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Multiparticle States with Spin-Independent Poincaré Transformation Properties

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We consider systems of noninteracting particles. We define, in terms of standard on-mass-shell states and direct-product helicity states, three complete classes of multiparticle states with spin-independent Poincaré transformation properties. No auxiliary spin group is used, and the new "scalar spin" state labels, which are eigenvalues of generalized covariant helicity operators, have a direct physical interpretation in terms of single-particle quantum numbers. In the two-particle case we compare our states with Jacob and Wick's c.m. angular momentum states.

I. INTRODUCTION

WHEN theories of particles with spin have been developed, it has been common practice to introduce spinors and finite-dimensional nonunitary representations of the homogeneous Lorentz group. In S-matrix theory, M-functions[1] are defined in terms of finite-dimensional representations $D_{\lambda'\lambda}^{\sigma}(L(p))$ of single-particle velocity transformations $L(p)$. One expresses the physical scattering amplitude as a series of D-function products with M-function coefficients. These D-functions form a complete set in which to expand S-matrix elements. However, there is no physical interpretation of the M-function "spin" labels in terms of single-particle quantum numbers.

The same criticisms apply in the case of field theory, where one constructs fields with simple homogeneous Lorentz group transformation properties. The characteristic field labels are related to eigenvalues of functions of auxiliary-spin-group generators, and not to single-particle quantum numbers alone.

Recently, in extensions of S-matrix theory in particular, one has encountered inconsistencies leading to the concept of families of particles.[2,3] The purpose of our investigation is to determine to what extent these inconsistencies arise from the use of auxiliary spin groups, and to what extent they are determined by the Poincaré invariance of the theory. In particular, we shall show that it is possible to derive a new partial-wave decomposition formula in terms of which the physical arguments for the introduction of "daughter" and "conspirator" trajectories in conventional Regge theory become more transparent. With the introduction of extended spin groups, in both S-matrix theory and field theory one obtains a simplification of the formal mathematical structure at the expense of clarity of physical interpretation. For this reason, we wish to construct analogous theories without auxiliary spin groups. Although this will initially involve us with more complicated mathematical structures, the physical significance of all parameters appearing in the equations will be unambiguous.

As a first step, we indicate how one may construct several special kinds of multiparticle state. These states will be labeled by eigenvalues of operators which are closely related to the covariant helicity operators of Feldman and Matthews.[4] We shall show in another paper[5] that by using these states one can expand scattering amplitudes in a complete set of functions with invariant-amplitude coefficients. The invariant amplitudes will be parametrized only by eigenvalues of physically meaningful single-particle observables. It will then be possible to compare our amplitudes with those obtained using auxiliary-spin-group decompositions.

In this paper we shall be concerned exclusively with on-mass-shell states of noninteracting particles. We neglect altogether the trivial though tedious complications of parity, time reversal, and charge conjugation.

In Sec. II we define basic single-particle states, and introduce our notation. We then proceed in Sec. III to define, in terms of direct products of basic single-particle states, complete sets of multiparticle state which have spin-independent Poincaré transformation properties. These "scalar-spin-component" states enable us to treat systems of particles with spin as if all particles had spin zero.

In Sec. IV we change the independent single-particle momentum variables, labeling the scalar-spin-component states to include a maximum number of Poincaré scalar momentum products. We then define a class of multiparticle states which have only four frame-dependent labels: a total momentum $\mathbf{p}$ and a spin component λ. These states have the same transformation properties as single-particle states.

In Sec. V we examine several exceptional types of state. We first of all construct two-particle states and compare them with those of Jacob and Wick.[6,7] We then define an interesting set of three-particle states in which each particle is treated in an identical way.

In Sec. VI we summarize the results of the previous three sections.

In Appendix A we derive some properties of Wigner

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rotation functions. In Appendix B we give a derivation of our formula for the construction of scalar-spin-component states. In Appendix C we determine the Jacobian of a momentum-space transformation. This is associated with the replacement of single-particle momentum variables with a set of parameters including a maximum number of Poincaré scalar momentum products.

Note that some of our transformation formulas will not hold for systems of particles with half-integral spins because we have for simplicity neglected factors of the form $(-1)^{2\sigma}$, where σ denotes the particle spin. We shall discuss elsewhere[5] the complications introduced by these phases.

II. SINGLE-PARTICLE STATES

In a quantum-mechanical system the states of a single particle are in one-to-one correspondence with the eigenvalues of a complete commuting set of particle observables. We shall first of all introduce complete sets of single-particle operators. We shall then construct eigenstates with specified relative phases. The results are well known. We wish, however, to aquaint the reader with the notation which we shall need when we come to construct multiparticle states in the next section.

A. Single-Particle Observables

The principle of relativity suggests that by changing our frame of reference we may infer how a free particle appears under space-time translation, rotation, or constant-velocity transformation. We introduce, as observables, Hermitian operators which generate these transformations: energy $\hat{p}_0$, momentum $\hat{\mathbf{p}}$, and relativistic angular momentum $\hat{J}_{\mu\nu}$, respectively. These single-particle operators have the following commutation relations:

$$-i[\hat{J}_{\mu\nu},\hat{J}_{\rho\sigma}]=g_{\mu\rho}\hat{J}_{\nu\sigma}+g_{\nu\sigma}\hat{J}_{\mu\rho}-g_{\mu\sigma}\hat{J}_{\nu\rho}-g_{\nu\rho}\hat{J}_{\mu\sigma}, \quad (1)$$

$$-i[\hat{J}_{\mu\nu},\hat{p}_\rho]=g_{\mu\rho}\hat{p}_\nu-g_{\nu\rho}\hat{p}_\mu, \quad (2)$$

and

$$-i[\hat{p}_\mu,\hat{p}_\lambda]=0, \quad (3)$$

where

$$g_{\mu\nu}=\begin{cases}\delta_{\mu\nu}, & \mu=0\\ -\delta_{\mu\nu}, & \mu=1,2,3.\end{cases} \quad (4)$$

The three-vector observables $\hat{\mathbf{p}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ which generate spatial translations, rotations, and boosts are given in terms of the operators $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$ by

$$\hat{\mathbf{p}}\leftrightarrow\hat{p}_i, \quad (5)$$

$$\hat{\mathbf{J}}\leftrightarrow\hat{J}_i=-\tfrac{1}{2}\epsilon_{ijk}\hat{J}_{jk}, \quad (6)$$

and

$$\hat{\mathbf{K}}\leftrightarrow\hat{K}_i=-\hat{J}_{0i}. \quad (7)$$

They have the following commutation relations implied by Eqs. (1)–(3):

$$\begin{aligned}&-i[\hat{J}_i,\hat{J}_j]=\epsilon_{ijk}\hat{J}_k, && -i[\hat{p}_i,\hat{K}_j]=+\delta_{ij}\hat{p}_0,\\ &-i[\hat{J}_i,\hat{p}_j]=\epsilon_{ijk}\hat{p}_k, && -i[\hat{K}_i,\hat{K}_j]=-\epsilon_{ijk}\hat{J}_k,\\ &-i[\hat{J}_i,\hat{K}_j]=\epsilon_{ijk}\hat{K}_k, && -i[\hat{p}_i,\hat{p}_j]=0.\end{aligned} \quad (8)$$

In order to enumerate the states of a particle, we must choose a complete set of commuting operators from the universal enveloping algebra of the ten operators $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$. All other operators which commute with these are associated with internal degrees of freedom, and we shall not be concerned with them here.

The universal enveloping algebra of $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$ has two invariant elements, the square $\hat{p}^2$ of the four-momentum $\hat{p}_\mu$ and the square $\hat{W}^2$ of the Pauli-Lubanski spin $\hat{W}^\mu$, where

$$\hat{W}^\mu=-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu\hat{J}_{\rho\sigma}. \quad (9)$$

In terms of rotational tensors we have

$$\hat{W}^i\leftrightarrow\hat{\mathbf{W}}=\hat{\mathbf{p}}\times\hat{\mathbf{K}}-\hat{p}_0\hat{\mathbf{J}}, \quad (10)$$

$$\hat{W}^0=\hat{\mathbf{p}}\cdot\hat{\mathbf{J}}. \quad (11)$$

We shall label single-particle states by the eigenvalues of a complete commuting set of operators. Each set must include the two invariant operators $\hat{p}^2$ and $\hat{W}^2$, or two independent functions of them.

If we wish our single-particle states to behave simply under homogeneous Lorentz transformations, we must first of all choose a complete set of operators from the universal enveloping algebra of the homogeneous Lorentz group generators $\hat{J}_{\mu\nu}$. One could take the complete commuting set of operators: $\hat{\mathbf{K}}^2$, $\hat{\mathbf{J}}\cdot\hat{\mathbf{K}}$, $\hat{\mathbf{J}}^2$, and $\hat{\mathbf{J}}_3$. A complete commuting set with respect to the Poincaré algebra generated by $\hat{J}_{\mu\nu}$ and $\hat{p}_\mu$ is obtained by adding the two Poincaré invariants $\hat{p}^2$ and $\hat{W}^2$. Eigenstates of these six operators are the relativistic equivalent of total angular momentum states. They have rather complicated transformation properties under space-time translations.

If we wish our single-particle states to behave simply under spatial translations, we choose a complete set of operators from the translation generators $\hat{\mathbf{p}}$, namely, the momentum triplet $\hat{\mathbf{p}}$ itself. To obtain a complete set with respect to the full Poincaré algebra, we add the invariants $\hat{p}^2$ and $\hat{W}^2$, and the helicity operator $\hat{h}$, where

$$\hat{h}=|\hat{\mathbf{p}}|^{-1}\hat{W}^0=|\hat{\mathbf{p}}|^{-1}\hat{\mathbf{J}}\cdot\hat{\mathbf{p}}. \quad (12)$$

The Pauli-Lubanski spin component $\hat{W}^0$ has been defined in Eq. (11) and the operator $|\hat{\mathbf{p}}|$ corresponds to the magnitude of the three-momentum $\hat{\mathbf{p}}$,

$$|\hat{\mathbf{p}}|=[\hat{\mathbf{p}}^2]^{1/2}. \quad (13)$$

In order to understand the significance of helicity in this context, we shall examine the part played by the intrinsic spin of the particle $\hat{\mathbf{S}}$. We recall[8] that the total

[8] F. Gürsey, in *High Energy Physics*, edited by C. de Witt and M. Jacob (Gordon and Breach, New York, 1966).

angular momentum operator $\hat{\mathbf{J}}$ may be expressed in terms of the intrinsic spin $\hat{\mathbf{S}}$ and a particle position operator $\hat{\mathbf{x}}$,

$$\hat{\mathbf{J}}=\hat{\mathbf{S}}+\hat{\mathbf{x}}\times\hat{\mathbf{p}}, \tag{14}$$

where the spin $\hat{\mathbf{S}}$ is given by

$$\hat{\mathbf{S}}=\frac{1}{m}\left[\hat{p}_0\hat{\mathbf{J}}-\hat{\mathbf{p}}\times\hat{\mathbf{K}}-\frac{\hat{\mathbf{p}}\cdot\hat{\mathbf{J}}\hat{\mathbf{p}}}{m+\hat{p}_0}\right]. \tag{15}$$

The helicity is seen to be equal to the component of spin $\hat{\mathbf{S}}$ in the direction of motion,

$$\hat{\mathbf{J}}\cdot\hat{\mathbf{p}}=\hat{\mathbf{S}}\cdot\hat{\mathbf{p}}. \tag{16}$$

The spin $\hat{\mathbf{S}}$ is simply related to the Pauli-Lubanski spin $\hat{W}^\mu$,

$$\hat{\mathbf{S}}\leftrightarrow\hat{S}_i=-m^{-1}L(-\hat{\mathbf{p}})^i{}_\mu\hat{W}^\mu, \tag{17}$$

where the matrix $L(p)^\nu{}_\mu$ is defined by

$$L(p)^\nu{}_\mu=L(-\mathbf{p})_\nu{}^\mu\leftrightarrow\begin{bmatrix} p_0/m & -\mathbf{p}/m \\ -\mathbf{p}/m & 1+\mathbf{pp}/m(m+p_0)\end{bmatrix}. \tag{18}$$

Using Eq. (17), we can obtain the spin commutation relations

$$-i[\hat{S}_i,\hat{S}_j]=\epsilon_{ijk}\hat{S}_k \tag{19}$$

from those of the Pauli-Lubanski spin operators,

$$-i[\hat{W}^\mu,\hat{W}^\nu]=\epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho\hat{W}_\sigma. \tag{20}$$

These follow from the fundamental equations (1)–(3). We note that the spin $\hat{\mathbf{S}}$ commutes with each momentum component operator

$$[\hat{\mathbf{S}},\hat{p}_i]=0, \tag{21}$$

and by construction its magnitude is a Poincaré scalar,

$$\hat{\mathbf{S}}^2=\hat{W}^2. \tag{22}$$

Equation (21) implies that we can choose any component of the spin $\hat{\mathbf{S}}$ to take the place of the helicity operator $\hat{h}$ in constructing a set of single-particle states.

Equations for the eigenvectors of a suitable complete set of commuting operators determine the single-particle states up to a complex normalization factor. Relative phases are chosen by convention. We now consider two specific examples.

B. Standard States

We shall refer to eigenstates of the single-particle operators $\hat{p}^2$, $\hat{\mathbf{S}}^2$, $\hat{\mathbf{p}}$, and $\hat{S}_3$ with eigenvalues m^2, $\sigma(\sigma+1)$, $\mathbf{p}$, and λ, as standard states. In defining the relative phases of these states we first of all consider rest states $|m,\sigma\colon\mathbf{0},\lambda\rangle$. For a rotation $\hat{R}(\phi,\theta,\psi)$ defined by

$$\hat{R}(\phi,\theta,\psi)=\exp(-i\hat{J}_3\phi)\exp(-i\hat{J}_2\theta)\exp(-i\hat{J}_3\psi), \tag{23}$$

we have

$$\langle m,\sigma\colon\mathbf{0},\lambda'|\hat{R}(\phi,\theta,\psi)|m,\sigma\colon\mathbf{0},\lambda\rangle=D_{\lambda'\lambda}{}^\sigma(\phi,\theta,\psi). \tag{24}$$

The function $D_{\lambda'\lambda}{}^\sigma(\phi,\theta,\psi)$ is a representation of the covering group of the rotation group $SU(2)$.

Let us now proceed to construct a state with arbitrary momentum $\mathbf{p}$. We first of all define a number of homogeneous Lorentz group elements associated with a four-momentum p_μ of the form

$$p_\mu\leftrightarrow m(\cosh\delta;\ \sinh\delta\sin\theta\cos\phi,\ \sinh\delta\sin\theta\sin\phi,\ \sinh\delta\cos\theta), \tag{25}$$

with

$$0\leqslant\delta<\infty,\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\phi<2\pi. \tag{26}$$

We define a boost $\hat{Z}(p)$, rotations $\hat{R}_3(p)$ and $\hat{R}(p)$, and a group element $\hat{L}(p)$ by

$$\hat{R}(p)=\hat{R}(\phi,\theta,-\phi),\quad \hat{R}_3(p)=\exp(i\hat{J}_3\phi), \tag{27}$$

$$\hat{Z}(p)=\exp(-i\hat{K}_3\delta), \tag{28}$$

and

$$\hat{L}(p)=\hat{R}(p)\hat{Z}(p)\hat{R}^{-1}(p). \tag{29}$$

With every element of the homogeneous Lorentz group $\hat{\Lambda}$, we associate a self-representation 4×4 matrix, which we shall denote by Λ. For example, in the case of the transormation $\hat{L}(p)$, we have the associated self-representation matrix $L(p)$ of Eq. (18).

We now define a single-particle state $|m;\sigma\colon\mathbf{p},\lambda\rangle$ in terms of a rest state $|m,\sigma\colon\mathbf{0},\lambda\rangle$ by the unitary transformation

$$|m,\sigma\colon\mathbf{p},\lambda\rangle=\hat{L}(p)|m,\sigma\colon\mathbf{0},\lambda\rangle. \tag{30}$$

Let us verify that the new state is an eigenstate of the momentum $\hat{\mathbf{p}}$ with eigenvalue $\mathbf{p}$ and of the spin operator $\hat{S}_3$ with eigenvalue λ. We have already noted that, in the self-representation, the operator $\hat{L}(p)$ takes the form $L(p)$ defined in Eq. (18). We have the momentum eigenvalue equation,

$$\begin{aligned}\hat{p}_\mu\hat{L}(p)|m,\sigma\colon\mathbf{0},\lambda\rangle&=\hat{L}(p)L(p)_\mu{}^\nu\hat{p}_\nu|m,\sigma\colon\mathbf{0},\lambda\rangle\\&=p_\mu\hat{L}(p)|m,\sigma\colon\mathbf{0},\lambda\rangle,\end{aligned} \tag{31}$$

and our state is indeed an eigenstate of the momentum operator $\hat{\mathbf{p}}$. For the spin component $\hat{S}_3$ we have, using Eq. (17),

$$\begin{aligned}\hat{S}_3\hat{L}(p)|m,\sigma\colon\mathbf{0},\lambda\rangle&=-m^{-1}L(-\hat{\mathbf{p}})^3{}_\mu\hat{W}^\mu\hat{L}(p)|m,\sigma\colon\mathbf{0},\lambda\rangle\\&=-m^{-1}\hat{L}(p)\hat{W}^3|m,\sigma\colon\mathbf{0},\lambda\rangle.\end{aligned} \tag{32}$$

We see from Eq. (10) for the Pauli-Lubanski operator $\hat{\mathbf{W}}$ that, when operating on rest states, the third component $\hat{W}_3$ is equivalent to the spin operator $m\hat{S}_3$, and

$$\hat{S}_3|m,\sigma\colon\mathbf{p},\lambda\rangle=\lambda|m,\sigma\colon\mathbf{p},\lambda\rangle \tag{33}$$

as required. This completes the proof that the states defined by Eq. (30) are standard states.

Under translations $\hat{a}$ defined by

$$\hat{a}=\exp(i\hat{p}\cdot a), \tag{34}$$

and general homogeneous Lorentz transformations $\hat{\Lambda}$ defined by

$$\hat{\Lambda}(\phi',\theta',\psi';\delta';\theta'',\phi'')=\hat{R}(\phi',\theta',\psi')\hat{Z}(\delta')\hat{R}(0,\theta'',\phi''), \tag{35}$$

the standard states behave in the following way:

$$\hat{a}|m,\sigma:\mathbf{p},\lambda\rangle=e^{ip\cdot a}|m,\sigma:\mathbf{p},\lambda\rangle\,, \quad (36)$$

$$\hat{\Lambda}|m,\sigma:\mathbf{p},\lambda\rangle=D^{\sigma}{}_{\lambda'\lambda}(W(\Lambda,p))|m,\sigma:\mathbf{p}',\lambda'\rangle\,, \quad (37)$$

where

$$p'=\Lambda p\,, \quad (38)$$

and the Wigner rotation $W(\Lambda,p)$ is defined by

$$W(\Lambda,p)=L^{-1}(\Lambda p)\Lambda L(p)\,. \quad (39)$$

C. Helicity States

We shall refer to eigenstates of the single-particle operators $\hat{p}^2$, $\hat{S}^2$, $\hat{\mathbf{p}}$, and $\hat{h}$, with eigenvalues m^2, $\sigma(\sigma+1)$, $\mathbf{p}$, and λ, as helicity states. We introduce a homogeneous Lorentz transformation $\hat{H}(p)$ associated with the four-momentum vector p_μ of Eq. (25),

$$\hat{H}(p)=\hat{L}(p)\hat{R}(p)\,, \quad (40)$$

where the operators $\hat{L}(p)$ and $\hat{R}(p)$ are defined by Eqs. (29) and (27). We define single-particle helicity states in terms of unitary transformations of standard rest states,

$$|m,\sigma:\mathbf{p},\lambda\rangle_h=\hat{H}(p)|m,\sigma:\mathbf{0},\lambda\rangle\,. \quad (41)$$

Let us verify that these states are indeed eigenstates of the helicity operator $\hat{h}$. We have, from the defining equation (12),

$$\begin{aligned}\hat{h}|m;\sigma:\mathbf{p},\lambda\rangle_h&=|\hat{\mathbf{p}}|^{-1}\hat{W}^0\hat{H}(p)|m,\sigma:\mathbf{0},\lambda\rangle\\&=|\mathbf{p}|^{-1}\hat{H}(p)[H(p)\hat{W}]_0|m,\sigma:\mathbf{0},\lambda\rangle\,. \quad (42)\end{aligned}$$

Now we recall that the operator $\hat{W}_\mu$ behaves like $(0;m\hat{\mathbf{S}})$ when acting on a rest state and the momentum $|\mathbf{p}|$ is related to the boost angle δ of $\hat{Z}(p)$ by

$$|\mathbf{p}|=m\sinh\delta\,. \quad (43)$$

Equation (42) then takes the form

$$\begin{aligned}\hat{h}|m,\sigma:\mathbf{p},\lambda\rangle_h&=\hat{H}(p)\hat{S}_3|m,\sigma:\mathbf{0},\lambda\rangle\\&=\lambda|m,\sigma:\mathbf{p},\lambda\rangle_h \quad (44)\end{aligned}$$

as required.

Note that the limit of the helicity matrix $H(p)$ as $\mathbf{p}\to\mathbf{0}$ is not in general well defined. This ambiguity corresponds to the rotational degree of freedom in taking the zero-momentum limit of expression (12) for $\hat{h}$. For this reason we prefer to take as our standard states eigenstates of the spin-component operator $\hat{S}_3$.

The helicity states behave in the following way under translation and homogeneous Lorentz transformations:

$$\hat{a}|m,\sigma:\mathbf{p},\lambda\rangle_h=e^{ip\cdot a}|m,\sigma:\mathbf{p},\lambda\rangle_h \quad (45)$$

and

$$\hat{\Lambda}|m,\sigma:\mathbf{p},\lambda\rangle_h=D^{\sigma}{}_{\lambda'\lambda}(W^h(\Lambda,p))|m,\sigma:\mathbf{p}',\lambda'\rangle_h\,, \quad (46)$$

where the transformed momentum p' is given by Eq. (38) and the Wigner rotation is

$$W^h(\Lambda,p)=H^{-1}(\Lambda p)\Lambda H(p)\,. \quad (47)$$

These expressions closely resemble Eqs. (36) and (37) for standard states. One can show from the defining equations (30) and (41) that helicity states are simple linear combinations of standard states,

$$|m,\sigma:\mathbf{p},\lambda\rangle_h=\langle\lambda'|\lambda\rangle_h|m,\sigma:\mathbf{p},\lambda'\rangle\,, \quad (48)$$

where the overlap coefficient is

$$\langle\lambda'|\lambda\rangle_h=D^{\sigma}{}_{\lambda'\lambda}(R(p))\,. \quad (49)$$

We conclude this section with the observation that our helicity states $|m,\sigma:\mathbf{p},\lambda\rangle$ differ by a phase from those of Wick.[7] The Wick boost operator $\hat{H}^W(p)$ is of the form

$$\hat{H}^W(p)=\hat{H}(p)\hat{R}_3(p)\,, \quad (50)$$

and helicity states are given by

$$|m,\sigma:\mathbf{p},\lambda\rangle_W=e^{-i\lambda\phi(p)}|m,\sigma:\mathbf{p},\lambda\rangle_h\,. \quad (51)$$

These states satisfy transformation equations similar to those for the states $|m,\sigma:\mathbf{p},\lambda\rangle_h$. Under homogeneous Lorentz transformations $\hat{\Lambda}$, we find

$$\hat{\Lambda}|m,\sigma:\mathbf{p},\lambda\rangle_W=D_{\lambda'\lambda}{}^{\sigma}(W^W(\Lambda,p))|m,\sigma:\mathbf{p}',\lambda'\rangle_W\,, \quad (52)$$

where

$$W^W(\Lambda,p)=[H^W(\Lambda p)]^{-1}\Lambda H^W(p)\,. \quad (53)$$

Our choice of helicity operator $\hat{H}(p)$ coincides with that of Jacob and Wick. The boost operator $\hat{H}^W(p)$ is not well defined if the vector p has zero first and second components. On the other hand, the operator $\hat{H}(p)$ is well defined if the momentum $\mathbf{p}$ is in the positive 3-direction.

Note that the definitions of all the single-particle states which we have considered are consistent with the Poincaré invariant normalization

$$\langle m,\sigma:\mathbf{p}',\lambda'|m,\sigma:\mathbf{p},\lambda\rangle=2p_0\delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{p}')\,. \quad (54)$$

III. SCALAR-SPIN-COMPONENT MULTIPARTICLE STATES

In Sec. IV we shall construct states labeled by a set of single-particle quantum numbers, and a maximum number of Poincaré scalar "internal variable" parameters. The first step in such a program is to replace the spin-component labels of direct-product states by eigenvalues of a complete set of Poincaré scalar spin operators. This we do in this section. We define a new class of states in terms of direct-product multiparticle states, and examine some of their properties in detail.

A. Multiparticle Observables

We consider a system of n free particles (i). We introduce, for each, space-time observables $\hat{J}_{i:\mu\nu}$ and $\hat{p}_{i:\mu}$ satisfying the commutation relations (1)–(3). Since the particles are independent, observables of different particles commute,

$$[\hat{p}_{i:\lambda},\hat{J}_{j:\mu\nu}]=[\hat{p}_{i:\lambda},\hat{p}_{j:\mu}]=[\hat{J}_{i:\mu\nu},\hat{J}_{j:\rho\sigma}]=0,\quad i\neq j;\ i,\ j=1,\ 2,\ \ldots,\ n. \quad (55)$$

We form the universal enveloping algebra of the Poincaré group generators $\hat{J}_{i:\mu\nu}$ and $\hat{p}_{i:\mu}$. The states of the multiparticle system are then characterized by the eigenvalues of a complete commuting set. The $2n$ operators of the form $\hat{p}_i^2$ and $\hat{W}_i^2$, which correspond to the masses m_i and spins σ_i of the n particles, commute with all Poincaré generators $\hat{J}_{i:\mu\nu}$ and $\hat{p}_{i:\mu}$. Thus, all multiparticle states will have $2n$ mass and spin labels m_i and σ_i. For simplicity of notation alone we shall suppress these parameters in the following sections.

B. Direct-Product States

The simplest n-particle states are those formed by taking the direct product of n single-particle states. We define standard multiparticle states $|\mathbf{p}_i,\lambda_i\rangle$ by

$$|\mathbf{p}_i,\lambda_i\rangle=\prod_{i=1}^{n}\{|m_i,\sigma_i:\mathbf{p}_i,\lambda_i\rangle\}. \tag{56}$$

They are eigenstates of the momentum operators $\hat{\mathbf{p}}_i$ and spin-component operators $\hat{S}_{i:3}$, with eigenvalues $\mathbf{p}_i$ and λ_i, respectively.

We define multiparticle helicity states $|\mathbf{p}_i,\lambda_i\rangle_h$ by

$$|\mathbf{p}_i,\lambda_i\rangle_h=\prod_{i=1}^{n}\{|m_i,\sigma_i:\mathbf{p}_i,\lambda_i\rangle_h\}. \tag{57}$$

These are eigenstates of the momentum operators $\hat{\mathbf{p}}_i$ and helicity operators $\hat{h}_i$ with eigenvalues $\mathbf{p}_i$ and λ_i. The Poincaré-invariant normalization of these states follows from that of the single-particle states $|m,\sigma:\mathbf{p},\lambda\rangle$ of Eq. (54),

$$\langle\mathbf{p}_i',\lambda_i'|\mathbf{p}_i,\lambda_i\rangle=\prod_{i=1}^{n}\{2p_{i:0}\delta_{\lambda_i'\lambda_i}\delta(\mathbf{p}_i-\mathbf{p}_i')\}. \tag{58}$$

In order to determine the space-time transformation properties of our multiparticle states, we observe that the generators of translations and homogeneous Lorentz transformations of the n-particle system are $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$, respectively, where

$$\hat{p}_\mu=\sum_{i=1}^{n}\hat{p}_{i:\mu}, \tag{59}$$

$$\hat{J}_{\mu\nu}=\sum_{i=1}^{n}\hat{J}_{i:\mu\nu}. \tag{60}$$

We now use the single-particle equations (36) and (37) and the commutation relations (55) to derive the multiparticle-state transformation formulas

$$\hat{a}|\mathbf{p}_i,\lambda_i\rangle=e^{ip\cdot a}|\mathbf{p}_i,\lambda_i\rangle, \tag{61}$$

$$\hat{\Lambda}|\mathbf{p}_i,\lambda_i\rangle=\prod_{j=1}^{n}\{D^{\sigma}{}_{\lambda_j'\lambda_j}(W(\Lambda,p_j))\}|\mathbf{p}_i',\lambda_i'\rangle, \tag{62}$$

where

$$p=\sum_{i=1}^{n}p_i \quad\text{and}\quad p_i'=\Lambda p_i. \tag{63}$$

The Wigner rotation function $W(\Lambda,p)$ has been defined in Eq. (39).

These transformation formulas are rather complicated. If the operators, the eigenvalues of which label multiparticle states, had had simple transformation properties under homogeneous Lorentz transformations, formula (62) would have been correspondingly simple. For this reason we wish to construct multiparticle states in such a way that the labels correspond to a maximum number of Poincaré scalar operators. We construct such states in three stages:

(1) We replace the spin-component operators $\hat{S}_{i:3}$ by a complete set of Poincaré scalar-spin-component operators $\hat{S}_i{}^q$, where

$$\hat{S}_i{}^q=\hat{q}_i\cdot\hat{W}_i/[(\hat{q}_i\cdot\hat{p}_i)^2-\hat{q}_i{}^2\hat{p}_i{}^2]^{1/2}. \tag{64}$$

We shall refer to such states as "states of type I," "q-spin states," or "scalar-spin-component states."

(2) We replace $3n$ momentum operators $\hat{\mathbf{p}}_i$ by $3n-6$ scalar-product operators of the form $\hat{p}_i\cdot\hat{p}_j$, three momentum operators $\hat{\mathbf{p}}$, and three additional nonscalar angle operators $\hat{\theta}$, $\hat{\phi}$, and $\hat{\psi}$. We shall refer to these states as "states of type II."

(3) We replace the three angle operators $\hat{\theta}$, $\hat{\phi}$, and $\hat{\psi}$ by a scalar spin $\hat{S}^q$, the magnitude of the spin $\hat{\mathbf{S}}^2$, and one spin component $\hat{S}_3$ or $\hat{h}$. We shall call these states "states of type III."

In the rest of this section we shall only be concerned with states of type I.

C. Scalar-Spin-Component States

We define a function $\Delta(p_1\cdots p_n)$ of the n four-momenta $p_1\cdots p_n$ by

$$\Delta(p_1\cdots p_n)=|\det(p_i\cdot p_j)|^{1/2}. \tag{65}$$

This function vanishes if the momenta $p_1, p_2, \ldots, p_n$ are not linearly independent. For one and two momenta, respectively, we have

$$\Delta(p)=m(p)=|p^2|^{1/2} \tag{66}$$

and

$$\Delta(p_1,p_2)=|(p_1\cdot p_2)^2-p_1{}^2p_2{}^2|^{1/2}. \tag{67}$$

Consider any set of four vector-momentum operators $\hat{q}_{i:\mu}$ defined by

$$\hat{q}_{i:\mu}=\textstyle\sum_j a_{ij}\hat{p}_{j:\mu}, \tag{68}$$

where the a_{ij} are constants, chosen so that (a) the eigenvalues of $\hat{q}_{i:\mu}$ are timelike four-vectors in the space of standard states and (b) for all i we have the relation

$$\Delta(q_i,p_i)\neq 0. \tag{69}$$

For each particle (i) we can then define a q-spin operator $\hat{S}_i{}^q$ by Eq. (64). We note that for a single-particle system no operator $\hat{S}^q$ exists since condition (b) cannot be satisfied; we have only one independent momentum $\hat{p}_\mu$.

The operators $\hat{S}_i{}^q$ commute with all momentum-component operators $\hat{p}_{i:\mu}$ but not with the spin-component operators $\hat{S}_{i:3}$ or $\hat{h}_i$. It is thus possible to construct a complete set of multiparticle states which are eigenstates of the scalar-spin-component operators $\hat{S}_i{}^q$ and momentum operators $\hat{\mathbf{p}}_i$ alone. We shall give here a definition of a class of scalar-spin-component states $|\mathbf{p}_i\colon \lambda_i\rangle_q$ and verify that they are indeed eigenstates of q-spin operators $\hat{S}_i{}^q$ with correpsonding eigenvalues λ_i. In Appendix B we show how such states may be constructed directly using the momentum and spin-eigenvalue equations.

We introduce a boost operator $\hat{M}(q_i)$ associated with each momentum q_i, where

$$\hat{M}(q_i)=\hat{L}_i(q_i)\hat{R}_i\,, \tag{70}$$

and the rotation R_i is arbitrary. We then define q-spin states $|\mathbf{p}_i\colon \lambda_i\rangle_q$ in terms of standard multiparticle direct-product states in the following way:

$$|\mathbf{p}_i\colon\lambda_i\rangle_q=\prod_{j=1}^{n}\{\hat{M}_j(q_j)\hat{H}_j(M^{-1}(q_j)p_j)\}|\mathbf{0},\lambda_i\rangle. \tag{71}$$

Let us now verify that these states are q-spin eigenstates. For particle (k) we have

$$\hat{q}_k\cdot\hat{W}_k|\mathbf{p}_i\colon\lambda_i\rangle_q=\prod_{j=1}^{n}\{\hat{M}_j(q_j)\hat{H}_j(M^{-1}(q_j)p_j)\}\times[q_k\cdot M(q_k)H(M^{-1}(q_k)p_k)\hat{W}_k]|\mathbf{0},\lambda_i\rangle. \tag{72}$$

We now use the definitions of $\hat{H}(p)$ and $\hat{M}(q)$, (40) and (70), respectively, to simplify the expression in square brackets,

$$\begin{aligned} q_k\cdot M(q_k)H(M^{-1}(q_k)p_k)\hat{W}_k &= M^{-1}(q_k)q_k\cdot H(M^{-1}(q_k)p_k)\hat{W}_k\\ &=\Delta(q_k)[Z(M^{-1}(q_k)p_k)\hat{W}_k]_0. \end{aligned} \tag{73}$$

Now the four-vector $M^{-1}(q_k)p_k$ has 0-component of the form

$$[M^{-1}(q_k)p_k]_0=\Delta(p_k)\cosh\delta\,, \tag{74}$$

where δ is the angle associated with the boost $Z(M^{-1}(q_k)p_k)$. Moreover, the four-vector $M^{-1}(q_k)q_k$ is by definition (70) of the form

$$[M^{-1}(q_k)q_k]_\mu=\delta_{\mu 0}\Delta(q_k)\,. \tag{75}$$

Thus the Poincaré scalar boost angle δ is given by

$$p_k\cdot q_k=\Delta(p_k)\Delta(q_k)\cosh\delta\,, \tag{76}$$

and

$$\sinh\delta=\Delta(p_k,q_k)/\Delta(p_k)\Delta(q_k)\,. \tag{77}$$

The spin W_μ is equivalent to the operator $(0;\Delta(p)\hat{\mathbf{S}})$ when operating on a rest state, and Eq. (73) takes the form

$$\Delta(q_k)(Z[M^{-1}(q_k)p_k]\hat{W}_k)_0|\mathbf{0},\lambda_i\rangle =\Delta(p_k,q_k)\hat{S}_{k:3}|\mathbf{0},\lambda_i\rangle. \tag{78}$$

We now substitute this expression into Eq. (72) and rearrange terms to obtain the q-spin eigenvalue equation

$$\hat{S}_k{}^q|\mathbf{p}_i\colon\lambda_i\rangle_q=\lambda_k|\mathbf{p}_i\colon\lambda_i\rangle_q. \tag{79}$$

The construction of q-spin eigenstates in the case where the four-vector operators $\hat{q}_{i:\mu}$ have spacelike eigenvalues is in general more complicated. We shall only consider a specific example in Sec. V.

It is interesting to note that, if we formally take $q_{i:\mu}$ to be of the form

$$q_{i:\mu}\leftrightarrow(1;\mathbf{0}) \tag{80}$$

in all Lorentz frames and set $\hat{R}_i=\hat{I}$, Eq. (71) defines the helicity states (41). The nonmanifestly covariant nature of the 4-tuples $q_{i:\mu}$ in this case is reflected in the nonmanifestly covariant nature of helicity under Lorentz transformations.

We now consider the behavior of these states under translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$. They have the following properties derivable from the defining equation (71):

$$\hat{a}|\mathbf{p}_i\colon\lambda_i\rangle_q=e^{ip\cdot a}|\mathbf{p}_i\colon\lambda_i\rangle_q\,, \tag{81}$$

$$\hat{\Lambda}|\mathbf{p}_i,\lambda_i\rangle_q=e^{-i\phi^j\lambda_j}|\mathbf{p}_i{}',\lambda_i\rangle_q\,, \tag{82}$$

where $p_i{}'=\Lambda p_i$ and the phases ϕ^j are given by

$$e^{-iJ_3\phi^j}=W^h(W^m(\Lambda,q_j),M^{-1}(q_j)p_j)\,, \tag{83}$$

where

$$W^m(\Lambda,q_j)=M^{-1}(\Lambda q_j)\Lambda M(q_j)\,. \tag{84}$$

These phases are rather complicated in general. We recall that in Eq. (70) the operator $\hat{M}(q_i)$ was not defined uniquely. We now take advantage of the degree of freedom in the choice of the operators $\hat{R}_i$ to eliminate the phases ϕ^j altogether.

For each particle (i) we introduce a four-momentum $\hat{r}_{i:\mu}$, where

$$\hat{r}_{i:\mu}=\sum_{j=1}^{n}b_{ij}\hat{p}_{j:\mu}\,, \tag{85}$$

and the constants b_{ij} are chosen so that (a′) the operators $\hat{r}_{i:\mu}$ have timelike eigenvalues and (b′) the eigenvalues r_i satisfy the equation

$$\Delta(p_i,q_i,r_i)\neq 0\,. \tag{86}$$

Condition (86) ensures that the vectors q_i and r_i are not collinear. We may thus define uniquely operators $\hat{M}(q_i,r_i)$ which are not trivially dependent on the four-vectors r_i,

$$\hat{M}(q_i,r_i)=\hat{L}(q_i)\hat{R}(-\mathbf{r}_i{}')\,, \tag{87}$$

where

$$\hat{R}(\mathbf{r}_i{}')=\hat{R}(r_i{}')=\hat{R}(L^{-1}(q_i)r_i)\,, \tag{88}$$

and operators $\hat{L}(p)$ and $\hat{R}(p)$ have been defined by Eqs. (29) and (27), respectively. We now express the rotations $\hat{R}(L^{-1}(q_i)r_i)$ in terms of boost functions alone.

We find that up to a 3-axis rotation on the right,

$$\hat{R}(-\mathbf{r}')=\hat{L}^{-1}(q)\hat{L}(r)\hat{H}(L^{-1}(r)q). \quad (89)$$

In order to see this, it is sufficient to note that the right-hand side is a rotation which leaves invariant a four-vector of the form (1; 0,0,0), and inverse takes the object $L^{-1}(q)r$ to a 4-tuple with "spacelike" component in the negative 3-direction alone.

If we replace the matrix $M(q_i,r_i)$ by $M(q_i,r_i)e^{iJ_3\phi}$ in Eq. (83) the phases ϕ^j do not change. We may thus substitute expressions (89) and (87) into Eq. (71) for the q-spin states $|\mathbf{p}_i\colon\lambda_i\rangle_q$. At the same time we replace the operator $\hat{H}(M^{-1}(q_j)p_j)$ of Eq. (71) by the Wick helicity operator $\hat{H}^W(M^{-1}(q_j)p_j)$. This has the effect of multiplying states by a phase.

We may now examine the modified expression (83) for the phases ϕ^j which are associated with homogeneous Lorentz transformations of these states,

$$e^{-iJ_3\phi^j}=W^W(W^h(W(\Lambda,r_j),L^{-1}(r_j)q_j),Q_j), \quad (90)$$

where the momenta Q_j are given by

$$Q_j=[H^W(L^{-1}(r_j)q_j)]^{-1}L^{-1}(r_j)p_j. \quad (91)$$

We show in Appendix A that the Wigner rotation $W^h(R,q)$ of a rotation R is a rotation about the 3-axis, provided the 3-vector $\mathbf{q}$ does not lie in the negative 3-direction. Moreover, the Wigner rotation $W^W(R_3,Q)$ of a 3-axis rotation R_3 is the identity, provided either the first or second component of the momentum Q is nonzero.

By construction the momenta $L^{-1}(r_j)q_j$ have at least one nonvanishing "space" component. We may rewrite expression (91) for momentum Q_j in the form

$$Q_j=R_3^{-1}(L^{-1}(r_j)q_j)Z^{-1}(L^{-1}(r_j)q_j)R^{-1}(L^{-1}(r_j)q_j)\times R(L^{-1}(r_j)p_j)Z(L^{-1}(r_j)p_j)L^{-1}(p_j)p_j, \quad (92)$$

which has nonvanishing first or second components provided

$$R(L^{-1}(r_j)q_j)\neq R(L^{-1}(r_j)p_j). \quad (93)$$

This condition is satisfied provided the vector q_j is not a linear cobmination of vectors p_j and r_j. This is in turn ensured by the constraint equation (86). We then learn from Eqs. (90) and (A11) that by construction all phases ϕ^j are zero.

One should note that, for a two-particle system with only two independent four-momenta, condition (86) cannot be satisfied. We shall examine this important exceptional case in Sec. V.

We have shown that for systems of at least three particles, it is possible to construct a set of scalar-spin-component states $|\mathbf{p}_i\colon\lambda_i\rangle_q$ according to the equation

$$|\mathbf{p}_i\colon\lambda_i\rangle_q=\prod_{j=1}^{n}\{\hat{H}_j(p_j,q_j,r_j)\}|\mathbf{0},\lambda_i\rangle, \quad (94)$$

where the boost functions $\hat{H}_j(p_j,q_j,r_j)$ are given by

$$\hat{H}(p_j,q_j,r_j)=\hat{L}(r_j)\hat{H}(L^{-1}(r_j)q_j)\times\hat{H}^W(H^{-1}(L^{-1}(r_j)q_j)L^{-1}(r_j)p_j). \quad (95)$$

These states behave in the following way under space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$:

$$\hat{a}|\mathbf{p}_i\colon\lambda_i\rangle_q=e^{ip\cdot a}|\mathbf{p}\colon\lambda_i\rangle_q, \quad (96)$$

$$\hat{\Lambda}|\mathbf{p}_i\colon\lambda_i\rangle_q=|\mathbf{p}_i'\colon\lambda_i\rangle_q, \quad (97)$$

where $p_i'=\Lambda p_i$.

From the defining equations (94), (56), and (30), we see that the q-spin states $|\mathbf{p}_i\colon\lambda_i\rangle_q$ may be expressed as linear combinations of standard direct-product states $|\mathbf{p}_i,\lambda_i\rangle$,

$$|\mathbf{p}_i\colon\lambda_i\rangle_q=\sum_{\lambda_i'}\langle\lambda_i'|\lambda_i\rangle_q|\mathbf{p}_i,\lambda_i'\rangle, \quad (98)$$

where the overlap coefficients are given by

$$\langle\lambda_i'|\lambda_i\rangle_q=\prod_{i=1}^{n}\{D_{\lambda_i'\lambda_i}{}^{\sigma_i}[L^{-1}(p_i)H(p_i,q_i,r_i)]\}. \quad (99)$$

In general each choice of vectors q_i and r_i satisfying Eq. (86) defines uniquely a class of multiparticle states with spin-independent Poincaré transformation properties. As the number n of particles, increases the number of classes of n-particle states increases very rapidly indeed. For each particle (i) in a system of n particles we have at least $n-1$ degrees of freedom in the choice of a vector q_i, and at least $n-2$ degrees of freedom in the choice of a vector r_i. The true number of degrees of freedom is actually greater because we may contract any three four-momenta with the tensor $\epsilon^{\mu\nu\rho\sigma}$ to form a new four-vector, in terms of which we may define vectors q_i, and r_i. Altogether this suggests that for $n\geqslant 2$, we have *at least* $n(2n-3)$ degrees of freedom in the construction of scalar-spin-component states.

IV. MULTIPARTICLE STATES WITH SINGLE-PARTICLE LABELS

In forming states of type I, we replace the frame-dependent operators $\hat{S}_{i:3}$, the eigenvalues of which labeled multiparticle direct-product states $|\mathbf{p}_i,\lambda_i\rangle$, by Poincaré scalar-component operators $\hat{S}_i{}^q$. We shall now replace the $3n$ momentum-state labels $\mathbf{p}_i$ by single-particle labels $\mathbf{p}$, σ, and λ, and a set of $(3n-5)$ Poincaré scalar parameters. This we do in two stages.

A. States of Type II

We replace the momenta $\mathbf{p}_i$ by a maximum number of Poincaré scalar momentum operators of the form s_{ij}, where

$$s_{ij}=(p_i+p_j)^2,$$

and

$$s_{0j}=(p-p_j)^2, \quad i,\ j=1,\ 2,\ \ldots,\ n. \quad (100)$$

Since all momentum operators $\hat{p}_{i:\mu}$ commute, the problem is essentially one of changing independent momentum variables $\mathbf{p}_i$.

We define a c.m. momentum $\bar{p}_2$ by

$$\bar{p}_2=L^{-1}(p)p_2, \quad (101)$$

and define angles θ and ϕ by

$$R(-\mathbf{p}_2)=R(\phi,\,\theta,\,-\phi). \tag{102}$$

We then tentatively introduce another c.m. momentum $p_3^\star$, where

$$p_3^\star=R^{-1}(-\mathbf{p}_2)L^{-1}(p)p_3. \tag{103}$$

Using arguments similar to those following Eq. (89), we can show that the components of vector $p_3^\star$ coincide with those of the vector $\tilde{p}_3$, where

$$\tilde{p}_3=H^{-1}(L^{-1}(p_2)p)L^{-1}(p_2)p_3. \tag{104}$$

We define an angle ψ by

$$R_3(\tilde{p}_3)=R_3(\psi). \tag{105}$$

For systems of at least three particles, we label our states by the complete set of n scalar spin components λ_i, and $3n$ momentum parameters[9] s, $\mathbf{p}$, ϕ, θ, ψ, s_{02}, s_{03}, s_{04}, s_{23}, s_{24} and s_{0k}, s_{2k}, s_{3k} for $5\leqslant k\leqslant n$, where

$$s=\Delta^2(p)=p^2. \tag{106}$$

We shall examine the two-particle case separately in Sec. V.

We define the new states of type II by

$$|s:\mathbf{p},\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q=J_n^{1/2}|\mathbf{p}_i:\lambda_i\rangle_q,$$

$$i=1,2,\ldots,n\begin{cases}j=0,\,k=2,3,\ldots,n\\ j=2,\,k=3,4,\ldots,n\\ j=3,\,k=5,6,\ldots,n\end{cases} \tag{107}$$

where the function J_n is a suitable normalization factor. If we choose this function to be equal to the modulus of the Jacobian of the momentum-variable transformation,

$$J_n=\left|[16\pi^2/\prod_{i=1}^{n}\{2p_{i:0}\}]\det\left[\frac{\partial(\mathbf{p}_i)}{\partial(p_\mu;\,\phi,\,\cos\theta,\psi:s_{jk})}\right]\right|, \tag{108}$$

the new states will have the normalization

$$\begin{aligned}{}_q\langle s':\mathbf{p}';\,\phi',\theta',\psi':s_{jk}',\lambda_i'|s:\mathbf{p};\,\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q\\ =\delta(s-s')2p_0\delta(\mathbf{p}-\mathbf{p}')16\pi^2\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')\\ \times\delta(\psi-\psi')\delta(s_{jk}-s_{jk}')\delta_{\lambda_i\lambda_i'}.\end{aligned} \tag{109}$$

In Appendix C we obtain closed-form expressions for these functions J_n,

$$J_3=\pi^2/2s, \tag{110}$$

$$J_4=\pi^2/32s\Delta(p,p_2,p_3,p_4), \tag{111}$$

and

$$\begin{aligned}J_n^{-1}=&\frac{32}{\pi^2}s\Delta(p_1,p_2)\\ &\times|\sum_{i=5}^{n}(\tilde{p}^\star{}_{4:1}\tilde{p}^\star{}_{i:2}-\tilde{p}^\star{}_{4:2}\tilde{p}^\star{}_{i:1})-\tilde{p}^\star{}_{4:2}\tilde{p}^\star{}_{3:1}|\\ &\times\prod_{i=5}^{n}16\Delta(p,p_2,p_3,p_i),\quad n\geqslant5\end{aligned} \tag{112}$$

where Poincaré scalar momentum components $\tilde{p}^\star{}_{i:1}$ and $\tilde{p}^\star{}_{i:2}$ are given by

$$\tilde{p}^\star{}_{i:1}=-\frac{(p^2p_i\cdot p_2-p_2\cdot pp_i\cdot p)(p^2p_2\cdot p_3-p_2\cdot pp_3\cdot p)+\Delta^2(p,p_2)(p^2p_i\cdot p_3-p_3\cdot pp_i\cdot p)}{s\Delta(p,p_2)\Delta(p,p_2,p_3)}, \tag{113}$$

$$\tilde{p}^\star{}_{i:2}=\frac{-\epsilon^{\mu\nu\rho\sigma}p_\mu p_{2:\nu}p_{3:\rho}p_{i:\sigma}}{\Delta(p,p_2,p_3)}=\frac{\pm\Delta(p,p_2,p_3,p_i)}{\Delta(p,p_2,p_3)}. \tag{114}$$

In order to determine the space-time transformation properties of states of type II, we must first of all know how c.m. momenta $\bar{p}_2$ and $\tilde{p}_3$ change under homogeneous Lorentz transformations $\hat{\Lambda}$. From the defining equation (101), we see that momentum $\bar{p}_2$ undergoes Wigner rotation,

$$\bar{p}_2\to\bar{p}_2'=W(\Lambda,p)\bar{p}_2. \tag{115}$$

From Eq. (104) for the c.m. momentum $\tilde{p}_3$, we find

$$\tilde{p}_3\to\tilde{p}_3'=W^h(W(\Lambda,p_3),L^{-1}(p_3)p)\tilde{p}_3. \tag{116}$$

Our states $|s:\mathbf{p};\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q$ thus behave in the following way under space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$:

$$\hat{a}|s:\mathbf{p};\,\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q=e^{ip\cdot a}|s:\mathbf{p};\,\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q, \tag{117}$$

$$\hat{\Lambda}|s:\mathbf{p};\,\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q=|s:\mathbf{p}';\,\phi',\theta',\psi':s_{jk},\lambda_i\rangle_q, \tag{118}$$

where $p'=\Lambda p$ and angles ϕ', θ', ψ' are defined by

$$R(\phi',\,\theta',\,-\phi')=W(\Lambda,p)R(\phi,\theta,\phi'') \tag{119}$$

for some angle ϕ'', and

$$R_3(\psi')=R_3(\tilde{p}_3')=W^h(W(\Lambda,p_2),L^{-1}(p_2)p)R_3(\psi). \tag{120}$$

We note that the overlap of states of type II and states of type I is proportional to the square root of the function J_n, defined by Eqs. (110)–(112),

$$\begin{aligned}{}_q\langle\mathbf{p}_i',\lambda_i|s:\mathbf{p};\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q\\ =J_n^{\frac12}\prod_i\{2p_{i:0}\delta(\mathbf{p}_i'-\mathbf{p}_i)\}.\end{aligned} \tag{121}$$

B. States of Type III

We now construct a complete set of multiparticle states labeled by eigenvalues $\mathbf{p}$ and λ of the momentum operator $\hat{\mathbf{p}}$ and spin-component operator $\hat{S}_3$, respectively, a number of Poincaré scalar momentum products s_{ij}, and scalar spin components λ_i. We define these

[9] In the case where $n\leqslant4$ see Eqs. (186) and (193).

states of type III by

$$|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=\frac{(2\sigma+1)^{1/2}}{16\pi^2}\int_0^{4\pi}\int_0^{\pi}\int_0^{2\pi}D_{\lambda\mu}{}^{\sigma*}(R(\phi,\theta,-\phi+\psi))\times|s:\mathbf{p};\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q\sin\theta d\phi d\theta d\psi \quad (122)$$

$$=(2\sigma+1)^{1/2}\hat{L}(p)\int D_{\lambda\mu}{}^{\sigma*}(R(\phi,\theta,\psi))\hat{R}(\phi,\theta,\psi)\times|s:\mathbf{0};0,0,0:s_{jk},\lambda_i\rangle_q d\mu(R), \quad (123)$$

where the invariant measure of the rotation group $[SU(2)]$ is of the form

$$d\mu(R)=(1/16\pi^2)\sin\theta d\phi d\theta d\psi,\quad 0\leqslant\phi<2\pi,\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\psi<4\pi. \quad (124)$$

The constants preceding the integral signs have been chosen so that these multiparticle states have a "single-particle" normalization

$${}_q\langle s',\sigma':\mathbf{p}',\lambda':s_{jk}',\lambda_i'|s,\sigma:\mathbf{p},\lambda:s_{jk},\lambda_i\rangle_q=\delta(s-s')\delta_{\sigma\sigma'}2p_0\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\delta(s_{jk}-s_{jk}')\delta_{\lambda_i\lambda_i'}. \quad (125)$$

From the defining equation (123) we may determine the behavior of these states under translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$:

$$\hat{a}|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=e^{ip\cdot a}|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q \quad (126)$$

and

$$\hat{\Lambda}|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=D_{\lambda'\lambda}{}^{\sigma}(W(\Lambda,p))|s,\sigma:\mathbf{p}',\lambda':\mu;s_{jk},\lambda_i\rangle_q. \quad (127)$$

The latter equation implies that our states are eigenstates of total spin operators $\hat{\mathbf{S}}^2$ and $\hat{S}_3$ with eigenvalues $\sigma(\sigma+1)$ and λ, respectively.

We now wish to determine the physical meaning of the parameter μ. Consider the effect of the transformation $e^{i\hat{S}^{p_2}\chi}$ on the state defined by Eq. (123). The Poincaré scalar operator $\hat{S}^{p_2}$ commutes with the homogeneous Lorentz group operators $\hat{L}(p)$ and $\hat{R}(\phi,\theta,\psi)$, and has the same effect as operators $\hat{S}_3$ and $\hat{J}_3$ when acting on the states $|s;\mathbf{0};0,0,0:s_{jk},\lambda_i\rangle_q$. This leads to the relation

$$\exp(i\hat{S}^{p_2}\chi)\,|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=(2\sigma+1)^{1/2}\hat{L}(p)\int D_{\lambda\mu}{}^{\sigma*}(R(\phi,\theta,\psi))\hat{R}(\phi,\theta;\psi-\chi)\times|s:\mathbf{0};0,0,0:s_{jk}\lambda_i\rangle_q d\mu(R). \quad (128)$$

We now use the invariance of the measure $d\mu(R)$ to replace the angle ψ by $\psi+\chi$ and so obtain the eigenvalue equation

$$e^{i\hat{S}^{p_2}\chi}|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=e^{i\mu\chi}|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q. \quad (129)$$

We thus identify μ with the eigenvalue of the Poincaré scalar-spin-component operator $\hat{S}^{p_2}$.

We may express these states of type III as linear combinations of states of type II. From the defining equation (122), we see directly that

$$|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q=\int\langle\phi,\theta,\psi|\sigma:\lambda:\mu\rangle\times|s:\mathbf{p};\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q d\mu(R), \quad (130)$$

where the overlap coefficient is given by

$$\langle\phi,\theta,\psi|\sigma:\lambda:\mu\rangle=(2\sigma+1)^{1/2}D_{\mu\lambda}{}^{\sigma}(\phi-\psi,-\theta,-\phi). \quad (131)$$

Alternatively we may express states of type II as linear combination of states of type III,

$$|s:\mathbf{p};\phi,\theta,\psi:s_{jk},\lambda_i\rangle_q=\sum_{\sigma,\lambda,\mu}\langle\sigma:\lambda:\mu|\phi,\theta,\psi\rangle|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q, \quad (132)$$

where

$$\langle\sigma:\lambda:\mu|\phi,\theta,\psi\rangle=(\langle\phi,\theta,\psi|\sigma:\lambda:\mu\rangle)^*. \quad (133)$$

These formulas will prove useful when we come to consider partial-wave decompositions of scattering amplitudes.

By analogy with the single-particle states $|m,\sigma:\mathbf{p},\lambda\rangle_h$ we may construct multiparticle helicity states $|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_{qh}$ which are eigenstates of the operator $\hat{h}$. Such states are related to the standard multiparticle states $|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_q$ by

$$|s,\sigma:\mathbf{p},\lambda:\mu;s_{jk},\lambda_i\rangle_{qh}=\sum_{\lambda'}\langle\lambda'|\lambda\rangle_h|s,\sigma:\mathbf{p},\lambda':\mu;s_{jk},\lambda_i\rangle_q, \quad (134)$$

where the overlap coefficient $\langle\lambda'|\lambda\rangle_h$ is given in Eq. (49).

V. TWO- AND THREE-PARTICLE STATES

The formalism which we have developed in Secs. III and IV applies in general to systems of at least three particles. We replaced spin-component-state labels by eigenvalues of scalar q-spin operators $\hat{S}^q$, defined in terms of momenta q, which were timelike in the physical region. We now examine two classes of state to which the foregoing theory does not apply. We first of all consider two-particle states of types I–III, and compare them with the Jacob-Wick[6,7] helicity states. We then construct a class of three-particle states labeled by eigenvalues of scalar q-spin operators $\hat{S}^q$, where the momentum q is spacelike in the physical region.

A. Two-Particle States

We have seen in Sec. III that in the case of a two-particle system, where we only have two independent four-momenta p_1 and p_2, the general formalism breaks down. In this case alone the scalar-spin-component operators $\hat{S}^q$ are uniquely determined up to a sign. For a momentum q of the form

$$q=\nu_1 p_1+\nu_2 p_2, \tag{135}$$

where ν_1 and ν_2 are nonzero constants, we find

$$\mathrm{sgn}(\nu_2)\hat{S}_1{}^q=\hat{S}_1{}^p=[\Delta(\hat{p}_1,\hat{p}_2)]^{-1}\epsilon^{\mu\nu\rho\sigma}\hat{p}_{2:\mu}\hat{p}_{1:\nu}\hat{J}_{1:\rho\sigma}, \tag{136}$$

$$\mathrm{sgn}(\nu_1)\hat{S}_2{}^q=\hat{S}_2{}^p=[\Delta(\hat{p}_1,\hat{p}_2)]^{-1}\epsilon^{\mu\nu\rho\sigma}\hat{p}_{1:\mu}\hat{p}_{2:\nu}\hat{J}_{2:\rho\sigma}. \tag{137}$$

These scalar p-spin component operators have been called "covariant helicity operators" by Matthews and Feldman.[4]

We define p-spin eigenstates using a modified form of Eq. (71):

$$|\mathbf{p}_1,\mathbf{p}_2{:}\ \lambda_1,\lambda_2\rangle_p=\hat{L}(p)\hat{H}_1(\bar{p}_1)\hat{H}_2(\bar{p}_2)|\mathbf{0},\mathbf{0};\ \lambda_1,\lambda_2\rangle, \tag{138}$$

where the c.m. momenta $\bar{p}_1$ and $\bar{p}_2$ are defined by

$$\bar{p}_1=L^{-1}(p)p_1 \quad\text{and}\quad \bar{p}_2=L^{-1}(p)p_2. \tag{139}$$

The operators $\hat{H}(\bar{p}_2)$ and $\hat{H}(\bar{p}_1)$ are not well defined if the vectors $\bar{\mathbf{p}}_2$ or $\bar{\mathbf{p}}_1$, respectively, lie in the negative 3-direction. Thus, our two-particle states $|\mathbf{p}_1,\mathbf{p}_2{:}\ \lambda_1,\lambda_2\rangle_q$ are not well defined if the first and second components of the c.m. momentum $\bar{p}_1$ are both zero.

From Eqs. (81) and (82) we see that, under homogenous Lorentz transformations $\hat{\Lambda}$, the states $|\mathbf{p}_1,\mathbf{p}_2{:}\ \lambda_1,\lambda_2\rangle_p$ behave in the following way:

$$\hat{\Lambda}|\mathbf{p}_1,\mathbf{p}_2{:}\ \lambda_1,\lambda_2\rangle_p=e^{-i(\phi^1\lambda_1+\phi^2\lambda_2)}|\mathbf{p}_1',\mathbf{p}_2'{:}\ \lambda_1,\lambda_2\rangle_p, \tag{140}$$

where

$$p_1'=\Lambda p_1 \quad\text{and}\quad p_2'=\Lambda p_2 \tag{141}$$

and the phase ϕ^i is determined by

$$e^{-iJ_3\phi^i}=W^h(W(\Lambda,p),L^{-1}(p)p_i). \tag{142}$$

Each of these two-particle states may be expressed as a sum of direct-product helicity states. From the defining equations (44), (57), and (138) we have

$$|\mathbf{p}_1,\mathbf{p}_2;\ \lambda_1,\lambda_2\rangle_p={}_h\langle\lambda_1',\lambda_2'|\lambda_1,\lambda_2\rangle_p\,|\mathbf{p}_1,\mathbf{p}_2,\lambda_1',\lambda_2'\rangle_h, \tag{143}$$

where the overlap coefficient is given by

$$\begin{aligned}{}_h\langle\lambda_1',\lambda_2'|\lambda_1,\lambda_2\rangle_p=&D_{\lambda_1'\lambda_1}{}^{\sigma_1}(W^h(L(p),\bar{p}_1))\\&\times D_{\lambda_2'\lambda_2}{}^{\sigma_2}(W^h(L(p),\bar{p}_2)).\end{aligned} \tag{144}$$

In the c.m. frame, the p-spin states $|\bar{\mathbf{p}}_1,\bar{\mathbf{p}}_2{:}\ \lambda_1,\lambda_2\rangle_p$ and helicity states $|\bar{\mathbf{p}}_1,\bar{\mathbf{p}}_2,\lambda_1,\lambda_2\rangle_h$ coincide.

Let us now construct states of type II. We change momentum variables from $\mathbf{p}_1$ and $\mathbf{p}_2$ to the total momentum parameters $\mathbf{p}$ and s, and the c.m. angles θ and ϕ associated with the c.m. momentum $-\bar{\mathbf{p}}_2$. The Jacobian of this transformation is well known, and is given by Eq. (C45).

The new states defined by

$$|s{:}\ \mathbf{p};\ \phi,\theta{:}\ \lambda_1,\lambda_2\rangle_p=[\pi\Delta(p_1,p_2)/s]^{1/2}|\mathbf{p}_1,\mathbf{p}_2;\ \lambda_1,\lambda_2\rangle_p \tag{145}$$

have the simple normalization

$$\begin{aligned}{}_p\langle s{:}\ \mathbf{p};\ \phi,\theta{:}\ \lambda_1,\lambda_2|s{:}\ \mathbf{p};\ \phi,\theta{:}\ \lambda_1,\lambda_2\rangle_p&=\delta(s-s')2p_0\delta(\mathbf{p}-\mathbf{p}')4\pi\delta(\phi-\phi')\\&\times\delta(\cos\theta-\cos\theta')\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}.\end{aligned} \tag{146}$$

They behave in the following way under homogeneous Lorentz transformations:

$$\begin{aligned}\hat{\Lambda}|s{:}\ \mathbf{p};\ \phi,\theta{:}\ \lambda_1,\lambda_2\rangle_p\\=e^{-i(\phi^1\lambda_1+\phi^2\lambda_2)}|s{:}\ \mathbf{p}';\ \phi',\theta'{:}\ \lambda_1,\lambda_2\rangle_p,\end{aligned} \tag{147}$$

where the phase angle ϕ^1 is defined by Eq. (142), the transformed momentum is $p'=\Lambda p$, and angles θ', ϕ' are determined by Eq. (119).

We may also construct two-particle states with single-particle labels:

$$\begin{aligned}&|s,\sigma{:}\ \mathbf{p},\lambda{:}\ \lambda_1,\lambda_2\rangle_p\\&=\frac{(2\sigma+1)^{1/2}}{4\pi}\int_0^{2\pi}\int_0^{\pi}D_{\lambda\ \lambda_1-\lambda_2}{}^{\sigma*}(R(\phi,\theta,-\phi))\\&\qquad\times e^{-2i\phi\lambda_2}|s{:}\ \mathbf{p};\ \phi,\theta{:}\ \lambda_1,\lambda_2\rangle_p\sin\theta d\theta d\phi \qquad (148)\\&=(2\sigma+1)^{1/2}\hat{L}(p)\int D_{\lambda\mu}{}^{\sigma*}(R(\phi,\theta,\psi))\\&\qquad\times[\hat{R}(\phi,\theta,\psi)\exp(\tfrac{1}{2}\pi iJ_2)]\\&\qquad\times|s{:}\ \mathbf{0};\ 0,\tfrac{1}{2}\pi{:}\ \lambda_1,\lambda_2\rangle_p d\mu(R), \qquad (149)\end{aligned}$$

where the invariant measure on the covering group $SU(2)$ of the rotation group $d\mu(R)$ has been defined in Eq. (124). We recall from the discussion preceding Eq. (129) that μ denotes the eigenvalue of the scalar-spin-component operator $\hat{S}^{p2}$. Now this operator is related to $\hat{S}_1{}^p$ and $\hat{S}_2{}^p$ by

$$\hat{S}^{p2}=\hat{S}_1{}^p-\hat{S}_2{}^p, \tag{150}$$

and this implies the condition

$$\mu=\lambda_1-\lambda_2. \tag{151}$$

One can alternatively show, using invariance of the rotation-group measure $d\mu(R)$, that the integral in Eq. (149) vanishes unless this constraint (151) is satisfied.

From the defining Eq. (149) one can show that under homogeneous Lorentz transformations these two-particle states behave in the same way as the single-particle states of Eq. (37),

$$\begin{aligned}\hat{\Lambda}|s,\sigma{:}\ \mathbf{p},\lambda{:}\ \lambda_1,\lambda_2\rangle_p=&D_{\lambda'\lambda}{}^{\sigma}(W(\Lambda,p))\\&\times|s;\ \sigma{:}\ \mathbf{p},\lambda'{:}\ \lambda_1,\lambda_2\rangle_p.\end{aligned} \tag{152}$$

This implies that parameter λ is the eigenvalue of the spin operator $\hat{S}_3$ and $\sigma(\sigma+1)$ is the eigenvalue of the effective spin operator $\hat{\mathbf{S}}^2$.

One may also construct helicity states. We replace the operator $\hat{L}(p)$ in Eq. (149) by the helicity operator $\hat{H}(p)$ of Eq. (40). The relation between such helicity states $|s,\sigma{:}\ \mathbf{p},\lambda{:}\ \lambda_1,\lambda_2\rangle_{ph}$ and the standard eigenstates of $\hat{S}_3$ is given by Eqs. (48) and (49):

$$|s,\sigma{:}\ \mathbf{p},\lambda{:}\ \lambda_1,\lambda_2\rangle_{ph}=D_{\lambda'\lambda}{}^{\sigma}(R(p))|s,\sigma{:}\ \mathbf{p},\lambda'{:}\ \lambda_1,\lambda_2\rangle_p. \tag{153}$$

They transform in a similar way to single-particle helicity states

$$|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{ph}=D_{\lambda'\lambda}{}^{\sigma}(W^h(\Lambda,p))\times|s,\sigma:\mathbf{p},\lambda':\lambda_1\lambda_2\rangle_{ph}. \quad (154)$$

We did not consider helicity states of a single particle (41) to be well defined when the momentum $\mathbf{p}$ was in the negative 3-direction. Similarly we do not consider c.m. helicity states, or two-particle states of the form (153), in which the total momentum $\mathbf{p}$ lies in the negative 3-direction, to be well defined.

We can also define Wick-type helicity states $|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{pW}$ if we replace the operator $\hat{L}(p)$ in Eq. (149) by the helicity operator[7] $\hat{H}^W(p)$. These states differ from the helicity states $|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{ph}$ by a phase

$$|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{pW}=e^{-i\lambda\phi(p)}|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{ph}, \quad (155)$$

and are not well defined if the first and second components of the total momentum $\mathbf{p}$ are both zero.

One may compare our states with those of Jacob and Wick.[6] In the c.m. frame the total momentum p is of the form

$$p_\mu\leftrightarrow\Delta(p)(1;0,0,0). \quad (156)$$

The operators $\hat{S}_1{}^p$ and $\hat{S}_2{}^p$ are then equivalent to helicity operators $\hat{h}_1$ and $\hat{h}_2$ when acting on c.m. frame states $|s:\mathbf{0};\phi,\theta:\lambda_1,\lambda_2\rangle_p$. Apart from a momentum-dependent normalization function $\mathfrak{N}$, we find that the Jacob-Wick helicity states $||\bar{\mathbf{p}}_1|,\sigma,\lambda;\lambda_1,\lambda_2\rangle_{JW}$ are simply related to our c.m. frame p-spin standard states of type III,

$$||\bar{\mathbf{p}}_1|,\sigma,\lambda;\lambda_1,\lambda_2\rangle_{JW}=\mathfrak{N}(-1)^{(\sigma_2-\lambda_2)}|s;\sigma:\mathbf{0},\lambda:\lambda_1,\lambda_2\rangle_p. \quad (157)$$

The Jacob-Wick parameter λ is the eigenvalue of the spin-component operators $\hat{S}_3$ or $\hat{J}_3$ or $(1/m)\hat{W}_3$, etc. The ambiguity arises because the states are defined only in the case of zero total momentum. Similarly the Jacob-Wick parameters λ_1 and λ_2 are the eigenvalues of helicity operators $\hat{h}_1$ and $\hat{h}_2$ or scalar-spin-component operators $\hat{S}_1{}^p$ and $\hat{S}_2{}^p$, etc. We shall see later that the real meaning of these parameters critically affects the crossing properties of associated partial-wave amplitudes.[5]

In a more recent paper,[7] Wick defines a class of helicity states $|p;\sigma,\lambda:\lambda_1\lambda_2\rangle_W$ for two-particle systems with nonzero total momentum $\mathbf{p}$, by

$$|p;\sigma,\lambda:\lambda_1\lambda_2\rangle_W=\left[\frac{\Delta(p_1,p_2)}{4s}\right]^{1/2}\left(\frac{2\sigma+1}{4\pi}\right)^{1/2}\hat{H}^W(p)\times\int D_{\mu\lambda}{}^{\sigma}(R^{-1}(\phi,\theta,0))\hat{R}(\phi,\theta,0)\times|\tilde{\mathbf{p}}_1,\tilde{\mathbf{p}}_2:\lambda_1,\lambda_2\rangle_p d\cos\theta d\phi, \quad (158)$$

where the momentum $\tilde{\mathbf{p}}_1$ lies in the positive 3-direction. These states coincide with our p-spin helicity states

$$|p;\sigma,\lambda:\lambda_1\lambda_2\rangle_W=|s,\sigma:\mathbf{p},\lambda:\lambda_1,\lambda_2\rangle_{pW}. \quad (159)$$

We should like to emphasize the point that in these states the parameter λ is the eigenvalue of the total helicity operator $\hat{h}$, and the parameters λ_1 and λ_2 are eigenvalues of the scalar p-spin operators $\hat{S}_1{}^p$ and $\hat{S}_2{}^p$ and not of the helicity operators $\hat{h}_1$ and $\hat{h}_2$.

B. Three-Particle States

Three-particle states of type I may be constructed in which each particle is treated in an identical way. We can in this case alone define two independent four-momenta p and Q, with

$$Q_\mu=\epsilon_\mu{}^{\nu\rho\sigma}p_{1:\nu}p_{2:\rho}p_{3:\sigma}, \quad (160)$$

which remain the same, up to a sign, on the interchange of any two single-particle momenta.

We have primarily been concerned with scalar spin components $\hat{S}^q$ associated with timelike momenta q. We now wish to show how to construct eigenstates of the spin-component operators $\hat{S}_1{}^Q$, $\hat{S}_2{}^Q$, and $\hat{S}_3{}^Q$ when the momentum Q is spacelike.

Let the momentum Q_μ be of the form

$$Q_\mu\leftrightarrow(\sinh\delta,\ -\cosh\delta\sin\theta\cos\phi,\ -\cosh\delta\sin\theta\sin\phi,\ -\cosh\delta\cos\theta)\Delta(Q). \quad (161)$$

We associate with it homogeneous Lorentz transformations $\hat{R}(Q)$, $\hat{R}_3(Q)$, and $\hat{H}(Q)$, where

$$\hat{R}(Q)=\exp(-i\hat{J}_3\phi)\exp(-i\hat{J}_2\theta)\exp(i\hat{J}_3\phi), \quad (162)$$

$$\hat{R}_3(Q)=\exp(-i\hat{J}_3\phi), \quad (163)$$

$$\hat{H}(Q)=\hat{R}(Q)\exp(-i\hat{K}_3\delta). \quad (164)$$

For timelike momenta p of the form (25) we also introduce the operator

$$\hat{N}(p)=\hat{L}(p)\hat{R}_3(p). \quad (165)$$

We may then define Q-spin component eigenstates by

$$|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3:\lambda_1,\lambda_2,\lambda_3\rangle_Q=\prod_{j=1}^{3}\{\hat{L}_j(p_j,Q,p)\}|\mathbf{0},\mathbf{0},\mathbf{0};\lambda_1,\lambda_2,\lambda_3\rangle, \quad (166)$$

where the operator $\hat{L}(p_j,Q,p)$ is of the form

$$\hat{L}(p_j,Q,p)=\hat{L}(p)\hat{H}(L^{-1}(p)Q)\times\hat{N}(H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_j). \quad (167)$$

This expression is similar to that for a timelike momentum Q which we gave in Eq. (95).

We now verify that the state defined by Eq. (166) is an eigenstate of operators $\hat{S}_1{}^Q$, $\hat{S}_2{}^Q$, and $\hat{S}_3{}^Q$ with eigenvalues λ_1, λ_2, and λ_3, respectively. From the defining equation (166) we have the relation

$$\hat{W}_k\cdot\hat{Q}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\lambda_1,\lambda_2,\lambda_3\rangle_Q=\prod_{j=1}^{3}\{\hat{L}_j(p_j,Q,p)\}L^{-1}(p_k,Q,p)Q\cdot\hat{W}_k\times|\mathbf{0},\mathbf{0},\mathbf{0};\lambda_1,\lambda_2,\lambda_3\rangle. \quad (168)$$

We wish to determine the form of the momentum $L^{-1}(p_k,Q,p)Q$. From the defining equations (164) and (167) of $\hat{H}(Q)$ and $\hat{L}(p_k,Q,p)$, we see that it is given by

$$L^{-1}(p_k,Q,p)=N^{-1}(H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_k)Q^{\circ}, \quad (169)$$

where Q° lies in the 3-direction,

$$Q_{\mu}^{\circ}\leftrightarrow\Delta(Q)(0;0,0,-1). \quad (170)$$

Now the c.m. momentum $H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_k$ has zero 3-component because vectors Q and p_k are orthogonal. The associated boost function must be of the form

$$N(H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_k) = e^{-iJ_3\phi}e^{-iJ_2\pi/2}e^{-iK_3\delta}e^{iJ_2\pi/2}, \quad (171)$$

which is an element of the little group of momentum Q°. We now recall that, on a rest state, operators $\hat{W}_{k:\mu}$ and $(0;\Delta(p_k)\hat{\mathbf{S}}_k)$ have the same eigenvalues. After substituting the new expressions for $L^{-1}(p_k,Q,p)Q$ and $\hat{W}_k$ into Eq. (168), we obtain the eigenvalue equation

$$\hat{W}_k\cdot\hat{Q}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q = \Delta(Q)\Delta(p_k)\lambda_k|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q \quad (172)$$

as required.

Under homogeneous Lorentz transformations $\hat{\Lambda}$ and translations $\hat{a}$, we see from the defining equation that these states behave in the following way:

$$\hat{a}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q = e^{i(p_1+p_2+p_3)\cdot a}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q, \quad (173)$$

$$\hat{\Lambda}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q = \prod_{j=1}^{3}\{e^{-iJ_3\phi^j\lambda_j}\}|\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q, \quad (174)$$

where $p_j'=\Lambda p_j$. The phases ϕ^j are given by

$$e^{-iJ_3\phi^j}=W^n(W^h(W(\Lambda,p),L^{-1}(p)Q), \times H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_j), \quad (175)$$

and the Wigner rotation $W^n(\Lambda,p)$ is of the form

$$W^n(\Lambda,p)=N^{-1}(\Lambda p)\Lambda N(p). \quad (176)$$

We show in Appendix A that the multiple Wigner rotation (175) is equal to the identity, and all phases ϕ^j are zero by construction.

We may express our three-particle states as linear combinations of direct-product standard states in the following way:

$$|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1,\lambda_2,\lambda_3\rangle_Q = \sum_{\lambda_1'\lambda_2'\lambda_3'}\langle\lambda_1',\lambda_2',\lambda_3'|\lambda_1,\lambda_2,\lambda_3\rangle_Q \times|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\colon\lambda_1',\lambda_2',\lambda_3'\rangle, \quad (177)$$

where the overlap coefficients are given by

$$\langle\lambda_1',\lambda_2',\lambda_3'|\lambda_1,\lambda_2,\lambda_3\rangle_Q=\prod_{j=1}^{3}\{D_{\lambda_j'\lambda_j}{}^{\sigma_j}(L^{-1}(p_j)L(p_j,Q,p))\}. \quad (178)$$

VI. SUMMARY AND CONCLUSIONS

We have characterized a system of noninteracting particles by the eigenvalues of complete sets of commuting-operator observables. Each state of the system is a function of individual particle masses m_i, spins σ_i, and a number of frame-dependent parameters. We took as standard states $|\mathbf{p}_i,\lambda_i\rangle$, eigenstates of momentum operators $\hat{\mathbf{p}}_i$, and spin-component operators $\hat{S}_{i:3}$. Our aim has been to construct a class of states labeled by a maximum number of frame-independent eigenvalues of Poincaré scalar observables.

In Secs. III and IV we have shown how, for systems of at least three particles, multiparticle states transforming like single-particle states may be constructed. They were labeled by four frame-dependent parameters $\mathbf{p}$ and λ corresponding to the total momentum $\hat{\mathbf{p}}$ and a component of the total effective spin $\hat{S}_3$. In order to do this, we first of all replaced the eigenvalues of spin-component operators $\hat{S}_{i:3}$, labeling standard direct-product states by eigenvalues of scalar-spin-component operators $\hat{S}_i{}^q$ of the form

$$\hat{S}_i{}^q=\hat{W}_i\cdot\hat{q}_i/\Delta(\hat{q}_i,\hat{p}_i). \quad (179)$$

In these new states of type I denoted by $|\mathbf{p}_i\colon\lambda_i\rangle_q$, we measure components of spins $\hat{\mathbf{S}}_i$ for each particle (i), relative to frames fixed by four-momenta p_i, q_i, and r_i. Such states are related to standard direct-product states $|\mathbf{p}_i,\lambda_i\rangle$ by

$$|\mathbf{p}_i,\lambda_i\rangle_q=\sum_{\lambda_i'}\langle\lambda_i'|\lambda_i\rangle_q|\mathbf{p}_i,\lambda_i'\rangle, \quad (180)$$

where the coefficients $\langle\lambda_i'|\lambda_i\rangle_q$ are given by Eq. (99). Under homogeneous Lorentz transformations $\hat{\Lambda}$, the scalar-spin-component state labels λ_i do not change,

$$\hat{\Lambda}|\mathbf{p}_i\colon\lambda_i\rangle_q=|\mathbf{p}_i'\colon\lambda_i\rangle_q, \quad (181)$$

and the transformed momenta $\mathbf{p}_i'$ are given by

$$p_i'=\Lambda p_i. \quad (182)$$

The construction of these q-spin eigenstates with spin-independent Poincaré transformation properties should prove useful in the generalization of spinless particle theories or models to include external particles with spin.

In Sec. IV we proceeded to construct scalar-spin-component states of type II. We replaced the frame-dependent three-momenta $\mathbf{p}_i$ by the square of the total momentum s, the total three-momentum $\mathbf{p}$, a number of scalar momentum products s_{ij}, where

$$s_{ij}=(p_i+p_j)^2,\quad s_{i0}=(p-p_i)^2,\quad i,j=1,2,\ldots,n \quad (183)$$

and three angle variables ϕ, θ, and ψ. These angles may be defined in terms of the c.m. momenta $\bar{p}_2$ and $\bar{p}_3$ of Eqs. (102) and (103),

$$R(\phi,\theta,-\phi)=R(-\bar{\mathbf{p}}_2), \quad (184)$$

$$R_3(\psi)=R_3(\tilde{p}_3). \quad (185)$$

The new states of type II, $|s\colon \mathbf{p};\phi,\theta,\psi\colon s_{jk},\lambda_i\rangle_q$, are simply related to the states $|\mathbf{p}_i\colon \lambda_i\rangle_q$ of type I,

$$|s\colon \mathbf{p};\phi,\theta,\psi\colon s_{jk},\lambda_i\rangle_q = J_n^{1/2}|\mathbf{p}_i\colon \lambda_i\rangle_q\,, \quad (186)$$

where the range of parameters i, j, and k is given in Eq. (193), and the coefficients J_n are defined by Eqs. (110)–(112).

Neither the momentum parameters s_{jk} nor the spin parameters λ_i change under homogeneous Lorentz transformations $\hat{\Lambda}$,

$$\hat{\Lambda}|s\colon \mathbf{p};\phi,\theta,\psi\colon s_{jk},\lambda_i\rangle_q = |s\colon \mathbf{p}';\phi',\theta',\psi'\colon s_{jk},\lambda_i\rangle_q\,. \quad (187)$$

The new angles ϕ', θ', and ψ' are defined by Eqs. (119) and (120) and the transformed momentum p' is given by $p'=\Lambda p$.

Using the Poincaré-invariant nature of a scattering operator $\hat{\mathrm{S}}$ one can show that matrix elements between these states are invariant amplitudes, functions of Poincaré scalars alone.[5]

Finally in Sec. IV we introduced states of type III labeled by a minimum number of frame-dependent parameters. They could be defined in terms of the scalar-spin-component states of type II, and overlap coefficients $\langle\phi,\theta,\psi|\sigma\colon \lambda\colon \mu\rangle$ of Eq. (131), by

$$|s,\sigma\colon \mathbf{p},\lambda\colon \mu;s_{jk},\lambda_i\rangle_q = \frac{1}{16\pi^2}\int_0^{4\pi}\int_0^{\pi}\int_0^{2\pi}\langle\phi,\theta,\psi|\sigma\colon \lambda\colon \mu\rangle \times|s\colon \mathbf{p};\phi,\theta,\psi\colon s_{jk},\lambda_i\rangle_q \sin\theta d\phi d\theta d\psi\,. \quad (188)$$

The new state parameters $\sigma(\sigma+1)$, λ, and μ are the eigenvalues of effective spin operators $\hat{\mathbf{S}}^2$, $\hat{S}_3$, and $\hat{S}^{p2}$, respectively. Under homogeneous Lorentz transformations $\hat{\Lambda}$ these states transform like single-particle states,

$$\hat{\Lambda}|s,\sigma\colon \mathbf{p},\lambda\colon \mu;s_{jk},\lambda_i\rangle_q = D_{\lambda'\lambda}{}^{\sigma}(W(\Lambda,p))|s,\sigma;\mathbf{p},\lambda'\colon \mu;s_{jk},\lambda_i\rangle_q\,, \quad (189)$$

and parameters σ and μ do not change. Matrix elements of a scattering operators $\hat{\mathrm{S}}$ between these states are partial-wave amplitudes.[5]

The multiparticle states of types I–III have the following normalizations:

$${}_q\langle\mathbf{p}_i'\colon \lambda_i'|\mathbf{p}_i\colon \lambda_i\rangle_q = \prod_{i=1}^{n}\{2p_{i:0}\delta(\mathbf{p}_i-\mathbf{p}_i')\delta_{\lambda_i\lambda_i'}\}\,, \quad (190)$$

$${}_q\langle s'\colon \mathbf{p}';\phi',\theta',\psi'\colon s_{jk}',\lambda_i'|s\colon \mathbf{p};\phi,\theta,\psi\colon s_{jk},\lambda_i\rangle_q = \delta(s-s')2p_0\delta(\mathbf{p}-\mathbf{p}')16\pi^2\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')\delta(\psi-\psi')\prod_{ijk}\{\delta(s_{jk}-s_{jk}')\delta_{\lambda_i\lambda_i'}\}\,, \quad (191)$$

$${}_q\langle s',\sigma'\colon \mathbf{p}',\lambda'\colon \mu';s_{jk}',\lambda_i'|s,\sigma\colon \mathbf{p},\lambda\colon \mu;s_{jk},\lambda_i\rangle_q = \delta(s-s')\delta_{\sigma\sigma'}2p_0\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\delta_{\mu\mu'}\prod_{ijk}\{\delta(s_{jk}-s_{jk}')\delta_{\lambda_i\lambda_i'}\}\,, \quad (192)$$

where the ranges of the indices i, j, and k are

$$i=1,2,\ldots,n;\begin{cases} j=0,\ k=2,3,\ldots,n; & \begin{cases} j=0,\ k=2,3,4; \\ j=2,\ k=3,4,\ n=4; \end{cases} \\ j=2,\ k=3,4,\ldots,n; & \\ j=3,\ k=5,6,\ldots,n,\ n\geqslant 5; & j=0,\ k=2,3,\ n=3. \end{cases} \quad (193)$$

In the two-particle case the procedure for constructing states of type III was no more complicated. However, states of types I and II did not have simple homogeneous Lorentz-group transformation properties. We could define a class of two-particle states of type I in terms of direct-product helicity states $|\mathbf{p}_i,\lambda_i\rangle_h$ and overlap coefficients ${}_h\langle\lambda_1',\lambda_2'|\lambda_1,\lambda_2\rangle_p$ of Eq. (144) by

$$|\mathbf{p}_1,\mathbf{p}_2\colon \lambda_1,\lambda_2\rangle_p = \sum_{\lambda_1'\lambda_2'}{}_h\langle\lambda_1',\lambda_2'|\lambda_1,\lambda_2\rangle_p \times|\mathbf{p}_1,\mathbf{p}_2,\lambda_1',\lambda_2'\rangle_h\,. \quad (194)$$

Under homogeneous Lorentz transformations $\hat{\Lambda}$ the scalar spin components λ_1 and λ_2 do not change, but the state is multiplied by a function of the phase ϕ^1 defined by Eq. (142),

$$\hat{\Lambda}|\mathbf{p}_1,\mathbf{p}_2\colon \lambda_1,\lambda_2\rangle_p = e^{-i(\phi^1\lambda_1-\phi^2\lambda_2)}|\mathbf{p}_1',\mathbf{p}_2'\colon \lambda_1,\lambda_2\rangle_p\,. \quad (195)$$

States of type II were defined by

$$|s\colon \mathbf{p};\phi,\theta\colon \lambda_1,\lambda_2\rangle_p = [\pi\Delta(p_1,p_2)/s]^{1/2}|\mathbf{p}_1,\mathbf{p}_2\colon \lambda_1,\lambda_2\rangle_p\,. \quad (196)$$

They also transformed with a frame-dependent phase,

$$\hat{\Lambda}|s\colon \mathbf{p};\phi,\theta\colon \lambda_1,\lambda_2\rangle_p = e^{-i(\phi^1\lambda_1-\phi^2\lambda_2)}|s\colon \mathbf{p}';\phi',\theta'\colon \lambda_1,\lambda_2\rangle_p\,, \quad (197)$$

and angles ϕ' and θ' were given by Eq. (119).

Finally we defined states of type III in terms of states of type II and an overlap coefficient $\langle\phi,\theta|\sigma\colon \lambda\rangle$,

$$|s,\sigma\colon \mathbf{p},\lambda\colon \lambda_1,\lambda_2\rangle_p = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\langle\phi,\theta|\sigma\colon \lambda\rangle e^{-2i\phi\lambda_2} \times|s\colon \mathbf{p};\phi,\theta\colon \lambda_1,\lambda_2\rangle_p \sin\theta d\phi d\theta\,, \quad (198)$$

where

$$\langle\phi,\theta|\sigma\colon \lambda\rangle = (2\sigma+1)^{1/2}D_{\lambda,\lambda_1-\lambda_2}{}^{\sigma*}(R(\phi,\theta,-\phi))\,. \quad (199)$$

Under translations and homogeneous Lorentz transformations these states transform in the same way as the multiparticle states of Eq. (188). We have shown in Sec. V that they are closely related to the helicity states of Wick,[7] and in the c.m. frame coincide, up to a normalization factor with the Jacob-Wick helicity states.[6]

The two-particle states of types I–III have the following normalizations:

$$_p\langle \mathbf{p}_1',\mathbf{p}_2' : \lambda_1',\lambda_2' | \mathbf{p}_1,\mathbf{p}_2 : \lambda_1,\lambda_2\rangle_p = 2p_{1:0}\delta(\mathbf{p}_1-\mathbf{p}_1')2p_{2:0}\delta(\mathbf{p}_2-\mathbf{p}_2')\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}, \quad (200)$$

$$_p\langle s' : \mathbf{p}' ; \phi',\theta' : \lambda_1',\lambda_2' | s : \mathbf{p} ; \phi,\theta : \lambda_1,\lambda_2\rangle_p = \delta(s-s')2p_0\delta(\mathbf{p}-\mathbf{p}')4\pi\delta(\phi-\phi') \times\delta(\cos\theta-\cos\theta')\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}, \quad (201)$$

$$_p\langle s',\sigma' : \mathbf{p}',\lambda' : \lambda_1',\lambda_2' | s,\sigma : \mathbf{p},\lambda : \lambda_1,\lambda_2\rangle_p = \delta(s-s')\delta_{\sigma\sigma'}2p_0\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}. \quad (202)$$

All these states were labeled by eigenvalues of q-spin operators $\hat{S}_i^q$ defined in terms of momentum operators $\hat{q}$ with timelike eigenvalues q in the physical region. In Sec. V B we gave an example to show that such a restriction is not really necessary. We constructed a special class of three-particle eigenstates of operators $\hat{S}_i^Q$, where the four-momentum Q was spacelike in the physical region. Such states had similar transformation properties and normalizations to those of the three-particle states of Eq. (180).

In another paper[5] we shall construct and analyze decompositions of general multiparticle scattering amplitudes using these states, without introducing auxiliary spin groups. The techniques developed will also enable us to generalize dynamical theories and models involving spinless particles to include external particles with spin.

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APPENDIX A: PROPERTIES OF SOME GENERALIZED WIGNER ROTATION FUNCTIONS

We shall examine two generalized Wigner rotation functions which are related to the phase changes of our scalar-spin-component states under homogeneous Lorentz transformations Λ.

We first of all consider the triple Wigner rotation function $W_1(\Lambda;\, p,q,r)$, where

$$W_1(\Lambda;\, p,q,r) = W^W(W^h(W(\Lambda,p),q),r). \quad (A1)$$

The ordinary Wigner function $W(\Lambda,p)$, defined by Eq. (39), is a rotation and, like the operator $\hat{L}(p)$, is well defined for all values of the momentum $\mathbf{p}$.

The function $W^h(R,q)$ of a rotation R and four-momentum q defined by Eq. (47) is of the form

$$W^h(R,q) = H^{-1}(Rq)RH(q). \quad (A2)$$

From the defining equation (40) for the boost function $\hat{H}(q)$, we see that it is given by

$$\hat{H}(q) = \hat{R}(\phi,\, \theta,\, -\phi)\hat{Z}(q), \quad (A3)$$

and it is well defined provided the vector q is nonzero and does not lie in the negative 3-direction. Now, since the product of two rotations is again a rotation, we may express the homogeneous Lorentz-group element $R(\alpha,\beta,\gamma)H(q)$ in the form

$$R(\alpha,\beta,\gamma)H(q) = R(\alpha',\beta',\gamma')Z(q), \quad (A4)$$

where angles α', β', and γ' are functions of parameters α, β, γ, θ, and ϕ. The boost function $H(Rq)$ is then given by

$$H(R(\alpha,\beta,\gamma)q) = R(\alpha',\, \beta',\, -\alpha')Z(q). \quad (A5)$$

The Wigner rotation $W^h(R,q)$ is obtained by substituting expressions (A4) and (A5) into Eq. (A2) and by making use of the zero commutator of operators $\hat{J}_3$ and $\hat{K}_3$,

$$W^h(R(\alpha,\beta,\gamma),q) = R_3(\alpha'+\gamma'). \quad (A6)$$

We have shown that the Wigner rotation $W^h(R,q)$ of an arbitrary rotation R and four-vector q is a rotation about the 3-axis.

Let us now examine the Wigner function $W^W(R_3,r)$ of a 3-axis rotation $R_3(\gamma)$ and a four-vector r defined by Eq. (53),

$$W^W(R_3,r) = [H^W(R_3(\gamma)r)]^{-1}R_3(\gamma)H^W(r). \quad (A7)$$

Like the operator $H^W(r)$, it is well defined unless both the first and second components of the vector r vanish. Now, if the boost function $H^W(r)$ defined by Eq. (50) is of the form

$$H^W(r) = R(\phi',\theta',0)Z(r), \quad (A8)$$

the transformation $H^W(Rr)$ will be given by

$$H^W(R_3(\gamma)r) = R(\gamma+\phi',\, \theta',\, 0)Z(r). \quad (A9)$$

On substituting these expressions into Eq. (A7), we discover that this Wigner rotation is equal to the identity

$$W^W(R_3,r) = I. \quad (A10)$$

We now combine Eqs. (A6) and (A10) to obtain an expression for the triple Wigner rotation $W_1(\Lambda,p,q,r)$,

$$W^W(W^h(W(\Lambda,p),q),r) = I. \quad (A11)$$

We now examine the triple Wigner rotation $W_2(\Lambda : p,q,r)$ of Eq. (175),

$$W_2(\Lambda;\, p,q,r) = W^n(W^h(W(\Lambda,p),q),r), \quad (A12)$$

where

$$q = L^{-1}(p)Q\,, \tag{A13}$$

$$r = H^{-1}(L^{-1}(p)Q)L^{-1}(p)p_1\,. \tag{A14}$$

The Wigner function $W(\Lambda,p)$ defined by Eq. (39) is a pure rotation. The function $W^h(R,q)$ of a rotation and spacelike four-vector q is given by Eq. (47),

$$W^h(R,q) = H^{-1}(Rq)RH(q)\,. \tag{A15}$$

By definition the vector Q is orthogonal to the vector p. Consequently the vector $L^{-1}(p)Q$ has zero timelike component and the operator $H(L^{-1}(p)Q)$ is a pure rotation of the form

$$H(L^{-1}(p)Q) = R(\phi,\,\theta,\,-\phi)\,. \tag{A16}$$

Since the product of two rotations is again a rotation we find

$$R(\alpha,\beta,\gamma)H(L^{-1}(p)Q) = R(\alpha',\beta',\gamma') \tag{A17}$$

for some angles α', β', and γ' which are functions of the parameters α, β, γ, θ, and ϕ. The function $H(R(\alpha,\beta,\gamma)q)$ is thus given by

$$H(R(\alpha,\beta,\gamma)q) = R(\alpha',\,\beta',\,-\alpha')\,, \tag{A18}$$

and on substituting this into Eq. (A15), we find

$$W^h(R(\alpha,\beta,\gamma),q) = R_3(\alpha'+\gamma')\,. \tag{A19}$$

We have shown that the Wigner rotation function $W^h(R,q)$ of a rotation R and spacelike vector q is a rotation about the 3-axis.

We finally consider the function $W^n(R_3(\gamma),r)$ of a 3-axis rotation $R_3(\gamma)$ and the four-vector r of Eq. (A14),

$$W^n(R_3(\gamma),r) = N^{-1}(R_3(\gamma)r)R_3(\gamma)N(r)\,. \tag{A20}$$

Since the vector p_1 is orthogonal to the vector Q, the vector r must have zero third component. The boost function $N(r)$ is then given by Eq. (171),

$$N(r) = e^{-iJ_3\phi}e^{-iK_1\delta}\,, \tag{A21}$$

and the function $N(R_3(\gamma)r)$ is of the form

$$N(R_3(\gamma)r) = e^{-iJ_3(\phi+\gamma)}e^{-iK_1\delta}\,. \tag{A22}$$

On substituting these expressions into Eq. (A20), we we find that the Wigner rotation $W^n(R_3(\gamma),r)$ is equal to the identity

$$W^n(R_3(\gamma),r) = I\,. \tag{A23}$$

We now combine Eqs. (A23) and (A19) to obtain an expression for the Wigner rotation $W_2(\Lambda;\,p,q,r)$,

$$W^n(W^h(W(\Lambda,p),q),r) = I\,. \tag{A24}$$

This formula holds provided the boost functions $\hat{H}(q)$ and $\hat{N}(r)$ are well defined. This is the case if the vector $\mathbf{q}$ does not vanish, or lie along the negative 3-axis.

APPENDIX B: SCALAR-SPIN-COMPONENT EIGENSTATES

We wish to construct states $|\mathbf{p}_i\colon \lambda_i\rangle_q$ which are eigenstates of momentum operators $\hat{p}_i$ and scalar-spin-component operators $\hat{S}_i{}^q$, with eigenvalues p_i and λ_i, respectively.

We construct eigenstates of the momentum operators $\hat{p}_i$ by boosting standard direct-product rest states. The most general form for such a state is

$$|\mathbf{p}_i,\lambda_i\rangle_r = \prod_{j=1}^{n}\,\{\hat{H}_j(p_j)\hat{R}_j{}^{-1}\}\,|\mathbf{0},\lambda_i\rangle\,, \tag{B1}$$

where $\hat{R}_j$ is an arbitrary rotation. The eigenvalue equation takes the form

$$\begin{aligned}\hat{p}_k|\mathbf{p}_i,\lambda_i\rangle_r &= \prod_{j=1}^{n}\,\{\hat{H}(p_j)\hat{R}_j{}^{-1}\}H(p_k)R_k{}^{-1}\hat{p}_k|\mathbf{0},\lambda_i\rangle\\ &= p_k|\mathbf{p}_i,\lambda_i\rangle_r\,.\end{aligned} \tag{B2}$$

We wish to determine the constraint we must impose on the rotations $\hat{R}_i$ if we are to obtain q-spin eigenstates.

Consider the effect of the operator $\hat{W}_k\cdot\hat{q}$ on the new states, where the momentum $\hat{q}$ has timelike eigenvlaues,

$$\hat{W}_k\cdot\hat{q}\,|\mathbf{p}_i,\lambda_i\rangle_r = \prod_{j=1}^{n}\,\{\hat{H}(p_j)\hat{R}_j{}^{-1}\}R_kH^{-1}(p_k)q\cdot\hat{W}_k|\mathbf{0},\lambda_i\rangle\,. \tag{B3}$$

On the rest state the eigenvalues of operators $\hat{W}_{k:\mu}$ and $(0;\,m_k\hat{\mathbf{S}}_k)$ coincide. Since the parameter λ_i is the eigenvalue of the operator $\hat{S}_{i:3}$ we shall have constructed q-spin eigenstates if the first and second components of the momentum $R_kH^{-1}(p_k)q$ are zero. In order to achieve this we choose R_k to be of the form of a Wigner rotation,

$$R_k = W^h(M^{-1}(q),p_k)\,, \tag{B4}$$

where the operator $M(q)$ is defined in Eq. (70). This evidently produces a momentum $R_kH^{-1}(p_k)q$ with the required properties. We now substitute this expression (B4) for R_k into Eq. (B3) and use arguments similar to those following Eq. (73) to show that

$$\hat{W}_k\cdot\hat{q}\,|\mathbf{p}_i\colon\lambda_i\rangle_q = \lambda_k\Delta(p_k,q)\,|\mathbf{p}_i\colon\lambda_i\rangle_q\,. \tag{B5}$$

On substituting from Eq. (B4) into Eq. (B1) we obtain an expression for the q-spin eigenstate $|\mathbf{p}_i\colon\lambda_i\rangle_q$,

$$|\mathbf{p}_i\colon\lambda_i\rangle_q = \prod_{j=1}^{n}\,\{\hat{M}_j(q)\hat{H}_j(M^{-1}(q)p_j)\}\,|\mathbf{0},\lambda_i\rangle\,. \tag{B6}$$

APPENDIX C: MOMENTUM SPACE TRANSFORMATION FUNCTIONS J_n

We wish to compare the phase-space volume elements

$$dV_n = \prod_{i=1}^{n}\left\{\frac{d\mathbf{p}_i}{2p_{i:0}}\right\}, \tag{C1}$$

$$dU_n = ds\frac{d\mathbf{p}}{2p_0}\frac{d\phi d(\cos\theta)d\psi}{16\pi^2}ds_{jk}\,, \tag{C2}$$

where the range of parameters j and k is given by Eq. (193). The function J_n is defined by

$$dV_n = \pm J_n dU_n\,, \tag{C3}$$

with

$$J_n = \left|[16\pi^2/\prod_{i=1}^{n}\{2p_{i:0}\}]\det\left[\frac{\partial(\mathbf{p}_i)}{\partial(p_\mu;\phi,\cos\theta,\psi:s_{jk})}\right]\right|. \tag{C4}$$

Direct computation of the Jacobian of the transformation is mathematically too tedious. We shall determine it indirectly, taking advantage of the Lorentz-transformation properties of individual-particle momentum-space volume elements.

We first of all consider particle (1) "off the mass shell," and interpret momentum component $p_{1:0}$ as an independent energy variable,

$$d\mathbf{p}_1/2p_{1:0} = d^4p_1\delta(p_1^2-m_1^2)\theta(p_{1:0})\,. \tag{C5}$$

We may then replace variables $p_{1:\mu}$ by the total four-momentum variables p_μ and obtain the expression

$$dV_n = \delta(p_1^2-m_1^2)\theta(p_{1:0})d^4p\,dV_n'\,, \tag{C6}$$

where

$$p_1 = p - \sum_{i=2}^{n} p_i\,, \tag{C7}$$

$$dV_n' = \prod_{i=2}^{n}\left\{\frac{d\mathbf{p}_i}{2p_{i:0}}\right\}. \tag{C8}$$

The differential volume elements $d\mathbf{p}_i/2p_{i:0}$ are Poincaré scalars. We may compute them in various Lorentz frames. We have already defined c.m. momenta $\bar{p}_2$ and $\tilde{p}_3$ by Eqs. (100) and (104). We now define momenta $\tilde{p}^\star{}_i$ by

$$\tilde{p}^\star{}_i = R_3^{-1}(\tilde{p}_3)R^{-1}(-\bar{\mathbf{p}}_2)L^{-1}(p)p_i\,, \tag{C9}$$

and examine the volume element

$$dV_n' = \frac{d\bar{\mathbf{p}}_2}{2\bar{p}_{2:0}}\frac{d\tilde{\mathbf{p}}_3}{2\tilde{p}_{3:0}}\prod_{i=4}^{n}\left\{\frac{d\tilde{\mathbf{p}}^\star{}_i}{2\tilde{p}^\star{}_{i:0}}\right\}. \tag{C10}$$

The components of momentum $\bar{p}_2$ are equal to those of momentum p_2 in a special c.m. frame. Similarly, the components of momentum $\tilde{p}_3$ are equal to those of momentum p_3 in a special c.m. frame in which momentum $-\bar{\mathbf{p}}_2$ lies in the 3-direction. Components of momenta $\tilde{p}_i^\star$ are equal to those of the momenta p_i in a c.m. frame in which momentum $-\bar{\mathbf{p}}_2$ lies in the 3-direction and momentum $\tilde{p}_3$ has zero component in the 2-direction. We may then define for all i corresponding c.m. momenta by

$$\bar{p}_i = L^{-1}(p)p_i\,, \tag{C11}$$

$$\tilde{p}_i = R^{-1}(-\bar{\mathbf{p}}_2)\bar{p}_i\,, \tag{C12}$$

$$\tilde{p}^\star{}_i = R_3^{-1}(\tilde{p}_3)\tilde{p}_i\,. \tag{C13}$$

Using arguments similar to those following Eq. (89), we may express the c.m. momenta in terms of single-particle momenta alone,

$$\tilde{p}_i = H^{-1}(L^{-1}(p_2)p)L^{-1}(p_2)p\,, \tag{C14}$$

$$\tilde{p}_i^\star = H^{-1}(p,p_2,p_3)p_i\,, \tag{C15}$$

where the operator $\hat{H}(p,p_2,p_3)$ is defined by Eq. (95).

All components $\tilde{p}^\star{}_{i:\mu}$ of the transformed momenta $\tilde{p}^\star{}_i$ are Poincaré scalars. To see this we note from the defining equation (C15) that, after a homogeneous Lorentz transformation $\hat{\Lambda}$, the transformed momentum $\tilde{p}^{\star\prime}$ is given by

$$\tilde{p}^\star{}_i{}' = H^{-1}(\Lambda p,\Lambda p_2,\Lambda p_3)\Lambda H(p,p_2,p_3)\tilde{p}^\star{}_i\,. \tag{C16}$$

From the definition (95) of the boost function $H(p,p_2,p_3)$, and Eq. (A11), we find

$$\tilde{p}^\star{}_i{}' = \tilde{p}^\star{}_i\,. \tag{C17}$$

The nonmanifestly covariant nature of c.m. momenta $\bar{p}_i$, $\tilde{p}_i$, and $\tilde{p}^\star{}_i$ arises because the homogeneous Lorentz transformations which relate them to manifestly covariant four-momenta p_i are themselves frame dependent.

We shall now derive explicit formulas relating the Poincaré scalar components $\tilde{p}^\star{}_{i:\mu}$ to the particle four-momenta $p_{i:\mu}$.

We define two spacelike momenta q_2 and r_3 by

$$q_2 = p_2 - p_2\cdot pp/p^2\,, \tag{C18}$$

$$r_3 = q_3 - q_3\cdot q_2q_2/q_2^2\,, \tag{C19}$$

where

$$q_3 = p_3 - p_3\cdot pp/p^2\,. \tag{C20}$$

The magnitudes of these four-momenta are given by

$$\Delta(q_2) = \Delta(p,p_2)/\Delta(p) \tag{C21}$$

and

$$\Delta(r_3) = \Delta(p,p_2,p_3)/\Delta(p,p_2)\,. \tag{C22}$$

Using the general definitions (C11), (C12), and (C15) of the c.m.-frame momenta, we find

$$\bar{p} \leftrightarrow \Delta(p)(1;0,0,0)\,, \tag{C23}$$

$$\bar{q}_2 \leftrightarrow \frac{\Delta(p,p_2)}{\Delta(p)}(0;0,0,-1)\,, \tag{C24}$$

$$\tilde{r}^\star{}_3 \leftrightarrow \frac{\Delta(p,p_2,p_3)}{\Delta(p,p_2)}(0;1,0,0) \tag{C25}$$

by construction. We now use these expressions to obtain explicit formulas for momentum components $\tilde{p}^\star$

From Eq. (C23) we find

$$\tilde{p}^{\star}{}_{i:0}=\tilde{p}_{i:0}=\bar{p}_{i:0}=\bar{p}\cdot\bar{p}_i/\Delta(p)=p\cdot p_i/\Delta(p)\,. \quad \text{(C26)}$$

From the mass-shell relations, we can obtain the magnitudes of the c.m. three-momenta $|\bar{\mathbf{p}}_i|$,

$$|\bar{\mathbf{p}}_i|=\Delta(p,p_i)/\Delta(p)\,. \quad \text{(C27)}$$

We use Eq. (C24) to determine the Poincaré scalar components $\tilde{p}_{i:3}$,

$$\tilde{p}^{\star}{}_{i:3}=\tilde{p}_{i:3}=\frac{p_i\cdot q_2}{\Delta(q_2)}=\frac{-p_i\cdot pp_2\cdot p+p^2p_i\cdot p_2}{\Delta(p)\Delta(p,p_2)}\,, \quad \text{(C28)}$$

and from Eq. (C25) we obtain an expression for the scalar components $\tilde{p}^{\star}{}_{i:1}$,

$$\tilde{p}^{\star}{}_{i:1}=-p_i\cdot r_3/\Delta(r_3)=-\frac{(p^2p_i\cdot p_2-p_2\cdot pp_ip)(p^2p_2\cdot p_3-p_2\cdot pp_3\cdot p)+\Delta^2(p,p_2)(p^2p_i\cdot p_3-p_3\cdot pp_i\cdot p)}{p^2\Delta(p,p_2,p_3)\Delta(p,p_2)}\,. \quad \text{(C29)}$$

Finally we note that the second component $\tilde{p}^{\star}{}_{i:2}$ is given by

$$\epsilon^{\mu\nu\rho\sigma}p_\mu p_{2:\nu}p_{3:\rho}p_{i:\sigma}=-\Delta(p)\Delta(q_2)\Delta(r_3)\tilde{p}_{i:2}\,, \quad \text{(C30)}$$

and up to a sign we have

$$\tilde{p}^{\star}{}_{i:2}=\Delta(p,p_2,p_3,p_i)/\Delta(p,p_2,p_3)\,. \quad \text{(C31)}$$

We are now in a position to transform the volume element dV_n' of Eq. (C7). For $i\geqslant 4$ we use expressions (C26), (C28), (C29), and (C31) for the momentum components $\tilde{p}^{\star}{}_{i:\mu}$ to change from variables $\mathbf{p}_i$ to variables s_{0i}, s_{2i}, and s_{3i}. Up to a sign we have

$$\frac{d\tilde{\mathbf{p}}^{\star}{}_i}{2\tilde{p}^{\star}{}_{i:0}}=\frac{d\tilde{p}^{\star}{}_{i:0}d\tilde{p}^{\star}{}_{i:1}d\tilde{p}^{\star}{}_{i:3}}{2\tilde{p}^{\star}{}_{i:2}}=\frac{ds_{0i}ds_{2i}ds_{3i}}{16\Delta(p,p_2,p_3,p_i)}\,. \quad \text{(C32)}$$

The first two components of momentum $\tilde{p}_3$ are of the form

$$\tilde{p}_{3:1}=p^{\star}\cos\psi,\quad \tilde{p}_{3:2}=p^{\star}\sin\psi\,. \quad \text{(C33)}$$

We make successive variable transformations and use the mass shell constraint for particle (3) to obtain the equation

$$\frac{d\tilde{\mathbf{p}}_3}{2\tilde{p}_{3:0}}=\frac{p^{\star}dp^{\star}d\tilde{p}_{3:3}}{2\tilde{p}_{3:0}}d\psi=\tfrac{1}{2}d\tilde{p}_{3:0}d\tilde{p}_{3:3}d\psi\,. \quad \text{(C34)}$$

We then express momentum components $\tilde{p}_{3:0}$ and $\tilde{p}_{3:3}$ as functions of Poincaré scalars s_{03} and s_{23}. Up to a sign, we find

$$d\tilde{\mathbf{p}}_3/2\tilde{p}_{3:0}=ds_{03}ds_{23}d\psi/8\Delta(p,p_2)\,. \quad \text{(C35)}$$

We now consider the c.m. phase space for particle (2). Angles θ and ϕ are defined by

$$-\bar{p}_{2:1}=|\bar{\mathbf{p}}_2|\sin\theta\cos\phi\,, \quad \text{(C36)}$$

$$-\bar{p}_{2:2}=|\bar{\mathbf{p}}_2|\sin\theta\sin\phi\,, \quad \text{(C37)}$$

$$-\bar{p}_{2:3}=|\bar{\mathbf{p}}_2|\cos\theta\,. \quad \text{(C38)}$$

We change to polar coordinates and use the mass-shell constraint to obtain the equation

$$\frac{d\bar{\mathbf{p}}_2}{2\bar{p}_{2:0}}=-\frac{|\bar{\mathbf{p}}_2|^2}{2\bar{p}_{2:0}}d|\bar{\mathbf{p}}_2|d\phi d\cos\theta=-\tfrac{1}{2}|\bar{\mathbf{p}}_2|d\bar{p}_{2:0}d\phi d\cos\theta\,. \quad \text{(C39)}$$

We now use relations (C26) and (C27) to derive the equation

$$\frac{d\bar{\mathbf{p}}_2}{2\bar{p}_{2:0}}=\frac{\Delta(p,p_2)}{4\Delta^2(p)}ds_{02}d\phi d\cos\theta\,. \quad \text{(C40)}$$

The volume element equation (C6) then takes the form[10]

$$dV_n=\pm\theta(p_{1:0})\delta(p_1{}^2-m_1{}^2)\frac{ds}{32s}\frac{d\mathbf{p}}{2p_0}\times ds_{02}ds_{03}ds_{23}d\phi d\cos\theta d\psi\prod_{i=4}^{n}\left\{\frac{ds_{0:i}ds_{2:i}ds_{3:i}}{16\Delta(p,p_2,p_3,p_i)}\right\}. \quad \text{(C41)}$$

We shall now consider separately the transformation formulas for 2-particle, 3-particle, and n-particle systems with $n\geqslant 4$.

For a two-particle system, Eq. (C41) is of the form

$$dV_2=\theta(p_{1:0})\delta(p_1{}^2-m_1{}^2)\frac{\Delta(p_1,p_2)}{4s}ds\frac{d\mathbf{p}}{2p_0}d\phi d\cos\theta ds_{02}\,, \quad \text{(C42)}$$

where

$$p_1{}^2=(p-p_2)^2=s_{02}\,. \quad \text{(C43)}$$

Thus the required volume element is

$$dV_2=\pi\frac{ds}{s}\Delta(p_1p_2)\frac{d\mathbf{p}}{2p_0}\frac{d\phi d\cos\theta}{4\pi}\,, \quad \text{(C44)}$$

$$J_2=\left|\frac{4\pi}{2p_{1:0}2p_{2:0}}\frac{\partial(\mathbf{p}_1,\mathbf{p}_2)}{\partial(p_\mu;\phi,\cos\theta)}\right|=\pi\frac{\Delta(p_1,p_2)}{s}\,. \quad \text{(C45)}$$

[10] When one determines the range of integration of scalar variables, one must take into account the multivalued nature of the momentum-space mapping.

For a three-particle system we have

$$dV_3=\theta(p_{1:0})\delta(p_1^2-m_1^2)\frac{ds}{32s}\frac{d\mathbf{p}}{2p_0}d\phi d\cos\theta d\psi ds_{02}ds_{03}ds_{23}\,,\tag{C46}$$

$$p_1^2=(p-p_2-p_3)^2=s_{23}+s_{02}+s_{03}-s-m_2^2-m_3^2\,.\tag{C47}$$

After removing the variable s_{23} we find

$$dV_3=\pi^2\frac{ds}{2s}\frac{d\mathbf{p}}{2p_0}\frac{d\phi d\cos\theta d\psi}{16\pi^2}ds_{02}ds_{03}\,,\tag{C48}$$

$$J_3=\left|\frac{16\pi^2}{2p_{1:0}2p_{2:0}2p_{3:0}}\frac{\partial(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}{\partial(p_\mu,\phi,\cos\theta,\psi;s_{02},s_{03})}\right|=\frac{\pi^2}{2s}\,.\tag{C49}$$

For a system with four or more particles we consider Eq. (C41) with momentum p_1 given by

$$p_1^2=q^2-2q\cdot p_4+p_4^2\,,\tag{C50}$$

where

$$q=p-p_2-p_3-\sum_{i=5}^{n}p_i\,.\tag{C51}$$

In the special c.m. frame the vector $\tilde{q}^\star$ is independent of the components of momentum $\tilde{p}^\star{}_{4:\mu}$. We wish to eliminate the variable s_{34} from Eq. (C41). We use the mass-shell constraint for particle (4) to obtain the relation

$$\frac{\partial(p_1^2)}{\partial\tilde{p}^\star{}_{4:1}}=\frac{2(\tilde{q}^\star{}_{4:1}\tilde{p}^\star{}_{4:2}-\tilde{q}^\star{}_{4:2}\tilde{p}^\star{}_{4:1})}{\tilde{p}^\star{}_{4:2}}\,,\tag{C52}$$

and from Eqs. (C29) and (99) we find

$$\frac{\partial\tilde{p}^\star{}_{4:1}}{\partial s_{34}}=\frac{\Delta(p,p_2)}{2\Delta(p,p_2,p_3)}\,.\tag{C53}$$

Up to a sign this leads to the volume element relation

$$dV_n=\pi^2\frac{ds}{32s}\frac{d\mathbf{p}}{2p_0}\frac{d\phi d\cos\theta d\psi}{16\pi^2}\times\frac{ds_{02}ds_{03}ds_{04}ds_{23}ds_{24}}{\Delta(p,p_2)(\tilde{q}^\star{}_{4:1}\tilde{p}^\star{}_{4:2}-\tilde{p}^\star{}_{4:1}\tilde{q}^\star{}_{4:2})}\times\prod_{i=5}^{n}\left\{\frac{ds_{0i}ds_{2i}ds_{3i}}{16\Delta(p,p_2,p_3,p_i)}\right\}.\tag{C54}$$

The function J_n is given by

$$J_n^{-1}=\frac{32s}{\pi^2}\Delta(p,p_2)\left|\sum_{i=5}^{n}(\tilde{p}^\star{}_{4:1}\tilde{p}^\star{}_{i:2}-\tilde{p}^\star{}_{4:2}\tilde{p}^\star{}_{i:1})-\tilde{p}^\star{}_{4:2}\tilde{p}^\star{}_{3:1}\right|\prod_{i=5}^{n}\{16\Delta(p,p_2,p_3,p_i)\}\,,\tag{C55}$$

where momenta $\tilde{p}^\star{}_{i:2}$ and $\tilde{p}^\star{}_{i:1}$ are given by Eqs. (C29) and (C31), repsectively. In the case where $n=4$, we simply omit the summation $\sum$ and product $\prod$ occurring in this formula. We note that

$$|\tilde{p}^\star{}_{3:1}|=\Delta(r_3)=\Delta(p,p_2,p_3)/\Delta(p,p_2)\,,\tag{C56}$$

and J_4 is given by

$$J_4=\pi^2/32s\Delta(p,p_2,p_3,p_4)\,.\tag{C57}$$