation similar to the (simpler) computation of $F_{\omega\pi}(0)$. However, in this case, the coupling of the ω to the nucleon must be used.

In the case of the $\rho\pi$ transition form factor, the representation of Fig. 8(c) suggests

$$F_{\rho\pi}(t)=g_{\omega\rho\pi}\lambda'/(m_\omega^2-t).$$

Now the isoscalar nucleon form-factor estimate can be considered (this is admittedly crude, since a single ω pole does not give a very good fit in the spacelike region).

[40] J. J. Sakurai, *Currents and Mesons* (Chicago U. P., Chicago, 1969).

[41] H. Harari, Phys. Rev. **155**, 1565 (1967).

[42] Y. Eisenberg, B. Haber, E. E. Ronat, A. Shapiro, and G. Yekutieli, Phys. Rev. Letters **25**, 764 (1970).

Comparing to Fig. 8(d),

$$F_N(t)=g_{\omega N\bar{N}}\lambda'/(m_\omega^2-t).$$

Using the parametrization of Hughes *et al.* and others[43] in terms of fits to experimental form-factor data, one obtains two simultaneous equations in two unknowns: $\lambda' g_{\omega N\bar{N}}$ and the corresponding ϕ coupling. Solving

$$\lambda' g_{\omega N\bar{N}}=1.9(1\pm0.554),$$

where $g_{\omega N\bar{N}}$ is the strong-interaction coupling constant, we obtain

$$F_{\rho\pi}(0)\cong(m_N^2/m_\omega^2)(\gamma_{\omega\rho\pi}/g_{\omega N\bar{N}}^2)^{1/2}(g_{\omega N\bar{N}}\lambda')\cong0.6\pm0.34.$$

The last method to be considered uses the formulas derived in the main text of this paper to obtain an expression for the residue at the ω pole. When this is required to agree with the pole-dominance value of the residue, an equation for $F_{\rho\pi}(0)$ results. This would be subject to corrections due to the finite width of the ω, but it should still be a rather good estimate. One takes, for t near m_ω^2,

$$F_K(t)\cong g_{\omega K\bar{K}}\lambda'/(m_\omega^2-t),\quad F_{\rho\pi}(t)\cong g_{\omega\rho\pi}\lambda'/(m_\omega^2-t),$$

using the representation of Figs. 8(e) and 8(c), respectively. Evaluating the latter equation at $t=0$ (and using the established smallness of the $\phi\rho\pi$ coupling to eliminate the ϕ term),

$$F_{\rho\pi}(0)=\lambda' g_{\omega\rho\pi}/m_\omega^2,$$

so that

$$\lambda'=m_\omega^2F_{\rho\pi}(0)/g_{\omega\rho\pi}.$$

Using (5.5) for $F_K(t)$, equating it to the pole-dominance value, and using the expression for λ', we obtain

$$m_\omega^2(g_{\omega K\bar{K}}/g_{\omega\rho\pi})F_{\rho\pi}(0)=\frac{\gamma_{\omega K\bar{K}}}{a_{11}+b_{11}m_\omega^2+A(m_\omega^2)}\{[a_{12}F_{\rho\pi}(0)+\tfrac{1}{2}(a_{22}+k(0))][a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]$$
$$-a_{12}(1-m_\omega^2/m_\phi^2)[\tfrac{1}{2}a_{12}+(a_{11}+A(0))F_{\rho\pi}(0)]\}.$$

Changing the g's to γ's, we obtain

$$F_{\rho\pi}(0)=\frac{1}{2}\frac{[a_{22}+k(0)][a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]-a_{12}^2(1-m_\omega^2/m_\phi^2)}{m_\omega^2[a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]/(6\gamma_{\omega K\bar{K}}\gamma_{\omega\rho\pi})^{1/2}-a_{12}[a_{11}+b_{11}m_\omega^2+A(m_\omega^2)]+a_{12}(1-m_\omega^2/m_\phi^2)[a_{11}+A(0)]}$$
$$=\begin{cases}-1.21/2(-1.433)=0.42, & a_{12}<0\\ -1.21/2(7.89)=-0.08, & a_{12}>0.\end{cases}$$

$a_{12}<0$ is chosen on the basis of the calculation by the first methods. Then the prediction is

$$F_{\rho\pi}(0)=0.42.$$

This clarifies the decay rate for $\rho\to\pi^0\gamma$ [whose current experimental upper limit is $F_{\rho\pi}(0)=0.98\pm0.12$] and predicts

$$\Gamma(\rho\to\pi^0\gamma)=44.1\text{ keV},$$
$$\Gamma(\rho\to\pi^0\gamma)/\Gamma(\rho\to\pi\pi)\simeq0.0005=0.05\%.$$

The value calculated for $F_{\rho\pi}(0)$ is sensitive to variations in the parameters and to $\gamma_{\omega\rho\pi}$ and $\gamma_{\omega K\bar{K}}$. A 10% change in $\gamma_{\omega\rho\pi}$ can produce a 15–20% effect in the value of $F_{\rho\pi}(0)$. In view of the uncertainty in the exact values of the parameters, $F_{\rho\pi}(0)$ should be written

$$F_{\rho\pi}(0)=0.42\pm0.06.$$

[43] E. B. Hughes, T. A. Griffy, M. R. Yearian, and R. Hofstadter, Phys. Rev. **139**, B458 (1965); C. W. Akerlof, K. Berkelman, G. Rouse, and M. Tigner, *ibid.* **135**, B810 (1964); S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).