pole" form of the nucleon form factor would suggest an effective cutoff in the range of the vector-meson masses. This would be another reason for taking $\Lambda = m_{K^*}$. For the range $B = 0.93$–1.80 MeV taken in Sec. III, $B\ln(\Lambda^2/m_{K^*}{}^2)$ takes values of (-0.29)–(-0.56) MeV for $\Lambda = m_\rho$ and 0.22–0.43 MeV for $\Lambda = m_\phi$. It is difficult to estimate realistic error bounds. The uncertainty lies partly in the determination of δ and γ, and partly in the dynamical assumptions including the value of Λ. From the above discussion we believe, perhaps too optimistically, that the result for δm_K given in Sec. III is correct within 1 MeV. Finally we mention the contribution to δm_K from the high-energy virtual processes. In place of the tadpole dominance, the pole dominance in the angular momentum plane has been used as a possible explanation of the octet enhancement.[2,3] The estimation due to Buccella *et al.* of the high-energy contribution from this viewpoint gives (-5.7)–(-7.1) MeV as the sum of the subtraction and the asymptotic contribution.[4] When combined with the low-energy contribution obtained in Sec. III, the above value well reproduces the experimental δm_K. Thus we conclude that the low-energy approach as employed in the present work offers a reliable method of calculation only for the low-energy contribution, and that the octet-enhancement mechanism cannot be made to appear in any reasonable way within the low-energy approach.

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Families of Indefinitely Rising Trajectories and Gell-Mann's Program

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Infinite-component wave equations giving rise to a linear mass spectrum and to families of parallel linear trajectories are considered. A general discussion is given of invariant equations for wave functions belonging to $SL(2,C)$ representations and the mass spectra that arise are examined. The simplest possibility corresponds to a higher-derivative equation that gives a linearly rising timelike spectrum that is free of continuum spacelike solutions. Discrete spacelike solutions are absent for the simplest choices of the $SL(2,C)$ representation. The currents and the commutators among the current components are calculated by setting up for the higher-derivative equation a Lagrangian formalism and a quantization procedure based on the action principle. An explicitly factorized model is considered, with respect to the internal symmetry group, and possible nonfactorized extensions are examined. A typical feature of the current commutators is the appearance of Schwinger terms which, besides satisfying known general requirements, also appear in commutators between time components of currents. An alternative interpretation of the physical system in terms of a bound-state equation is presented. The interpretation, in terms of a bound system in two space dimensions, leads to a extension to three space dimensions, again formulable as an infinite-component wave equation. The system describes a family of parallel linearly rising trajectories spaced by one unit of angular momentum. No continuum spacelike spectrum is present, and discrete spacelike solutions are absent for physical choices of the representation of the internal spin group.

I. INTRODUCTION

INFINITE-COMPONENT wave equations have been widely discussed in the literature.[1,2] They have appeared of interest also in connection with Gell-Mann's program of saturation of current commutation rela-

[1] The results described in the present paper were summarized in R. Casalbuoni, R. Gatto, and G. Longhi, Nuovo Cimento Letters **2**, 159 (1969); **2**, 166 (1969).

[2] For general references, see Y. Nambu, Phys. Rev. **160**, 1171 (1967); C. Fronsdal, *ibid.* **171**, 1811 (1968); L. O'Raifeartaigh, in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy*, edited by A. Perlmutter *et al.* (Benjamin New York, 1968); R. C. Hwa, Nuovo Cimento **56A**, 107 (1968); **56A**, 127 (1968).

tions.[3] Difficulties with such an approach have been repeatedly pointed out,[2,3] among which is the appearance of unwanted solutions besides those of the timelike spectrum. The recent success of the Veneziano model[4] has suggested the possible physical relevance of indefinitely rising linear trajectories. It therefore appears of interest to look for infinite-component wave equations which lead to a linearly rising mass spectrum or, better, to families of parallel linearly rising trajectories. In this paper we shall present a systematic discussion, within a certain formalism, of such possibilities. We shall first discuss the simplest possibility—an infinite-component wave equation leading to a linearly rising trajectory. The solutions of such an equation do not contain spacelike states for physical choices of the representation of the internal spin group. In regard to the problem of saturation of the current algebra, we shall verify, by quantizing the field and deriving the current commutation relations, that the proposed solution is related to Gell-Mann's program. Additional Schwinger terms may occur in the commutation relations; the appearance of such terms is related to our choice of canonical variables in developing the quantization procedure, and does not, of course, impair in any way the physical interpretation of the theory.

We shall also derive an interpretation of our model in terms of a composite system, of hydrogenlike character. The interpretation clarifies physically the reason for the absence of spacelike states. In addition, the composite-particle interpretation will suggest a generalization to a model which gives indefinitely rising parallel trajectories, again expressible by an infinite-component wave equation and essentially unchanged as regards the absence of unwanted solutions.

The internal spin group for the infinite-component wave function, in the search for an equation giving rise to a linear trajectory, will be chosen to be $SL(2,C)$. Starting from the most general form for such an equation, we shall look, in Sec. II, for the simplest cases, finally obtaining an equation with linear mass spectrum, and other possible alternatives, which will be discussed in detail.

The discussion of the solutions, classified according to the various series of representations of the little groups $SU(2)$, $SU(1,1)$, $E(2)$, and $SL(2,C)$ of the Poincaré group, for the different orbits of the four-momentum, is given in Sec. III. The particular choice of the Majorana representations of $SL(2,C)$ will be considered, and a possible simplification of the wave equation will be pointed out.

In Sec. IV, we shall perform the necessary work to relate the results on the infinite-component wave equation to the problem of saturation of current algebra at infinite momentum. The particular feature of the wave equation, viz., that of being of higher order in the field derivatives, will force us to develop a formally complex treatment based on the action principle. We shall also deal with the possibility of defining alternative expressions for the currents by performing suitable contact transformations.

The program of deriving the current commutators will be completed in Sec. V, where a quantization procedure for such a higher-order Lagrangian will be applied to our infinite-component equation. The current commutation relations will be derived for a factorized model; some suggestions towards a nonfactorized model will be advanced. Depending on our quantization procedure Schwinger terms appear, which modify the Gell-Mann commutators. We are referring to the commutators among time components, which are generally assumed to be free of such terms. The Schwinger terms that appear have vanishing vacuum expectation values, and they do not alter the once-integrated commutation relations.

The interpretation in terms of a hydrogenlike composite system is described in Sec. VI. The interpretation is based on a description of the invariant operator, appearing in the infinite-component wave equation, as a differential operator in a two-dimensional angular momentum space. The extension to three dimensions, in Sec. VII, leads directly to the generalization at which we were aiming—a system of an infinite number of parallel trajectories, linearly rising and spaced by one unit of angular momentum. An infinite-component wave equation can again be derived, employing the group $O(4,1)$ and its representations. The solutions have features similar to those of the system described in Sec. VI, which, however, describes a single trajectory.

Appendices A and B summarize the main results and notation used in the text regarding the Poincaré group and $SL(2,C)$. The explicit forms of the Schwinger terms are reported in Appendix C.

II. DISCUSSION OF INFINITE-COMPONENT WAVE EQUATION

In this section we shall present a general discussion of the mass spectrum of the following invariant infinite-component wave equation, written here in momentum space:

$$[f(p^2)-\tfrac{1}{4}p^2-W]\phi(p)=0. \qquad (2.1)$$

In Eq. (2.1), $f(p^2)$ is an arbitrary nonsingular function of p^2, and W is the Pauli-Lubanski invariant.[5] A short summary of the main concepts relevant for the discussion is given in Appendices A and B. It will be assumed that $\phi(p)$ belongs to an infinite-dimensional representa-

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[3] M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles, 1967*, edited by H. Filthuth (Wiley, New York, 1968).

[4] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[5] Our metric is that used, for instance, in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1964).

TABLE I. Solutions of Eq. (2.1) in relation to the sign of the function $f(p^2)$.

$f(p^2)>0$	spacelike continuum solutions [principal series of $SU(1,1)$]	no solutions	lightlike continuum solutions [principal series of $E(2)$]	timelike discrete[a,b] solutions [of definite spin of $SU(2)$]
$f(p^2)=0$	no solutions	there are in general solutions[a,c]	lightlike discrete[a] solutions [discrete series of $E(2)$][d]	no solutions
$f(p^2)<0$	spacelike discrete solutions[a,b] [discrete series of $SU(1,1)$]	no solutions[b]	no solutions[b]	no solutions
	$p^2<0$	$p_\mu=0$	$p^2=0,\ p_\mu\neq0$	$p^2>0$

[a] $f(p^2)>0$ for $p^2>0$, $f(p^2)=0$ for $p^2=0$ ($p^\mu\neq0$ or $p^\mu=0$), $f(p^2)<0$ for $p^2<0$.
[b] $f(p^2)>0$ for $p^2>0$, $f(p^2)<0$ for $p^2=0$ ($p^\mu\neq0$ or $p^\mu=0$), $f(p^2)<0$ for $p^2<0$.
[c] Only if $\phi(p)$ belongs to a representation of $SL(2,C)$ which has some unitary irreducible component (see Sec. III).
[d] See the discussion of Eq. (2.6).

tion of[6] $SL(2,C)$ (besides transforming according to a unitary representation of the Poincaré group).

The mass spectrum of Eq. (2.1), for p_μ belonging to the various orbits (a), (b), (c), and (d), is given by the following implicit equations (see Appendix A).

(a) p^μ timelike, in the standard frame $p^\mu=(m,0,0,0)$:

$$f(m^2)/m^2=(j+\tfrac{1}{2})^2. \qquad (2.2)$$

(b) p^μ lightlike, in the standard frame $p^\mu=(\omega,0,0,-\omega)$:

$$f(0)/\omega^2=f^2. \qquad (2.3)$$

(c) p^μ vacuumlike, $p^\mu=(0,0,0,0)$:

$$f(0)=0. \qquad (2.4)$$

(d) p^μ spacelike, in the standard frame $p^\mu=(0,0,0,q)$:

$$f(-q^2)/(-q^2)=(j+\tfrac{1}{2})^2. \qquad (2.5)$$

In case (b) one sees from Eq. (2.3) that solutions that transform according to the discrete series of $E(2)$ for which $f=0$ can be present only if $f(p^2)=p^2g(p^2)$. Equation (2.1) then takes the form

$$p^2[g(p^2)-\tfrac{1}{4}-W/p^2]\phi(p)=0, \qquad (2.6)$$

and one finds two types of solutions for $p^2=0$. One type, which is always present, corresponds to the vanishing of the first factor in Eq. (2.6); in the mass spectrum it represents a discrete point. The other type is for

$$\lim_{p^2\to0}\frac{f(p^2)}{p^2}=g(0)=\tfrac{1}{4}+\lambda^2, \qquad (2.7)$$

where λ is the helicity. When Eq. (2.7) holds, the square bracket in Eq. (2.6) vanishes (see Appendix A for a discussion of the eigenvalues of W/p^2 in the lightlike case and for the discrete series). Equation (2.7) shows that solutions of the second type can be considered as limits of solutions for timelike or spacelike p^μ.

In Table I the existence of solutions of Eq. (2.1) is related to the nature of $f(p^2)$. One sees that the possibility $f(p^2)<0$ everywhere for $p^2>0$ has to be excluded. In fact, we require the existence of physical states, that is, the existence of definite spin, for timelike p^μ. Of course, one could have an oscillating $f(p^2)$, with $f(p^2)>0$ only inside some intervals of the positive p^2 axis. The mass spectrum would, however, consist of distinct intervals. Excluding such a possibility, which does not appear to be realistic, one is forced to assume that $f(p^2)>0$ for all $p^2>0$, or for p^2 larger than some positive value, apart from at most some isolated points where $f(p^2)=0$.

A situation with $f(p^2)>0$ for $p^2<0$ and $p^2=0$ can be excluded by the requirement that no continuum solutions be present.

Summarizing, we are left with two possibilities: (a) $f(p^2)>0$ for $p^2>0$, $f(p^2)=0$ for $p^2=0$ ($p^\mu\neq0$ or $p^\mu=0$), $f(p^2)<0$ for $p^2<0$; (b) $f(p^2)>0$ for $p^2>0$, $f(p^2)<0$ for $p^2=0$ ($p^\mu\neq0$ or $p^\mu=0$), $f(p^2)<0$ for $p^2<0$. These possibilities are denoted as (a) and (b) in Table I. We have excluded the possibility $f(p^2)\equiv0$ for any $p^2<0$ which would require either vanishing $f(p^2)$ or nonanalytical $f(p^2)$. Of the two possibilities (a) and (b), only (a) is compatible with a trajectory regular at $m^2=0$. In fact, by inspection of Eq. (2.2), we see that alternative (b) would give $j\to\infty$ for $m^2\to0$; for instance, for linear trajectories, $m^2(j)=aj+b$; Eq. (2.2) gives $f(m^2)=m^2(j+\frac{1}{2})^2=m^2[(1/a)m^2+\frac{1}{2}-b/a]^2$.

In Sec. VI we shall present an interpretation of the wave equation (2.1) as a nonrelativistic limit of a Bethe-Salpeter equation for a system bound by a Coulomb potential,[7] with the coupling constant depending on the total energy.[8] In such an interpretation, the coupling constant is proportional to $[f(p^2)/p^2]^{1/2}$ [see Eq. (6.11)]. The following cases are then excluded, for a real $f(p^2)$.

(i) $f(p^2)/p^2<0$. The coupling constant would have to be pure imaginary.

(ii) $f(p^2)\neq0$ for $p^2=0$. The coupling constant would become infinite.

[6] For a detailed study of the group $SL(2,C)$ and of the invariant wave equations, see M. A. Naimark, *Les Représentations Linéaires du Groupe de Lorentz* (Dunod, Paris, 1962), and I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of Rotations and Lorentz Groups and Their Applications* (Pergamon, London, 1963).

[7] G. Bisiacchi, P. Budini, and G. Calucci, Phys. Rev. **172**, 1508 (1968).

[8] G. Tiktopoulos, Phys. Letters **28B**, 185 (1969).

If we wish to exclude both situations (i) and (ii) as unphysical, the only possibility which remains is alternative (a).

We can summarize the preceding requirements by writing

$$f(p^2)=p^2g(p^2)\,, \tag{2.8}$$

where $g(p^2)$ is a regular, real non-negative function of p^2, for p^2 real. It appears from Eq. (2.8) and from Table I that the presence of discrete spacelike solutions is inevitable. On the other hand, as we shall see in detail in Sec. III, such solutions are present only for particular representations of $SL(2,C)$. Moreover, all values of λ are *a priori* possible for the lightlike solutions which appear when Eq. (2.7) is satisfied, depending on the chosen $SL(2,C)$ representation, by suitably varying the value of $g(0)$, unless $g(0)<\frac{1}{4}$.

We shall now specialize our discussion to $g(p^2)$ =polynomial in p^2. To illustrate such a choice we consider the possibility of deriving Eq. (2.1) from a Lagrangian. For polynomial $g(p^2)$ the Lagrangian will be a function of the field ϕ, of its first derivatives $\phi_{,\mu}$, and generally of its higher derivatives $\phi_{,\mu\nu}$, etc. There are detailed discussions of such Lagrangians,[9] and theories of this kind seem in principle admissable. On the other hand, for a nonpolynomial $g(p^2)$, one would have a Lagrangian for which there may in general appear a "lack of propagation character," as Pais and Uhlenbeck[9] have shown in the exponential case. Such a situation would not be completely deprived of physical meaning; also, there have been no detailed studies for other forms of functional behavior. Nevertheless, the choice of a polynomial $g(p^2)$ appears suggestive because its simplicity and it seems free of evident inconsistencies.

For polynomial $g(p^2)$ we have, taking into account the conditions after Eq. (2.8),

$$g(p^2)=(1/\alpha^2)\prod_{n_i}(p^2-\alpha_i)^{n_i}(p^2-\alpha_i^*)^{n_i}\,. \tag{2.9}$$

The polynomial in Eq. (2.9) is of degree $2N=2\sum_i n_i$ in p^2. The mass spectrum, from Eq. (2.2), is given by

$$\prod_{n_i}(m^2-\alpha_i)^{n_i}(m^2-\alpha_i^*)^{n_i}=\alpha^2(j+\tfrac{1}{2})^2,\quad \sum_i n_i=N. \tag{2.10}$$

The mass spectrum will in general consist of different branches: one indefinitely increasing with j, with an asymptotic behavior

$$m^2\propto j^{1/N}\,, \tag{2.11}$$

and additional branches with increasing or decreasing behavior. The decreasing branches only allow for a limited number of values of j ($m^2>0$).

The minimum degree for the polynomial is $2N=2$, i.e., $N=1$. In fact, for $N=0$ one has $f(p^2)=(1/\alpha^2)p^2$, and the wave equation becomes

$$[p^2-\tfrac{1}{4}p^2\alpha^2-\alpha^2W]\phi(p)=0\,. \tag{2.12}$$

Equation (2.12) allows at most for one spin value, given by $1=\alpha^2(j+\frac{1}{2})^2$.

For $N=1$ one has

$$[p^2(p^2-\beta)(p^2-\beta^*)-\alpha^2W-\tfrac{1}{4}\alpha^2p^2]\phi(p)=0\,, \tag{2.13}$$

and the mass spectrum, for $p^2=m^2>0$, is

$$m^2=\pm[\alpha^2(j+\tfrac{1}{2})^2-(\mathrm{Im}\beta)^2]^{1/2}+\mathrm{Re}\beta\,. \tag{2.14}$$

In particular, for $\beta=\beta^*$ one obtains a linear mass spectrum in j,

$$m^2=\pm\alpha(j+\tfrac{1}{2})+\beta\,. \tag{2.15}$$

We shall discuss this case in detail in Sec. III.

Let us apply the considerations we have developed to some equations which have been discussed in the literature.

We shall first consider the Majorana equation[10]

$$(p_\mu\Gamma^\mu-M_0)\phi(p)=0\,, \tag{2.16}$$

where M_0 is a constant, $\phi(p)$ belongs to either one of the two Majorana representations $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$, and Γ^μ is the four-vector operator which can be defined within such representations (see Appendix B).

Equation (2.16) can be rewritten in the general form of Eq. (2.1). To such purpose we note that for both Majorana representations the $SL(2,C)$ Casimir operator $C_1=\frac{1}{2}J_{\mu\nu}J^{\mu\nu}$, where $J_{\mu\nu}$ are the $SL(2,C)$ generators, takes on the value $C_1=-\frac{3}{4}$ (see Appendix B) and, using the relations in Eq. (B11), valid for the Majorana representations, one has

$$(p_\mu\Gamma^\mu)^2=W+\tfrac{1}{4}p^2\,. \tag{2.17}$$

Multiplying Eq. (2.16) by $(p_\mu\Gamma^\mu+M_0)$, one obtains

$$(M_0^2-\tfrac{1}{4}p^2-W)\phi(p)=0\,, \tag{2.18}$$

which is of the form of Eq. (2.1), specialized to $f(p^2)=M_0^2>0$. Table I tells us that Eq. (2.18) has continuum spacelike solutions, continuum lightlike solutions, and discrete timelike solutions—a well-known result for the Majorana equation.

As a second example, let us consider the equation proposed by Leutwyler[11]:

$$(p^2-2\mu p_\nu\Gamma^\nu-M_0^2)\phi(p)=0\,, \tag{2.19}$$

with $\phi(p)$ belonging to either one of the two Majorana representations and μ and M_0 constants. With the same procedure used before one finds

$$[(1/4\mu^2)(p^2-M_0^2)^2-\tfrac{1}{4}p^2-W]\phi(p)=0\,, \tag{2.20}$$

[9] See A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950), where exhaustive references may be found. More recently, a complete work on this subject has been published by A. O. Barut and G. H. Mullen, Ann. Phys. (N. Y.) **20**, 203 (1962). See also M. Borneas, Acta Phys. Polon. **24**, 471 (1963); Phys. Rev. **186**, 1299 (1969).

[10] E. Majorana, Nuovo Cimento **9**, 335 (1932); for a detailed work on the Majorana equation, see W. Rühl, Commun. Math. Phys. **6**, 312 (1967).

[11] H. Bebié, F. Ghielmetti, V. Gorgé, and H. Leutwyler, Phys. Rev. **177**, 2133 (1969).

which is of the form of Eq. (2.1), provided that

$$f(p^2)=(1/4\mu^2)(p^2-M_0{}^2)^2.$$

Clearly $f(p^2)\geqslant 0$ for any p^2, and Table I tells us that Eq. (2.20) also has continuum spacelike, continuum lightlike, and discrete timelike solutions.

III. WAVE EQUATION WITH LINEAR TRAJECTORIES

In this section we shall concentrate on the discussion of the wave equation (2.13), obtained in Sec. II as the simplest polynomial possibility. In addition, we assume β real, such that the mass spectrum is linear, as given by Eq. (2.15). The wave equation (2.13) for real β is

$$[p^2(p^2-\beta)^2-\tfrac{1}{4}\alpha^2p^2-\alpha^2W]\phi(p)=0. \quad (3.1)$$

Here we shall examine its solutions.

States of timelike momentum. In the rest system, $p^\mu=(m,0,0,0)$ and $W=m^2j(j+1)$ (see Appendix A). The mass spectrum, from Eq. (3.1), is

$$m^2=\pm\alpha(j+\tfrac{1}{2})+\beta. \quad (3.2)$$

One obtains two branches, as shown in Fig. 1. In general, the decreasing branch only allows for a finite number of physical states; such states are absent if the intercept $\alpha(0)$ satisfies the condition

$$\alpha(0)\geqslant -1-j_{\min}, \quad (3.3)$$

where $j_{\min}$ is the minimum value of j contained in the trajectory. For the case illustrated in Fig. 1, $j_{\min}=0$ (this is a "bosonic" case) and the condition in Eq. (3.3) is not satisfied. Condition (3.3) is satisfied by the intercepts of the known physical trajectories.

States of spacelike momentum. In the standard frame, $p^\mu=(0,0,0,q)$ and $W=-q^2j(j+1)$ (see Appendix A). It follows from Eq. (3.1) that

$$\alpha^2(j+\tfrac{1}{2})^2=(q^2+\beta)^2\geqslant 0. \quad (3.4)$$

States of the principal series of $SU(1,1)$ are absent, since for such states, $j+\frac{1}{2}$ is pure imaginary. States of the discrete series, i.e., with $j+\frac{1}{2}\leqslant 0$ and $2j=$integer, are in general present, depending on the chosen $SL(2,C)$ representation. In particular, they are absent if the chosen representation has $j_0=0$ or $j_0=\frac{1}{2}$.[12] We note that such spacelike states of the discrete series can be avoided, for trajectories with $j_{\min}>\frac{1}{2}$, simply by choosing the $SL(2,C)$ representation as one which is the direct product of an infinite-dimensional representation with $j_0=0$ or $j_0=\frac{1}{2}$ and a suitable finite-dimensional representation. Such reducible $SL(2,C)$ representations are not unitary.

States of lightlike momentum. In the standard frame, $p^\mu=(\omega,0,0,-\omega)$ and $W=\omega^2f^2$ (see Appendix A).

[12] N. Mukunda, J. Math. Phys. 8, 2210 (1967); 9, 50 (1968).

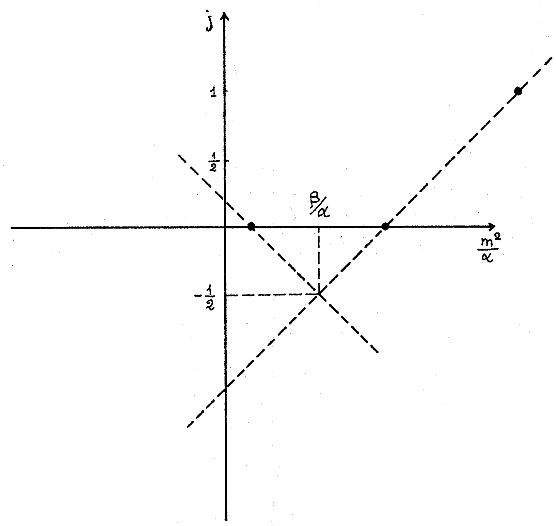

FIG. 1. Mass spectrum of the wave equation (3.1). The two dashed lines correspond to the two branches of Eq. (3.2); the circles indicate the physical states.

Equation (3.1) implies

$$f^2=0. \quad (3.5)$$

States of the principal series of $E(2)$ are therefore excluded: Only states of the discrete series can be present. For such states of the discrete series one has $w_\mu=-\lambda p_\mu$, where λ is the helicity. Furthermore, the limit for $p^2\to 0$ of W/p^2 exists and it is equal to $+\lambda^2$. Substituting $W/p^2\to\lambda^2$ into Eq. (3.1), one finds in the limit

$$p^2(\beta^2-\tfrac{1}{4}\alpha^2-\alpha^2\lambda^2)\phi(p)=0. \quad (3.6)$$

Equation (3.6) shows that there are lightlike solutions; for an irreducible representation of $SL(2,C)$, the helicity of such solutions is[13]

$$\lambda=\pm j_0. \quad (3.7)$$

The values of α and β do not appear in Eq. (3.7). The lightlike solution of helicity $\pm j_0$, Eq. (3.7), is an isolated point in the mass spectrum. Besides such solutions there are additional lightlike solutions corresponding to the vanishing of the parenthesis in Eq. (3.6). They appear for special values of the ratio α/β. One has

$$\lambda=\pm[(\beta/\alpha)^2-\tfrac{1}{4}]^{1/2}. \quad (3.8)$$

Conditions $\lambda=$integer or $=$half-integer define such special values of α/β. These additional solutions are limits for $m^2\to 0$ of the timelike solutions. For instance, for $\beta/\alpha=-\frac{1}{2}$ the $j=0$ state disappears from the timelike spectrum and reappears as a lightlike state.[14] Such a situation can be illustrated in a Chew-Frautschi plot by noting that the relation between j and λ for lightlike

[13] S. Ström, Arkiv. Fys. 34, 215 (1967); R. Delbourgo, K. Koller, and P. Mahanta, Nuovo Cimento 52A, 1254 (1967); Y. Frishman and C. Itzykson, Phys. Rev. 180, 1556 (1969).

[14] This possibility may be of interest for describing the pion since it provides for a continuous limit to zero mass.

states is given by

$$W/p^2=+\lambda^2=j(j+1). \qquad (3.9)$$

Therefore, the values $j=0$ and $j=-1$ both correspond to $\lambda=0$. This is illustrated in Fig. 2.

States of zero momentum. Equation (3.1) is satisfied, for zero momentum, by any $\phi(p)$ belonging to a generally reducible $SL(2,C)$ representation which contains some unitary irreducible components (see Appendix A).

So far we have only assumed that the wave function $\phi(p)$ belongs to some unspecified, generally reducible representation R of the spin group, which has been identified with $SL(2,C)$. More detailed statements can be made after choosing R. Some general requirements for the choice of the representation R may be the following.

(a) the requirement that there exists a bilinear Hermitian invariant form, such as to make possible the existence of a Lagrangian from which the wave equation (3.1) is derived.

(b) the requirement of invariance under the parity operation.

For a generally reducible $SL(2,C)$ representation R, (a) is satisfied provided the component $(j_0,-j_1^*)$ exists together with (j_0,j_1) in the decomposition of R into its irreducible components; similarly, (b) requires the presence of $(j_0,-j_1)$ together with (j_0,j_1) (see Appendix B). If R is chosen to be irreducible it clearly can only be of the type $(j_0,0)$ or of the type $(0,j_1)$, with j_0 integer or half-integer and j_1 real or pure imaginary. In particular, R can be chosen to be $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$, the two Majorana representations, which are unitary.

Let us identify R with $(0,\frac{1}{2})$ or with $(\frac{1}{2},0)$. In Sec. II we discussed two examples of infinite-component wave equations, namely, Eqs. (2.18) and (2.20) which, using

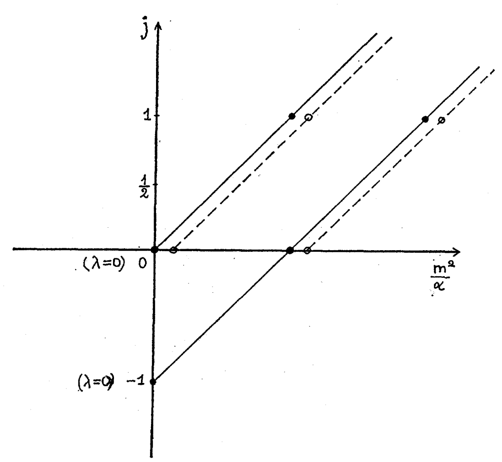

FIG. 2. Representation of the $\lambda=0$ (λ is the helicity) lightlike states on the plane (m^2,j). The isolated state with $\lambda=\pm j_0$ is not shown.

relations (B11), could be reduced to lower-degree equations in p^2, viz., Eqs. (2.16) and (2.19), respectively. Equation (3.1), for $R=(0,\frac{1}{2})$ or $R=(\frac{1}{2},0)$, cannot similarly be reduced since the reduced form would contain the variable $\sqrt{p^2}$ and therefore be essentially of nonlocal character. Also we note that the above choice of R does not allow for the construction of a pseudoscalar form, bilinear in ϕ.[15] For either choice, $R=(0,\frac{1}{2})$ or $R=(\frac{1}{2},0)$, one can write Eq. (3.1) in the form

$$[p^2(p^2-\beta)^2-\tfrac{1}{2}\alpha^2\{\Gamma_\mu,\Gamma_\nu\}p^\mu p^\nu]\phi(p)=0. \qquad (3.10)$$

There are no spacelike discrete solutions, but vacuum-like solutions cannot be discarded, since the Majorana representations are unitary.

For reducible R, the simplest choice is a direct product of a Majorana representation and a Dirac representation. It is nonunitary. Both requirements (a) and (b) above are satisfied in this case also. For such a choice, Eq. (3.1) can be reduced to the lower-degree equation[16]

$$[\gamma_\mu p^\mu(p^2-\beta)-\alpha p_\mu\Gamma^\mu]\phi(p)=0. \qquad (3.11)$$

One verifies that multiplication of Eq. (3.11) by $[\gamma_\mu p^\mu(p^2-\beta)+\alpha p_\mu\Gamma^\mu]$ gives Eq. (3.10) by taking Eqs. (B11) into account. There are no spacelike solutions and, furthermore, the $p_\mu=0$ solutions are absent because R is now nonunitary. There exists a pseudoscalar form bilinear in ϕ.

IV. LAGRANGIAN FORMALISM. CURRENTS

In Sec. III we have discussed the conditions to be satisfied by the $SL(2,C)$ representation R, to which $\phi(p)$ belongs, in order to try to construct a Lagrangian from which Eq. (3.1) can be derived. We shall here assume that R is indeed such that it satisfies those conditions; and we shall determine a possible Lagrangian and the related conserved currents.[17] For the physical interpretation of the currents in the frame of an infinite-component wave equation and for their connection with measurable quantities such as form factors, etc., we shall again refer to the papers quoted in Ref. 2, in particular, to Nambu's work.

A conserved current can be derived directly from the wave equation (3.1) by a standard procedure; one verifies directly the lack of divergence of the current

$$\begin{aligned}V_\mu(p',p)&=\phi^\dagger(p')[(p+p')_\mu\{(p^2+p'^2-\beta)^2-p'^2p^2\\&\quad-\alpha^2(\tfrac{1}{4}+\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu})\}\\&\quad+\tfrac{1}{2}\alpha^2(p+p')^\lambda\{J_{\mu\nu},J_\lambda{}^\nu\}]\phi(p) \qquad (4.1)\end{aligned}$$

[15] The only psuedoscalar that can be constructed out of the $SL(2,C)$ generators is the Casimir $C_2=\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}$; however, C_2 is zero for a Majorana representation [see Eq. (B3)].

[16] For a third-order equation analogous to Eq. (3.11), see A. O. Barut, P. Cordero, and G. C. Ghirardi, Nuovo Cimento **66A**, 36 (1970); these authors propose a unified treatment of leptons based on a third-order equation.

[17] We denote by $\phi^\dagger$ the adjoint of ϕ in the proper metric. The bilinear Hermitian invariant form is $\phi^\dagger A\phi$, where A is the matrix giving the metric; for the $J_{\mu\nu}$ generators, one has $J_{\mu\nu}{}^\dagger A=AJ_{\mu\nu}$.

or, in a slightly different form,

$$V_\mu(p',p)\\ =\phi^\dagger(p')\{(p+p')_\mu[p'^4+p^4+p^2p'^2-2\beta(p'^2+p^2)]\\ -\gamma(p+p')_\mu+T_{\mu\nu}(p+p')^\nu\}\phi(p)\,, \quad (4.1')$$

with

$$\gamma=\alpha^2(\tfrac{1}{4}+\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu})-\beta^2\,, \quad (4.2)$$

$$T^{\mu\nu}=\tfrac{1}{2}\alpha^2\{J^\mu{}_\lambda,J^{\nu\lambda}\}\,. \quad (4.3)$$

The current (4.1) can of course be derived from a suitable Lagrangian—not necessarily the simplest one. For the moment, to build up our formalism, it will be convenient to start from the following Lagrangian:

$$\mathcal{L}(x)=\phi_{,\mu}{}^\dagger(x)(\gamma g^{\mu\nu}-T^{\mu\nu})\phi_{,\nu}(x)\\ +2\beta\phi_{,\mu\nu}{}^\dagger(x)\phi^{,\mu\nu}(x)-\phi_{,\mu\nu\rho}{}^\dagger(x)\phi^{,\mu\nu\rho}(x)\,, \quad (4.4)$$

where $\phi_{,\mu}(x)=\partial\phi(x)/\partial x^\mu$, etc. The current derived from (4.4), from the action principle, is different from the current (4.1). We shall, however, show later on that the Lagrangian (4.4) can be reduced, by means of a contact transformation consisting in the addition of a divergence, to a Lagrangian which gives directly the current (4.1).

The wave equation for ϕ follows from the Lagrangian (4.4) by application of the action principle.[18]

The action is defined as

$$W=\int_{\sigma_0}^{\sigma}\mathcal{L}(x)d^4x\,,$$

where σ_0 and σ are spacelike surfaces that limit the integration volume from "below" and from "above," respectively. The "local" variation of the field is

$$\delta_0\phi(x)=\phi'(x)-\phi(x)\,, \quad (4.5)$$

and the "total" variation[18] is

$$\delta\phi(x)=\phi'(x')-\phi(x)=\delta_0\phi(x)+[\partial_\mu\phi(x)]\delta x^\mu\,. \quad (4.6)$$

The variation δW is

$$\delta W=\int_{\sigma_0}^{\sigma}\delta_0\mathcal{L}(x)d^4x+\int_{\sigma_0}^{\sigma}\mathcal{L}(x)\delta(d^4x)\,. \quad (4.7)$$

The Lagrangian $\mathcal{L}(x)$ in Eq. (4.4) contains third-order derivatives of the field; however, the following method is general. From

$$\delta(d^4x)=(\partial_\mu\delta x^\mu)d^4x \quad (4.8)$$

one has, after partial integration,

$$\delta W=\int_{\sigma_0}^{\sigma}\left\{\left[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}-\partial_\mu\pi^\mu(x)\right]\delta_0\phi(x)+\text{H.c.}\right\}d^4x\\ +F(\sigma)-F(\sigma_0)\,, \quad (4.9)$$

[18] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1951).

where

$$F(\sigma)=\int_\sigma\{\mathcal{L}(x)\delta x^\mu+[\pi^\mu(x)\delta_0\phi(x)+\pi^{\mu\nu}(x)\delta_0\phi_{,\nu}(x)\\ +\pi^{\mu\nu\rho}\delta_0\phi_{,\nu\rho}(x)+\text{H.c.}]\}d\sigma_\mu\,, \quad (4.10)$$

and

$$\pi^\mu(x)=\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu}(x)}-\partial_\nu\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu\nu}(x)}+\partial_\nu\partial_\rho\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu\nu\rho}(x)}\,,$$

$$\pi^{\mu\nu}(x)=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}(x)}-\partial_\rho\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu\nu\rho}(x)}\,, \quad (4.11)$$

$$\pi^{\mu\nu\rho}(x)=\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu\nu\rho}(x)}\,.$$

In all these equations the derivatives of the Lagrangian with respect to higher-order gradients of the field is understood to be symmetrized in the following sense [see Eq. (16) of Barut and Mullen[9]]:

$$\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu1\cdots\mu n}}\\ =\frac{1}{n!}\left[\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu1\cdots\mu n}}+\frac{\partial\mathcal{L}(x)}{\partial\phi_{,\mu2\mu1\cdots\mu n}}+(\text{all permutations})\right].$$

A more convenient form for $F(\sigma)$, obtained by use of Eq. (4.6), is

$$F(\sigma)=\int_\sigma[T^\mu{}_\nu(x)\delta x^\nu+j^\mu(x)]d\sigma_\mu(x)\,, \quad (4.12)$$

where

$$T^\mu{}_\nu(x)=\mathcal{L}(x)g^\mu{}_\nu-[\pi^\mu(x)\phi_{,\nu}(x)+\pi^{\mu\rho}(x)\phi_{,\rho\nu}(x)\\ +\pi^{\mu\rho\lambda}(x)\phi_{,\rho\lambda\nu}(x)+\text{H.c.}]\,, \quad (4.13)$$

$$j^\mu(x)=\pi^\mu(x)\delta\phi(x)+\pi^{\mu\nu}(x)\delta\phi_{,\nu}(x)\\ +\pi^{\mu\nu\rho}(x)\delta\phi_{,\nu\rho}(x)+\text{H.c.} \quad (4.14)$$

According to the action principle,

$$\delta W=F(\sigma)-F(\sigma_0)\,. \quad (4.15)$$

In particular, for a variation of the field that vanishes on the contour of the integration region, one has

$$F(\sigma)=F(\sigma_0)=0\,.$$

This equation must be satisfied for arbitrary $\delta_0\phi(x)$, implying the field equations

$$\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}-\partial_\mu\pi^\mu(x)=0\,, \quad (4.16)$$

$$\frac{\partial\mathcal{L}(x)}{\partial\phi^\dagger(x)}-\partial_\mu\pi^{\mu\dagger}(x)=0\,. \quad (4.17)$$

One verifies that Eq. (4.17), with $\mathcal{L}(x)$ as given in Eq. (4.4), coincides with Eq. (3.1), in its configuration-space form

$$(\Box^3+2\beta\Box^2-\gamma\Box)\phi(x)+T^{\mu\nu}\phi_{,\mu\nu}(x)=0. \qquad (4.18)$$

The current $j^\mu(x)$, as given from Eq. (4.14), where it is specialized for a gauge transformation of the first kind, $\delta\phi(x)=-i\alpha\phi(x)$, is

$$j_\mu(x)=\alpha J_\mu(x) \qquad (4.19)$$

where, for the Lagrangian in Eq. (4.4),

$$\begin{aligned} J_\mu(x)=-i\phi^\dagger(x)\{[&\overrightarrow{\Box}^2\overrightarrow{\partial}_\mu-\overleftarrow{\Box}^2\overleftarrow{\partial}_\mu \\ &+(\overrightarrow{\partial}_\nu\overleftarrow{\partial}^\nu)(\overleftarrow{\Box}\overleftarrow{\partial}_\mu-\overrightarrow{\Box}\overrightarrow{\partial}_\mu)+(\overleftarrow{\partial}_\nu\overleftarrow{\partial}_\rho\overrightarrow{\partial}^\nu\overrightarrow{\partial}^\rho)\overleftrightarrow{\partial}_\mu^{(-)}] \\ &+2\beta[\overrightarrow{\Box}\overrightarrow{\partial}_\mu-\overleftarrow{\Box}\overleftarrow{\partial}_\mu-(\overleftarrow{\partial}_\nu\overrightarrow{\partial}^\nu)\overleftrightarrow{\partial}_\mu^{(-)}] \\ &-\gamma\overleftrightarrow{\partial}_\mu^{(-)}+T_{\mu\nu}\overleftrightarrow{\partial}^{(-)\nu}\}\phi(x)=\phi^\dagger(x)I_\mu\phi(x), \qquad (4.20) \end{aligned}$$

with $\overleftrightarrow{\partial}_\mu^{(-)}=\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu$.

In configuration space, the current of Eq. (4.1) takes on the form

$$\begin{aligned} V_\mu(x)=-i\phi^\dagger(x)[&(\overrightarrow{\Box}^2+\overrightarrow{\Box}\overleftarrow{\Box}+\overleftarrow{\Box}^2)\overleftrightarrow{\partial}_\mu^{(-)} \\ &+2\beta(\overrightarrow{\Box}+\overleftarrow{\Box})\overleftrightarrow{\partial}_\mu^{(-)}-\gamma\overleftrightarrow{\partial}_\mu^{(-)}+T_{\mu\nu}\overleftrightarrow{\partial}^{(-)\nu}]\phi(x). \qquad (4.21) \end{aligned}$$

$V_\mu(x)$ is clearly different from $J_\mu(x)$. The difference is, however, the divergence of an antisymmetric tensor, namely,

$$J_\mu(x)=V_\mu(x)+\partial^\nu V_{\nu\mu}(x), \qquad (4.22)$$

where

$$\begin{aligned} V_{\nu\mu}(x)=i\phi^\dagger(x)[&\overleftarrow{\partial}_\nu(\overrightarrow{\Box}+\overleftarrow{\Box})\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu(\overrightarrow{\Box}+\overleftarrow{\Box})\overrightarrow{\partial}_\nu \\ &+\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\rho\overrightarrow{\partial}^\rho\overrightarrow{\partial}_\nu-\overleftarrow{\partial}_\nu\overleftarrow{\partial}_\rho\overrightarrow{\partial}^\rho\overrightarrow{\partial}_\mu+2\beta(\overleftarrow{\partial}_\nu\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu)]\phi(x). \qquad (4.23) \end{aligned}$$

Obviously, the additional term in (4.22), $\partial^\nu V_{\nu\mu}(x)$, is divergenceless and does not contribute to the total charge.

The Lagrangian $\mathcal{L}'(x)$,

$$\mathcal{L}'(x)=\phi^\dagger(x)(-\gamma\overrightarrow{\Box}+T^{\mu\nu}\overrightarrow{\partial}_\mu\overrightarrow{\partial}_\nu+2\beta\overrightarrow{\Box}^2+\overrightarrow{\Box}^3)\phi(x), \qquad (4.24)$$

gives rise to a current, by using a formula analogous to Eq. (4.14), which coincides with the current (4.1′) in configuration space. The Lagrangian $\mathcal{L}'(x)$ differs from $\mathcal{L}(x)$, defined in Eq. (4.4), by a divergence. It can be obtained from $\mathcal{L}(x)$ by a contact transformation, which leaves the field equations invariant but modifies the current. The two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ provide for two equivalent descriptions of the same physical system.

A more general class of Lagrangians, containing the Lagrangian (4.4) as a special case, can be obtained by adding to the symmetric tensor $T^{\mu\nu}$ an antisymmetric tensor, which will have to be proportional to $J^{\mu\nu}$, the only algebraic antisymmetric tensor at our disposal. The addition leaves the equations of motion unchanged—it is equivalent to adding a divergence. The new Lagrangian $\mathcal{L}''(x)$ is

$$\begin{aligned} \mathcal{L}''(x)=\phi_{,\mu}{}^\dagger(x)&(\gamma g^{\mu\nu}-T^{\mu\nu}+ibJ^{\mu\nu})\phi_{,\nu}(x) \\ &+2\beta\phi_{,\mu\nu}{}^\dagger(x)\phi^{,\mu\nu}(x)-\phi_{,\mu\nu\rho}{}^\dagger(x)\phi^{,\mu\nu\rho}(x), \qquad (4.25) \end{aligned}$$

with a real arbitrary parameter b. From the invariance of $\mathcal{L}''(x)$ under a gauge transformation of the first kind, one obtains the local conservation of the more general current

$$V_\mu(x)+\partial^\nu V_{\nu\mu}(x)+\partial^\nu\mathcal{V}_{\nu\mu}(x), \qquad (4.26)$$

where

$$\mathcal{V}_{\nu\mu}(x)=-b\phi^\dagger(x)J_{\nu\mu}\phi(x). \qquad (4.27)$$

V. QUANTIZATION AND CURRENT COMMUTATION RELATIONS IN FACTORIZED MODEL

In this section we shall derive the current commutation relations in a model of the kind called "factorized."[3]

The current commutation rules are derived by quantizing the field ϕ. The action principle provides for the commutation relations among the canonical variables. The commutation relations among the currents are then obtained by explicit calculation.

The field ϕ is a free field; therefore, the set of one-particle states of the field, if complete, will saturate exactly the algebra of its currents at infinite momentum. The one-particle states are fully determined by the wave equation (3.1).

The above program will be seen to be related to Gell-Mann's program, in the factorized case.[3] The algebra of the currents will in fact be found to differ from Gell-Mann's algebra[19] by the appearance of Schwinger terms. Such Schwinger terms are such as not to modify the once-integrated commutation relations (i.e., charge-current commutators), and they have vanishing vacuum expectation values. They follow from our choice of canonical variables.

The completeness of the set of states, which is required for saturating the current algebra, is defined with respect to the Poincaré invariant norm

$$(\phi,\phi)=\int_\sigma \phi^\dagger(x)I^\mu\phi(x)d\sigma_\mu(x), \qquad (5.1)$$

with I^μ defined in Eq. (4.20). The integral in (5.1) is over an arbitrary spacelike surface; it is, however, independent of the actual choice of σ since the current is conserved.

The vacuumlike solutions with $p^\mu=0$ have a vanishing Poincaré norm, defined as in Eq. (5.1).[20] The $SL(2,C)$ representation R, to which ϕ belongs, must therefore be chosen in such a way that the appearance of such states is avoided. We have seen that this can be done for Eq. (3.1) by choosing a nonunitary representation R.

The completeness relation, with the scalar product defined according to Eq. (5.1), has a nonvanishing contribution from the isolated lightlike state of the spectrum with $\lambda=\pm j_0$ [see Eq. (3.7)].[20]

The Lagrangian (4.4) is the simplest one, among those discussed in Sec. IV, which leads to the field equation (3.1). We shall use such a Lagrangian to

[19] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[20] C. Fronsdal, Phys. Rev. **182**, 1564 (1969).

deduce the commutation rules. To this end, it will be necessary to introduce a sufficient number of Lagrangian coordinates for the description of the mechanical system specified by the Lagrangian (4.4). The specification of the Lagrangian coordinates is essentially a problem of specifying a set of initial conditions, in the classical sense. It was solved in classical mechanics for Lagrangians containing higher time derivatives of the coordinate $q(t)$ by Ostrogradsky.[21] We shall employ here a generalization of the Ostrogradsky formalism to relativistic field theory.[9]

Let us define the canonical covariant coordinates (the treatment is limited here to Lagrangians involving at most third-order derivatives)

$$\phi_1=\phi\,,\quad \phi_2=n_\lambda D^\lambda\phi\,,\quad \phi_3=n_\lambda n_\rho D^\lambda D^\rho\phi\,, \tag{5.2}$$

where n is the normal to the spacelike surface σ, and

$$D^\lambda=n^\lambda(n_\rho\partial^\rho)\,. \tag{5.3}$$

Expression (4.12) for $F(\sigma)$ can be rewritten for $\delta x^\mu=0$ in the form

$$F(\sigma)=\sum_{r=1}^{3}\int_\sigma [\pi_r{}^\mu\delta\phi_r+\text{H.c.}]d\sigma_\mu\,, \tag{5.4}$$

where we have explicitly introduced the variations of the fields of Eq. (5.2). In Eq. (5.4) we have also introduced

$$\pi_1{}^\mu=\pi^\mu-\nabla_\rho\pi^{\mu\rho}+\nabla_\nu\nabla_\rho\pi^{\mu\nu\rho}\,, \tag{5.5}$$

$$\pi_2{}^\mu=\pi^{\mu\rho}n_\rho-2n_\nu\nabla_\rho\pi^{\mu\nu\rho}\,, \tag{5.5$'$}$$

$$\pi_3{}^\mu=\pi^{\mu\nu\rho}n_\nu n_\rho\,, \tag{5.5$''$}$$

with

$$\nabla_\rho=\partial_\rho-D_\rho\,, \tag{5.6}$$

which is a space gradient for $n=(1,0,0,0)$.

In going from Eq. (4.12) to Eq. (5.4), terms of the kind

$$\int d\sigma_\mu\nabla_\rho(\cdots)\,,$$

arising from partial integrations, have been neglected, under the usual assumption that the fields vanish fast enough asymptotically in any spacelike direction.

The canonical variables conjugate to the fields ϕ_r $(r=1,2,3)$ are

$$\pi_r=\pi_r{}^\mu n_\mu\,. \tag{5.7}$$

Schwinger's quantization procedure gives the following commutation rules:

$$[\phi_r(x),\delta\phi_s(y)]_\sigma=0\,, \tag{5.8}$$

$$[\phi_r(x),\pi_s(y)]_\sigma=i\delta_{rs}\delta(x,y)\,, \tag{5.9}$$

where the symbol $[\cdots,\cdots]_\sigma$ denotes a commutator for arguments x,y both lying on the spacelike surface σ, and $\delta(x,y)=n_\mu\delta^\mu(x-y)$ is the covariant generalization of $\delta^3(x-y)$ to a spacelike surface orthogonal to n_μ.

Schwinger has shown that for consistency, from Eqs. (5.8) and (5.9), one deduces

$$[\pi_r(x),\delta\pi_s(y)]_\sigma=0\,. \tag{5.10}$$

The result follows from a contact transformation which interchanges the variables ϕ_r and π_r.

Let us add a few remarks about Eqs. (5.8) and (5.9). First, we could have chosen anticommutation rather than commutation relations. The choice does not affect, however, the current commutators, in which we are interested: In fact, the currents are bilinear expressions. Second, Eqs. (5.8) and (5.9) are also valid for renormalized fields. Indeed, they follow from the identity

$$\delta\phi_s(x)=-i\sum_{r=1}^{3}\int\{[\phi_s(x),\pi_r{}^\mu(y)]\delta\phi_r(y)+\pi_r{}^\mu(y)[\phi_s(x),\delta\phi_r(y)]\}d\sigma_\mu(y) \tag{5.11}$$

(or from the corresponding identity with anticommutators) and cannot therefore tolerate a change of scale. Finally, one can convince oneself that Eqs. (5.8) and (5.9) cannot contain Schwinger terms on their right-hand side. The only additional solutions of Eq. (5.11) are provided by commutation rules like those of Eq. (5.8) with the addition of a nonvanishing q number on the right-hand side, in such a way that it compensates in Eq. (5.11) for a similar q number appearing in Eq. (5.9). This possibility corresponds to the so-called parastatistics, and we will exclude it here.

Assuming the validity of Eqs. (5.8) and (5.9), we are in the position of calculating the commutators among the currents $J_\mu{}^i(x)$ (i being an index related to an unspecified internal symmetry group), which we define as

$$J_\mu{}^i(x)=\phi^\dagger(x)I_\mu\gamma^i\phi(x)\,. \tag{5.12}$$

In Eq. (5.12), γ^i represents the generators of the internal symmetry group in the representation to which the field ϕ belongs. The form (5.12) of $J_\mu{}^i(x)$ is the direct extension of the current of the preceding section to a non-Abelian symmetry group, under the assumption of factorization.

The calculation of the commutators is straightforward although rather lengthy. For the components $J^i(x)=J_\mu{}^i(x)n^\mu$, we obtain

$$\begin{aligned}[J^i(x),J^j(y)]_\sigma=&\,if^{ijk}J^k(x)\delta(x,y)\\&+S_\mu{}^{(1)ij}(x,y)\nabla^\mu(y)\delta(x,y)\\&+S_{\mu\nu}{}^{(2)ij}(x,y)\nabla^\mu(y)\nabla^\nu(y)\delta(x,y)\\&+S_{\mu\nu\rho}{}^{(3)ij}(x,y)\nabla^\mu(y)\nabla^\nu(y)\nabla^\rho(y)\delta(x,y)\\&+S_{\mu\nu\rho\lambda}{}^{(4)ij}(x,y)\nabla^\mu(y)\nabla^\nu(y)\nabla^\rho(y)\nabla^\lambda(y)\delta(x,y)\,,\end{aligned} \tag{5.13}$$

where the coefficients f^{ijk} are the structure constants of the (compact) internal symmetry group and ∇_μ is defined as in Eq. (5.6).

[21] M. Ostrogradsky (1850), as quoted in E. T. Whittaker, *Analytical Dynamics* (Cambridge U.P., Cambridge, England, 1937), p. 265.

The Schwinger terms $S^{(\alpha)ij}(x,y)$ ($\alpha=1,\ldots,4$) are q numbers and have the following properties: They have vanishing vacuum expectation value and they do not contribute to the once-integrated commutation relations, which are

$$[Q^i,J^j(x)]_\sigma = if^{ijk}J^k(x) \tag{5.14}$$

(see Appendix C). In Eq. (5.14), Q^i is the charge associated with $J^i(x)$,

$$Q^i=\int d\sigma_\mu(x)J^{i\mu}(x)\,. \tag{5.15}$$

The explicit expressions of the Schwinger terms are reported in Appendix C. They exhibit an apparent noncanonical structure: They are sums of products of nonconjugate variables.

We note that the current $J^i(x)$ can be decomposed in such a way as to exhibit the existence of a part that is responsible for the appearance of the Schwinger terms. The decomposition is

$$J^i(x)=\tilde{J}^i(x)+\bar{J}^i(x)\,, \tag{5.16}$$

where

$$\tilde{J}^i(x)=-i\sum_{r=1}^{3}[\pi_r{}^\mu(x)\gamma^i\phi_r(x)-\text{H.c.}]n_\mu\,, \tag{5.16$'$}$$

$$\bar{J}^i(x)=-i\nabla_\rho[\pi^{\mu\rho}(x)-2\nabla_\nu\pi^{\mu\nu\rho}(x)\gamma^i\phi_1(x) +2\pi^{\mu\nu\rho}(x)n_\nu\gamma^i\phi_r(x) +\nabla_\nu(\pi^{\mu\nu\rho}(x)\gamma^i\phi_1(x))-\text{H.c.}]n_\mu\,. \tag{5.16$''$}$$

One obtains

$$[\tilde{J}^i(x),\tilde{J}^j(y)]_\sigma=if^{ijk}\tilde{J}^k(x)\delta(x,y)\,. \tag{5.17}$$

The current $\bar{J}^i(x)$ is associated to a vanishing charge; it has a noncanonical form in the sense described before. In a conventional Lagrangian theory without higher derivatives, the quantities $\pi^{\mu\rho}$ and $\pi^{\mu\nu\rho}$ would be absent; the term $\bar{J}^i(x)$ would then also be absent, whereas $\tilde{J}^i(x)$ would still satisfy Eq. (5.17). Thus the appearance of $\bar{J}^i(x)$ in the theory with higher derivatives lies at the origin of the Schwinger terms in Eq. (5.13).

Fronsdal has pointed out[20] that the addition of particular terms to Gell-Mann's commutation relations allows for the absence of spacelike states in the saturation problem.[19] The Schwinger terms we have discussed here have no relation to such additional terms considered by Fronsdal.[20] Rather, they must be attributed to the presence of higher derivatives in the Lagrangian (4.4), as we have shown through the decomposition in Eq. (5.16).

We conclude this section by speculating on the possibility of constructing a simple nonfactorizable model on the basis of the Lagrangian formalism developed here.

We already observed in Sec. IV that the expression for the current can be changed by performing a contact transformation. In particular, the Lagrangian in Eq. (4.25) gives the conserved current of Eq. (4.26). Such current differs by a divergence from the current in Eq. (4.20): The difference is the divergence of the tensor $\mathcal{V}_{\nu\mu}$ in Eq. (4.27). Let us call

$$J_\mu{}'(x)=J_\mu(x)-b\partial^\nu(\phi^\dagger(x)J_{\nu\mu}\phi(x))\,,$$

with b real. A nonfactorized model can be obtained by introducing a current

$$J_\mu{}'^i(x)=\phi^\dagger(x)I_\mu\gamma^i\phi(x)-b\partial^\nu(\phi^\dagger(x)J_{\nu\mu}\delta^i\phi(x))\,,$$

where δ^i acts on the internal space. The imposition of the charge-current commutation relations (5.14) and the charge-charge relations leads to

$$[\gamma^i,\gamma^j]=if^{ijk}\gamma^k\,,\quad [\gamma^i,\delta^j]=if^{ijk}\delta^k\,.$$

On the other hand, using the canonical commutation relations, one sees that the additive tensor term (whose associated charge is zero) commutes with itself for any δ^i. In particular, it is consistent to choose

$$[\delta^i,\delta^j]=0\,.$$

In this way one obtains a nonfactorized model of the kind considered by Hamprecht and Kleinert.[22]

VI. ALTERNATIVE INTERPRETATION IN TERMS OF BOUND SYSTEM IN TWO SPACE DIMENSIONS

In this section we present an interpretation of the model developed before in terms of a bound system. The trajectories defined from the infinite-component wave equation will be interpreted as analytical interpolations of bound states of a particular dynamical system. Such an approach seems useful in at least two respects. First, it will offer a natural procedure for extending the model that we have discussed so far to a richer model that produces families of trajectories. This will be considered in Sec. VII. Second, we think that a description in terms of a composite system may offer a direct approach for the extension to a nonfactorized model that is in a way different, in general, from the model suggested at the end of Sec. V. It will in fact be sufficient to attribute a definite behavior under the internal symmetry group to the constituent particles.

We shall concentrate on the simplest possibility, that of bound states of two particles, supposing a nonrelativistic relative motion. The latter assumption, which is *a priori* artificial, has a number of possible justifications,[23] but essentially it is adopted here because of the great simplification it provides.[7]

The interpretation is based on the observation that the operator W/p^2, in the rest system of the bound state, $\mathbf{p}\equiv\mathbf{p}_R$ with $\mathbf{p}_R=0$, coincides with the modulus

[22] B. Hamprecht and H. Kleinert, Phys. Rev. **180**, 1410 (1969).
[23] G. Morpurgo, Physics 2, 95 (1965).

squared of the intrinsic angular momentum. More precisely, it can be interpreted as the squared modulus of the relative orbital angular momentum when the constituent particles are spinless, assuming for the $SL(2,C)$ representation R, to which $\phi(p)$ belongs, an irreducible representation with $j_0=0$. For spin-$\frac{1}{2}$ constituents, the natural choice for R will be the direct product of an irreducible representation with $j_0=0$ and two Dirac representations [that is, $(\frac{1}{2},\frac{3}{2})\oplus(\frac{1}{2},-\frac{3}{2})$]. The representation with $j_0=0$ will describe the states of internal orbital motion. It will be unitary, and therefore its value of j_1 will be either pure imaginary or real, satisfying $0<j_1<1$, as in the Majorana representation $(0,\frac{1}{2})$ (see Appendix B).[24]

More definitely, we shall write the operator W/p^2 as a differential operator defined on the unit sphere and, correspondingly, the wave function ϕ will be interpreted as a function $\phi(p_R,\mathbf{u})$ defined on the unit sphere, $\mathbf{u}$ denoting a unit vector in three space dimensions. The interpretation is possible only if the solutions are all timelike. By considering in place of Eq. (3.1) the corresponding equation divided by p^2,

$$[(p_R^2-\beta)^2-\tfrac{1}{4}\alpha^2-\alpha^2W/p_R^2]\phi(p_R,\mathbf{u})=0\,, \quad (6.1)$$

one can eliminate the isolated lightlike and vacuumlike solutions found in Eq. (3.7). A correct choice of the representation, on the other hand, allows for eliminating the spacelike solutions of the discrete spectrum, as we have seen.

Let us consider the operator

$$\mathfrak{D}=(\tfrac{1}{4}+W/p_R^2)^{1/2} \quad (6.2)$$

in the space of the solutions $\phi(p_R,\mathbf{u})$. The operator $\mathfrak{D}$ is well-defined: In fact, $\frac{1}{4}+(W/p_R^2)$ is positive definite in the space of the solutions of Eq. (6.1). We can rewrite Eq. (6.1) in the form

$$\phi(p_R,\mathbf{u})=[(p_R^2-\beta)/\alpha]\mathfrak{D}^{-1}\phi(p_R,\mathbf{u}) \quad (6.3)$$

or, equivalently,

$$\phi(p_R,\mathbf{u})=\frac{p_R^2-\beta}{\alpha}\mathfrak{D}^{-1}\int d\mathbf{v}\,\phi(p_R,\mathbf{v})\delta(\mathbf{u}-\mathbf{v})$$
$$=\frac{p_R^2-\beta}{\alpha}\int d\mathbf{v}\,\phi(p_R,\mathbf{v})\sum_{j,m}(j+\tfrac{1}{2})^{-1}\times Y_{jm}(\mathbf{v})Y_{jm}^*(\mathbf{u})\,. \quad (6.4)$$

The equations

$$\sum_{j,m}Y_{jm}(\mathbf{v})Y_{jm}^*(\mathbf{u})=\delta(\mathbf{u}-\mathbf{v})\,, \quad (6.5)$$

$$\mathfrak{D}Y_{jm}(\mathbf{u})=(j+\tfrac{1}{2})Y_{jm}(\mathbf{u}) \quad (6.6)$$

have been used. Recalling that

$$1/|\mathbf{u}-\mathbf{v}|=2\pi\sum_{j,m}(j+\tfrac{1}{2})^{-1}Y_{jm}(\mathbf{v})Y_{jm}^*(\mathbf{u})\,, \quad (6.7)$$

one can rewrite Eq. (6.4) as

$$\phi(p_R,\mathbf{u})=\frac{p_R^2-\beta}{2\pi\alpha}\int d\mathbf{v}\frac{\phi(p_R,\mathbf{v})}{|\mathbf{u}-\mathbf{v}|}\,. \quad (6.8)$$

Let us now consider the Schrödinger equation for a hydrogen atom in two space dimensions. In momentum space the equation is

$$(\mathbf{k}^2+\eta^2)\psi(\mathbf{k})=\frac{2\mu(Ze^2)}{\pi}\int d^2q\frac{\psi(\mathbf{q})}{|\mathbf{q}-\mathbf{k}|}\,, \quad (6.9)$$

where $\eta^2=-2\mu E$ and E is the binding energy; μ is the reduced mass. One can transform Eq. (6.9) by making a stereographic projection from the plane, where $\mathbf{k}$ lies, to the unit sphere in three dimensions. Let us call $\mathbf{u}$ the unit vector corresponding to $\mathbf{k}$. The equation one obtains[25] is

$$\psi(\mathbf{u})=\frac{\mu(Ze^2)}{2\pi\eta}\int d\mathbf{v}\frac{\psi(\mathbf{v})}{|\mathbf{u}-\mathbf{v}|}\,. \quad (6.10)$$

Comparison of Eqs. (6.8) and (6.10) shows that the wave equation (6.1) can in fact be interpreted as the wave equation of a two-dimensional hydrogen atom, provided that the coupling constant Ze^2 is taken to be an increasing function of the energy E, namely,

$$Ze^2=\frac{\eta}{\mu}\frac{p_R^2-\beta}{\alpha}\,, \quad (6.11)$$

where p_R^2 is now the total mass of the bound system. On the other hand, the eigenvalues of a two-dimensional hydrogen atom are given by the formula[25]

$$E_j=-\mu(Ze^2)^2/[2(j+\tfrac{1}{2})^2]\,. \quad (6.12)$$

Inserting, in Eq. (6.12), Eq. (6.11), one verifies that the mass spectrum of Eq. (6.1) has indeed been exactly reobtained.

The treatment so far is noncovariant: We have fixed, from the beginning, the reference frame as that with $\mathbf{p}_R=0$. A covariant treatment can be performed following the line of the work by Bisiacchi *et al.*[7]

Finally, we remark that the interpretation of a hydrogenlike atom with energy-dependent coupling that is proposed here is to be regarded as a mechanism to simulate the appearance of an increasing number of open channels with increasing energy.

[24] For a system composed by scalar particles this contradicts the requirements on the representation of $SL(2,C)$ (see Sec. III) to avoid vacuumlike states.

[25] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966); E. Kyriakopoulos, Phys. Rev. **174**, 1846 (1968).

VII. BOUND MODEL IN THREE SPACE DIMENSIONS. FAMILIES OF TRAJECTORIES

We shall consider here the extension of the bound model discussed in Sec. VI to a model in three space dimensions. The extension will be made by simply writing down the equation analogous to Eq. (6.8) for three dimensions and developing an equivalent formulation in terms of an infinite-component wave equation. The model will be seen to describe a whole family of parallel linear trajectories.

We shall here proceed in a way opposite that of Sec. VI. Let us start from the Schrödinger equation of the three-dimensional hydrogen atom

$$(\mathbf{k}^2+\eta^2)\psi(\mathbf{k})=\frac{\mu(Ze^2)}{\pi^2}\int d^3q\frac{\psi(\mathbf{q})}{|\mathbf{q}-\mathbf{k}|^2}, \qquad (7.1)$$

where $\eta^2=-2\mu E$, μ is the reduced mass, and E is the binding energy.

We can transform Eq. (7.1) by stereographically projecting each vector $\mathbf{k}$ in three dimensions into a corresponding unit vector $\mathbf{u}$ of the unit sphere in four dimensions.[25] The transformed equation, which is quite analogous to Eq. (6.8) but is for three dimensions instead of two, is

$$\psi(\mathbf{u})=\frac{\mu(Ze^2)}{2\pi^2\eta}\int d\mathbf{v}\frac{\psi(\mathbf{v})}{|\mathbf{u}-\mathbf{v}|^2}. \qquad (7.2)$$

As in Sec. VI [see Eq. (6.11)], we put

$$Ze^2=\frac{\eta}{\mu}\frac{m^2-\beta'}{\alpha'}, \qquad (7.3)$$

where m is the total mass of the composite system and α' and β' are parameters. The energy spectrum for a three-dimensional hydrogen atom is

$$E_n=-\mu(Ze^2)^2/(2n^2).$$

Inserting Eq. (7.3) into this equation, one gets

$$(m^2-\beta')^2=n^2\alpha'^2. \qquad (7.4)$$

We shall concentrate on that solution of Eq. (7.4) which gives an m^2 increasing with n. The other solution will be discarded in the interpretation in terms of an infinite-component wave equation, which will be given below. The mass spectrum is

$$m^2=\alpha' n+\beta', \qquad (7.5)$$

where n is the principal quantum number. We shall put $n=n_r+j$, where n_r is the radial quantum number, $n_r=1,2,\ldots$, and j is the orbital quantum number. Equation (7.5) becomes

$$m^2=\alpha'(n_r+j)+\beta'. \qquad (7.5')$$

We consider the trajectories obtained from (7.5′) for fixed n_r and varying j. Such trajectories are parallel and separated by one unit on the (m^2,j) plane. The highest trajectory (to be called the "mother" trajectory) has $n_r=1$; it will be written in the form used in Sec. II,

$$m^2=\alpha(j+\tfrac{1}{2})+\beta, \qquad (7.6)$$

by introducing $\alpha=\alpha'$ and $\beta=\beta'+\frac{1}{2}\alpha'$. Equation (7.5′) becomes

$$m^2=\alpha(j+n_r-\tfrac{1}{2})+\beta. \qquad (7.7)$$

Equation (7.2) becomes

$$\psi(\mathbf{u})=\frac{m^2-\beta+\frac{1}{2}\alpha}{2\pi^2\alpha}\int d\mathbf{v}\frac{\psi(\mathbf{v})}{|\mathbf{u}-\mathbf{v}|^2}. \qquad (7.8)$$

The linearity of the trajectories has again been obtained at the price of an energy-dependent coupling constant, as shown in Eq. (7.3).

We shall now obtain from Eq. (7.8) an infinite-component wave equation. To such purpose, we use a formula analogous to Eq. (6.7) but for $\mathbf{u},\mathbf{v}$ in three dimensions instead of two[26]:

$$\frac{1}{|\mathbf{u}-\mathbf{v}|^2}=2\pi^2\sum_{n,j,m}\frac{1}{n}Y_{njm}(\mathbf{u})Y_{njm}^*(\mathbf{v}). \qquad (7.9)$$

In Eq. (7.9) the integer n is the principal hydrogenlike quantum number, j and m are the orbital and magnetic quantum numbers, respectively, and $Y_{njm}(\mathbf{u})$ are the hyperspherical harmonics on the four-dimensional unit sphere. The functions $Y_{njm}(\mathbf{u})$ are eigensolutions, with eigenvalue n, of the operator $(D^T+1)^{1/2}$, where D^T is the modulus square of the angular momentum in four dimensions.[7,25] Inserting Eq. (7.9) into Eq. (7.8), applying the operator $(D^T+1)^{1/2}$ to both sides of the resulting equation, and using the completeness of the hyperspherical harmonics, one obtains

$$[(m^2-\beta+\tfrac{1}{2}\alpha)^2-\alpha^2(D^T+1)]\psi(\mathbf{u})=0. \qquad (7.10)$$

The wave function ψ is defined on the unit four-dimensional sphere. We shall interpret ψ, in analogy to what was done in Sec. VI as a function of the mass and of a spin variable. Correspondingly, D^T will act as a matrix.

In order to write Eq. (7.10) in a general Lorentz frame, we note that D^T can be expressed through the invariants $\tilde{W}$ and $\tilde{p}^2$ of $IO(4,1)$. The invariant $\tilde{W}$ of $IO(4,1)$ is

$$\tilde{W}=\tfrac{1}{2}\tilde{p}_a\tilde{p}^aJ_{bc}J^{bc}-\tilde{p}_a\tilde{p}_bJ^{ac}J^b{}_c \quad (a,b,c=0,1,\ldots,4), \qquad (7.11)$$

and $\tilde{p}^2=\tilde{p}_a\tilde{p}^a$. Specifically, D^T is $\tilde{W}/\tilde{p}^2$ in the $IO(4,1)$ reference frame where $\tilde{p}=p_R\equiv(m,0,0,0,0)$. However, $\tilde{W}/\tilde{p}^2$ is an invariant for $IO(4,1)$, and therefore in any

[26] There is a misprint of a factor $2\pi^2$ in Eq. (A2) of Ref. 7 [see J. Schwinger, J. Math. Phys. 5, 1606 (1964)].

frame of $IO(4,1)$, Eq. (7.10) has the form

$$[\tilde{p}^2(\tilde{p}^2-\beta+\tfrac{1}{2}\alpha)^2-\alpha^2\tilde{p}^2-\alpha^2\tilde{W}]\psi(\tilde{p})=0. \quad (7.12)$$

In a general Lorentz frame we shall identify $\tilde{p}$ with $p\equiv(p_0,p_1,p_2,p_3,0)$; thus

$$\tilde{W}=W-p^2\Gamma_\mu\Gamma^\mu+p_\mu p_\nu\Gamma^\mu\Gamma^\nu, \quad (7.13)$$

where

$$\Gamma_\mu=J_{\mu 4}.$$

Finally we have been led to the wave equation

$$[p^2(p^2-\beta+\tfrac{1}{2}\alpha)^2-\alpha^2p^2-\alpha^2\tilde{W}]\psi(p)=0, \quad (7.14)$$

where $\tilde{W}$ is given in Eq. (7.13).

With respect to the choice of the $O(4,1)$ representation to which $\psi(p)$ belongs, it must be noted that those representations which are suitable for the description of a hydrogenlike atom are such that the quadratic invariant $J_{ab}J^{ab}$ is negative, and the biquadratic invariant $-w_aw^a$ vanishes[7]; w^a is

$$w^a=\tfrac{1}{8}\epsilon^{abcde}J_{bc}J_{de}.$$

We already saw that the eigenvalue of $\tilde{W}/\tilde{p}^2+1$ is n^2. Thus the mass spectrum of Eq. (7.14) is formally analogous to that for Eq. (3.1), except for the substitution of $(j+\frac{1}{2})^2$ in place of n^2. In particular, the spacelike solutions have negative n^2, as in the case of $SL(2,C)$ in connection with Eq. (3.1), where $(j+\frac{1}{2})^2$ was negative for such solutions. Spacelike solutions are thus avoided.

VIII. CONCLUSIONS

We shall mention here some results of the present work which appear of particular interest.

We have seen that the requirement of a linear mass spectrum automatically eliminates the continuum spacelike solutions. The question of the presence of a discrete spacelike spectrum is directly related to the choice of the representation. The discrete spacelike spectrum can be eliminated under quite natural choices of such a representation.

In fact, an additional result has been obtained: The requirement that certain unphysical solutions be absent, together with the simplifying assumption of a polynomial Lagrangian, leads to the conclusion that the lowest-degree equation satisfying such conditions possesses a linear mass spectrum.

The quantization procedure we have adopted, based on a choice of canonical variables originally due to Ostrogradsky,[9,21] is not to be considered as unique. The appearance of Schwinger terms in the commutators between time components of the conserved current essentially follows from that choice. For instance, it can be shown that a suitable choice of the canonical variables can lead to the disappearance of such Schwinger terms.[27] The particular form, in Eq. (5.13), of the Schwinger terms is thus to be considered as related to the special quantization procedure we have adopted here.

The reinterpretation, in Sec. VI, of the wave equation discussed in this work in terms of a two-dimensional compound system, can be considered as the realization of a simplest model of a quantum-mechanical bound state, having linear trajectories (see Ref. 8). The extension to a three-dimensional system, in Sec. VII, does not destroy the linearity, but rather implies the existence of families of trajectories (daughters), bringing the model in closer contact with other models (apart from questions of degeneracy).

The model presented here is of course rather far from providing a complete picture, but we consider that its value mostly resides in underlining some still interesting aspects of the infinite-wave-component approach, especially in relation to current algebra and to linear trajectories.

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APPENDIX A: ALGEBRA OF POINCARÉ GROUP AND LITTLE GROUPS

The Poincaré algebra is

$$[M^{\mu\nu},M^{\rho\sigma}]=i(M^{\mu\rho}g^{\nu\sigma}+M^{\nu\sigma}g^{\mu\rho}-M^{\nu\rho}g^{\mu\sigma}-M^{\mu\sigma}g^{\nu\rho}),$$
$$[M^{\mu\nu},P^\rho]=i(P^\mu g^{\rho\nu}-P^\nu g^{\rho\mu}), \quad [P^\mu,P^\nu]=0. \quad (A1)$$

Introducing

$$M_k=\tfrac{1}{2}\epsilon_{klm}M_{lm}, \quad N_k=M_{k0}, \quad (A2)$$

one has

$$[M_l,M_m]=-[N_l,N_m]=i\epsilon_{lmk}M_k,$$
$$[M_l,N_m]=[N_l,M_m]=i\epsilon_{lmk}N_k. \quad (A3)$$

The Casimir operators are $P^2=P_\mu P^\mu$ and $W=-w_\mu w^\mu$, where w_μ is the Pauli-Lubansky four-vector

$$w_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma. \quad (A4)$$

Thus

$$W=\tfrac{1}{2}P^2M_{\mu\nu}M^{\mu\nu}-P^\rho P^\sigma M_{\rho\mu}M_\sigma{}^\mu. \quad (A5)$$

The vector w_μ has three independent components (because $w^\mu P_\mu=0$); they generate the little groups associated with P_μ. For the eigenvalues p_μ of P_μ, one has (a) p_μ timelike, (b) p_μ spacelike, (c) p_μ lightlike, and (d) p_μ vacuumlike.

(*a*) p_μ *timelike*. In the standard frame, $p^\mu=(m,0,0,0)$,

$$w^\mu\equiv(0,m\mathbf{M}), \quad (A6)$$

and the little group is $SU(2)$, generated by $\mathbf{M}$. The eigenvalues of W are

$$W=m^2j(j+1), \quad (A7)$$

where $j(j+1)$ is an eigenvalue of $\mathbf{M}^2$.

(*b*) p_μ *spacelike*. Choosing the standard frame such

[27] R. Casalbuoni and G. Longhi, Nuovo Cimento **70A**, 329 (1970).

that $p^\mu \equiv (0,0,0,q)$, one has

$$w^\mu \equiv (qM_3,\, qN_2,\, -qN_1,\, 0)\,. \quad \text{(A8)}$$

The little group is $SU(1,1)$ with the algebra

$$[N_1,N_2]=-iM_3,\ [M_3,N_1]=iN_2,\ [N_2,M_3]=iN_1. \quad \text{(A9)}$$

The eigenvalues of W are

$$W=-q^2 j(j+1)\,, \quad \text{(A10)}$$

where $j(j+1)$ is an eigenvalue of the $SU(1,1)$ Casimir operator $Q=M_3{}^2-N_1{}^2-N_2{}^2$. The unitary representations of $SU(1,1)$ are infinite dimensional. They belong to the following three classes.

(i) *Principal series.* $0\leqslant j+\frac{1}{2}<i\infty$ on the imaginary axis and, for the eigenvalue m of M_3,
$m=0, \pm 1, \pm 2, \ldots,$ integer representation
$=\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots,$ half-integer representation.

(ii) *Discrete series.* $j=-\frac{1}{2}, -1, -\frac{3}{2}, \ldots$ and
$m=-j, -j+1, -j+2, \ldots,$ "plus" representation
$=j, j-1, j-2, \ldots,$ "minus" representation.

(iii) *Supplementary series.* $-\frac{1}{2}<j\leqslant 0$ and
$m=0, \pm 1, \pm 2, \ldots.$

(*c*) p_μ *lightlike.* In the standard frame, $p^\mu \equiv (\omega,0,0,-\omega)$

$$w^\mu \equiv (-\omega M_3, \omega F_2, \omega F_1, \omega M_3)\,, \quad \text{(A11)}$$

with

$$F_1=N_1+M_2\,,\quad F_2=M_1-N_2\,. \quad \text{(A12)}$$

The little group is $E(2)$, with the algebra

$$[F_1,F_2]=0\,,\quad [F_2,M_3]=iF_1\,,\quad [M_3,F_1]=iF_2\,. \quad \text{(A13)}$$

The eigenvalues of W are

$$W=\omega^2 f^2\,, \quad \text{(A14)}$$

where f^2 is an eigenvalue of the $E(2)$ Casimir operator $F_1{}^2+F_2{}^2$. The unitary representations belong to the following two classes.

(*i*) *Principal series.* $0<f<\infty$ and
$m=0, \pm 1, \pm 2, \ldots,$ integer representation
$=\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots,$ half-integer representation.

(*ii*) *Discrete series.* $f=0$ and
$m=0, \pm\frac{1}{2}, \pm 1, \ldots.$

In the discrete series, $F_1=0$, $F_2=0$, and from (A11)

$$w^\mu=-p^\mu M_3\,. \quad \text{(A15)}$$

In a general frame,

$$w^\mu = p^\mu \mathbf{M}\cdot\mathbf{p}/|\mathbf{p}|\,,$$

giving as eigenvalues

$$w^\mu=-\lambda p^\mu\,, \quad \text{(A16)}$$

where λ is the helicity. One has $W=0$, from (A14) for $f=0$; however, one can still define the eigenvalues of W/p^2, as limits from timelike p^μ or from spacelike p^μ, obtaining, in both cases,

$$W/p^2=\lambda^2\,. \quad \text{(A17)}$$

(*d*) p_μ *vacuumlike* ($p^\mu\equiv 0$). The little group is $SL(2,C)$ (see Appendix B).

APPENDIX B: REPRESENTATIONS OF $SL(2,C)$

The $SL(2,C)$ generators $J_{\mu\nu}$ obey the algebra

$$[J^{\mu\nu},J^{\rho\sigma}]=i(J^{\mu\rho}g^{\nu\sigma}+J^{\nu\sigma}g^{\mu\rho}-J^{\nu\rho}g^{\mu\sigma}-J^{\mu\sigma}g^{\nu\rho})\,. \quad \text{(B1)}$$

The irreducible representations of $SL(2,C)$ can be specified by the pair (j_0,j_1), where j_0 is an integer or a half-integer, and j_1 is an arbitrary complex number. The representations (j_0,j_1) and $(-j_0,-j_1)$ are equivalent; therefore, j_0 can be restricted to $j_0\geqslant 0$. j_0 and j_1 are related to the eigenvalues of the two $SL(2,C)$ Casimir operators

$$C_1=\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}\,,\quad C_2=\tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma} \quad \text{(B2)}$$

by

$$C_1=j_0{}^2+j_1{}^2-1\,,\quad C_2=2ij_0j_1\,. \quad \text{(B3)}$$

j_0 is the minimum spin value contained in the representation (j_0,j_1). If $|j_1|=j_0+n$, $n=1,2,\ldots$, then the representation is finite dimensional and nonunitary, and its spin content is

$$j_0, j_0+1, \ldots, |j_1|-1\,.$$

If $|j_1|\neq j_0+n$, then the representation is infinite dimensional and the spin content is $j_0, j_0+1, \ldots.$

Finite-dimensional representations are sometimes specified by giving (l_1l_2), with l_1 and l_2 integers or half-integers, defined as

$$l_1=\tfrac{1}{2}(|j_0+j_1|-1)\,,\quad l_2=\tfrac{1}{2}|j_1-j_0-1|\,. \quad \text{(B4)}$$

Unitary representations are as follows.

(*a*) *Principal series.* j_1 pure imaginary.

(*b*) *Supplementary series.* $j_0=0$, j_1 real with $0<|j_1|<1$.

A generally reducible $SL(2,C)$ representation R allows for a bilinear, Hermitian, invariant, nondegenerate form if and only if it contains, for each irreducible component (j_0,j_1), also $(j_0,-j_1{}^*)$; in particular, if R is irreducible, i.e., $R\equiv(j_0,j_1)$, either j_1 is pure imaginary, or $j_0=0$ and j_1 is real. Under parity $(j_0,j_1)\to(j_0,-j_1)$. Therefore for an irreducible R, definite parity implies either $(j_0,0)$ or $(0,j_1)$. For general R, a four-vector operator Γ_μ exists if and only if R contains, for each (j_0,j_1), at least one of the components of

$$(0,2)\otimes(j_0,j_1)=(j_0,\, j_1-1)\oplus(j_0,\, j_1+1)\oplus(j_0-1,\, j_1)\oplus(j_0+1,\, j_1)\,. \quad \text{(B5)}$$

Within any such R, the representative of Γ_μ can be calculated from the relation

$$[J^{\mu\nu},\Gamma^\rho]=i(\Gamma^\mu g^{\rho\nu}-\Gamma^\nu g^{\rho\mu}). \qquad \text{(B6)}$$

In particular,

$$\Gamma_0|\tau,j,m\rangle=\sum_{\tau'} c_j^{\tau,\tau'}|\tau',j,m\rangle, \qquad \text{(B7)}$$

where $\tau\equiv(j_0,j_1)$, j and m specify the eigenstates of $\mathbf{J}^2$ and J_3 (one defines $J_i=\frac{1}{2}\epsilon_{ijk}J_{jk}$), and

$$c_j^{\tau,\tau'}=c^{\tau,\tau'}[(j\mp j_0)(j\pm j_0+1)]^{1/2} \quad \text{for } \tau'\equiv(j_0\pm1,\, j_1) \qquad \text{(B8)}$$

$$=c^{\tau,\tau'}[(j\mp j_1)(j\pm j_1+1)]^{1/2} \quad \text{for } \tau'\equiv(j_0,\, j_1\pm1). \qquad \text{(B8')}$$

The coefficients $c^{\tau,\tau'}$ are left undetermined. The representatives of Γ_i can then be determined from

$$[K_i,\Gamma_0]=i\Gamma_i, \quad K_i=J_{i0}. \qquad \text{(B9)}$$

An irreducible R which allows for the existence of Γ_μ coincides with either one of the two (unitary) Majorana representations $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$, as is evident from (B5). For them, putting $c^{\tau,\tau}=1$, one has

$$\Gamma_0|\tau,j,m\rangle=(j+\tfrac{1}{2})|\tau,j,m\rangle. \qquad \text{(B10)}$$

Finally we recall that by suitably normalizing Γ_μ one has, in a Majorana representation,[28]

$$[\Gamma_\mu,\Gamma_\nu]=-iJ_{\mu\nu}, \quad \{\Gamma_\mu,\Gamma_\nu\}=-\{J_{\mu\lambda},J_\nu{}^\lambda\}-g_{\mu\nu}. \qquad \text{(B11)}$$

APPENDIX C: EXPRESSIONS FOR SCHWINGER TERMS

The Schwinger terms in Eq. (5.13), calculated by formal use of the canonical commutation relations, are found to be

$$\begin{aligned}S_\rho{}^{(1)ij}(x,y)=i\{&(\partial_\rho\tau_r(y))\gamma^j\gamma^i\phi_r(y)\\&+3(\nabla_\sigma\nabla^\sigma\partial_\rho\tau_3(y))\gamma^j\gamma^i\phi_1(y)\\&+\nabla^\nu[(\partial_\rho\partial_\nu\tau_3(y))\gamma^j\gamma^i\phi_1(y)]\\&-\tau_r(x)\gamma^i\gamma^j(\nabla_\rho\phi_r(y))+\text{H.c.}\}\\&+(x\leftrightarrow y,\, i\leftrightarrow j),\end{aligned}$$

$$\begin{aligned}S_{\rho\nu}{}^{(2)ij}(x,y)=i\{&(\partial_\rho\partial_\nu\tau_3(y))\gamma^j\gamma^i\phi_1(y)\\&-\tau_r(x)\gamma^i\gamma^j\phi_r(x)g_{\rho\nu}\\&-\tau_3(x)\gamma^i\gamma^j(\partial_\rho\partial_\nu\phi_1(y))\\&+2(\partial_\rho\tau_3(y))\gamma^j\gamma^i(\partial_\nu\phi_1(y))+\text{H.c.}\}\\&-(x\leftrightarrow y,\, i\leftrightarrow j),\end{aligned}$$

$$\begin{aligned}S_{\rho\nu\lambda}{}^{(3)ij}(x,y)=i\{&\tau_3(x)\gamma^i\gamma^j(\partial_\rho\phi_1(y))g_{\nu\lambda}\\&+2\tau_3(y)\gamma^j\gamma^i(\partial_\rho\phi_1(x))g_{\nu\lambda}\\&-2(\partial_\rho\tau_3(y))\gamma^j\gamma^i\phi_1(x)g_{\nu\lambda}+\text{H.c.}\}\\&+(x\leftrightarrow y,\, i\leftrightarrow j),\end{aligned}$$

$$\begin{aligned}S_{\rho\nu\lambda\sigma}{}^{(4)ij}(x,y)=i\{&2\tau_3(x)\gamma^i\gamma^j\phi_1(y)g_{\rho\nu}g_{\lambda\sigma}\\&-2\tau_3(y)\gamma^j\gamma^i\phi_1(x)g_{\rho\nu}g_{\lambda\sigma}+\text{H.c.}\},\end{aligned}$$

where a summation convention in r for $r=1,2$ is adopted and

$$\tau_1(x)=2\beta\phi_2{}^\dagger(x)-\pi_3(x),$$
$$\tau_2(x)=-2\phi_3{}^\dagger(x), \quad \tau_3(x)=-\phi_2{}^\dagger(x).$$

One has the symmetry rules

$$S_\rho{}^{(1)ij}(x,y)=S_\rho{}^{(1)ji}(y,x),$$
$$S_{\rho\nu}{}^{(2)ij}(x,y)=-S_{\rho\nu}{}^{(2)ji}(y,x),$$
$$S_{\rho\nu\lambda}{}^{(3)ij}(x,y)=S_{\rho\nu\lambda}{}^{(3)ji}(y,x),$$
$$S_{\rho\nu\lambda\sigma}{}^{(4)ij}(x,y)=-S_{\rho\nu\lambda\sigma}{}^{(4)ji}(y,x).$$

From (5.13), the expression for the commutators, one sees that only those parts of the tensors S which are totally symmetrical in the tensor indices, ρ, ν, etc., contribute to the commutators. The commutation rules (5.14) are easily verified using

$$[Q^i,\phi_r(x)]=-\gamma^i\phi_r(x), \quad [Q^i,\phi_r{}^\dagger(x)]=\phi_r{}^\dagger(x)\gamma^i,$$
$$[Q^i,\pi_r(x)]=\pi_r(x)\gamma^i, \quad [Q^i,\pi_r{}^\dagger(x)]=-\gamma^i\pi_r{}^\dagger(x).$$

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[28] See, for instance, A. Böhm, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Gordon and Breach, New York, 1968), Vol X B, pp. 483–526.