that

$$I=\int_{-1}^{1} I_0(R\sin\theta')d\cos\theta'=2\int_0^{\pi/2} I_0(R\sin\theta)\sin\theta d\theta$$

$$=(2\pi/R)^{1/2}I_{1/2}(R)$$

$$=2R^{-1}\sinh R=2\frac{\sinh[2s(\sigma^2+\sigma'^2+2\sigma\sigma'\cos\theta_{fi})^{1/2}]}{2s(\sigma^2+\sigma'^2+2\sigma\sigma'\cos\theta_{fi})^{1/2}}. \quad \text{(A2)}$$

The same result can also obtained by brute force as follows: Expand $\exp(x\cos\theta')$ and $I_0(y\sin\theta')$ separately in Legendre series with the aid of Secs. 3.9 and 3.13.5 of Ref. 5, integrate term by term, and note that the result coincides with (A2) when the latter is expanded according to the formula[14]

$$\frac{I_{1/2}(R)}{\sqrt{R}}=\left(\frac{2\pi}{xy}\right)^{1/2}\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\times P_n(\cos\theta)I_{n+1/2}(x)I_{n+1/2}(y). \quad \text{(A3)}$$

[14] Reference 5, Sec. 3.9.

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Superconvergence Sum Rules and Exchange Degeneracy in Cross-Duality Model

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Superconvergence conditions and finite-energy sum rules in the cross-duality model are verified and discussed. Exotic channels give exchange degeneracies among branch-point trajectories, which induce degeneracies on their generating pole trajectories. The Pomeranchuk singularity is also briefly considered.

I. INTRODUCTION

RECENTLY,[1,2] explicit constructions were given for two amplitudes, with narrow resonances and cuts, satisfying crossing symmetry and asymptotic behavior. The method of construction is explained in Ref. 1.

One of these amplitudes, called in Ref. 1 the direct-duality model, consists of the Veneziano amplitude[3] plus a term that behaves asymptotically like a Regge cut while it contains an infinite number of square-root branch points in the direct channel starting at the lowest normal threshold. Explicitly the amplitude $M(s,t,u)$ for a two-particle $\to$ two-particle equal-mass spinless process is given as

$$M(s,t,u)=A(s,t)+A(u,t)+A(s,u) \quad (1.1)$$

where, for the direct-duality construction,

$$A(s,t)=C_1\int_0^1 y^{-\alpha(t)-1}(1-y)^{-\alpha(s)-1}dy+C_2\int_0^1\frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha_c(s)-1}}{[1-\ln(1-y)]^{3/2}}dy, \quad (1.2)$$

with C_1 and C_2 being arbitrary constants, $\alpha(t)$ the leading Regge trajectory, and $\alpha_c(t)$ the branch point of the leading Regge cut. This amplitude, except for a slight difference in the integrand of the second term in (1.2), was first suggested by Matveev, Stoyanov, and Tavkhelidze.[4]

The other amplitude, called the cross-duality model, is given by

$$A(s,t)=D\int_0^1\frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha(s)-1}}{(1-\ln y)^{3/2}}dy+(s\leftrightarrow t), \quad (1.3)$$

where D is an arbitrary constant. In the physical region of s-channel scattering ($s>4m^2$, $t<0$) the first term in (1.3) may be written as

$$\sum_{n=0}^{\infty}\frac{g_n(t)}{\alpha(s)-n}, \quad (1.4)$$

where the coefficients $g_n(t)$ are polynomials of order n in t. This term behaves like a Regge cut as $s\to\infty$. The second term in (1.3) has a cut for $s>4m^2$ and behaves like a Regge pole as $s\to\infty$.

Both amplitudes (1.2) and (1.3) give expressions for the discontinuities of the partial-wave amplitude across the Regge cut that vanish at the branch point,[1] in agreement with the general result of Bronzan and Jones.[5] Away from the branch point the discontinuities decay exponentially. Our method of construction imposes, in

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[2] M. O. Taha, Nuovo Cimento Letters **3**, 861 (1970).
[3] G. Veneziano, Nuovo Cimento **57A**, 1395 (1968).
[4] V. A. Matveev, D. T. Stoyanov, and A. N. Tavkhelidze, Phys. Letters **32B**, 61 (1970).
[5] J. B. Bronzan and C. E. Jones, Phys. Rev. **160**, 1494 (1967).

both cases, the equations

$$\alpha(t)=b(t-t_R)\,, \tag{1.5}$$

$$\alpha_c(t)=a(t-t_0)\,, \tag{1.6}$$

where $t=t_R$ is the lowest resonance (assumed narrow and spinless), $t=t_0$ is the lowest normal threshold, and a, b are arbitrary constants. Equation (1.6) is discussed in Ref. 1 when coupled to the assumption that the leading Regge cut is generated by the exchange of two Regge-pole trajectories.

The construction and general analytic properties of both amplitudes (1.2) and (1.3) are discussed in Ref. 1. It is our purpose in this paper to investigate further the cross-duality model because of its novel features and because it does not appear to possess any obvious disadvantage relative to the direct-duality model. In particular, we discuss superconvergence, finite-energy sum rules (FESR), exchange degeneracy, and the Pomeranchuk pole.

In Sec. II expressions for the discontinuities $D_1(s,t)$ and $D_2(s,t)$ of the two terms in (1.3), across the t cut, are obtained. The way in which the superconvergence conditions and FESR are satisfied is then explicitly checked. It is found that the identities obtained are similar to those of the Veneziano model. This section is then concluded with a brief discussion related to duality, double-counting, and bootstrap conditions. A continuous-parameter version of the FESR on D_1 is also given.

In Sec. III we discuss exotic channels under cross-duality, without using the specific amplitude (1.3). The presence of an exotic channel implies exchange-degeneracy relations among the trajectories of the leading branch points. These are then seen to impose exchange degeneracies among the Regge-pole trajectories that generate the leading cuts. This discussion leads us to some considerations on the Pomeranchuk singularity in Sec. IV. In the presence of an exotic channel our picture seems to be consistent with the part of the Freund-Harari[6] conjecture that associates the background with the Pomeranchukon contribution.

II. SUPERCONVERGENCE SUM RULES

In the cross-duality model,[2] the scattering amplitude $M(s,t,u)$ for an equal-mass spinless two-particle $\leftrightarrow$ two-particle process is given by

$$M(s,t,u)=A(s,t)+A(u,t)+A(s,u)\,, \tag{2.1}$$

where

$$A(s,t)=T(s,t)+T(t,s)\,, \tag{2.2}$$

$$T(s,t)=D\int_0^1 \frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha(s)-1}}{(1-\ln y)^{3/2}}dy\,. \tag{2.3}$$

[6] P. G. O. Freund, Phys. Rev. Letters **20**, 235 (1968); H. Harari, *ibid.* **20**, 1395 (1968).

In Eq. (2.3), D is a constant while $\alpha(s)$ and $\alpha_c(s)$ are the leading Regge-pole and branch-point trajectories. These are linear functions given by

$$\alpha(s)=b(s-s_R)\,, \tag{2.4}$$

$$\alpha_c(s)=a(s-s_0)\,, \tag{2.5}$$

where s_R is the position of the lowest-lying (spinless) resonance and $s_0(=4m^2)$ is the lowest threshold.

Define the functions $D_1(s,t)$ and $D_2(s,t)$ by

$$D_1(s,t)=(1/2\pi i)\Delta_t T(s,t)\,, \tag{2.6}$$

$$D_2(s,t)=(1/2\pi i)\Delta_t T(t,s)\,, \tag{2.7}$$

where Δ_t denotes the discontinuity across the t cut. The asymptotic behavior in the t channel is then given by[1]

$$D_1(s,t)\sim\frac{D(\alpha_c(t))^{\alpha(s)}}{\Gamma(\alpha(s)+1)}\,,\quad t\to\infty \tag{2.8}$$

$$D_2(s,t)\sim\frac{D(\alpha(t))^{\alpha_c(s)}}{\Gamma(\alpha_c(s)+1)(\ln t)^{3/2}}\,,\quad t\to\infty\,. \tag{2.9}$$

Thus each of the two terms on the right-hand side of (2.2) is independently superconvergent. The general superconvergence sum rules take the form

$$\int_{t_0}^{\infty} t^n D_1(s,t)dt=0\,,\quad \alpha(s)<-n-1 \tag{2.10}$$

$$\int_{t_0}^{\infty} t^n D_2(s,t)dt=0\,,\quad \alpha_c(s)<-n-1 \tag{2.11}$$

where n is any non-negative integer. It is our purpose to check in detail that the model satisfies these sum rules and to analyze the content of their FESR versions. For simplicity, we consider the case $n=0$ only; the results easily generalize to all positive moments. For $n=0$ the functions $D_1(s,t)$ and $D_2(s,t)$ will be discussed separately.

A. Superconvergence of $T(s,t)$

We start by calculating the discontinuity $D_1(s,t)$ of the function $T(s,t)$. From Eqs. (2.3) and (2.5), $T(s,t)$ can be written in the form

$$T(s,t)=\int_0^{\infty} e^{xt}F(s,x)dx\,,\quad \mathrm{Re}\,t<t_0 \tag{2.12}$$

where

$$F(s,x)=\frac{D}{a}e^{-t_0x}\frac{(1-e^{-x/a})^{-\alpha(s)-1}}{(1+x/a)^{3/2}}\,,\quad x>0\,. \tag{2.13}$$

$F(s,x)$ is then the inverse Laplace transform of $T(s,-t)$ with respect to t, so that

$$F(s,x)=\lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_{c-i\lambda}^{c+i\lambda} e^{-xt}T(s,t)dt\,,\quad c<t_0\,. \tag{2.14}$$

The contour of integration in (2.14) may be transformed into one around the t cut, together with a semicircle at infinity. For $\alpha(s)+1<0$ the semicircle gives a vanishing contribution, so that (2.14) becomes

$$F(s,x)=\int_{t_0}^{\infty} e^{-xt}D_1(s,t)dt\,,\quad x>0\,. \tag{2.15}$$

From this equation, one sees that $D_1(s,t)$ is the inverse Laplace transform of the function $F(s,x)$ in (2.13):

$$D_1(s,t)=\mathfrak{L}^{-1}\{F(s,x)\}\,. \tag{2.16}$$

Inserting (2.13) into (2.16) and using inverse Laplace transform tables,[7] we obtain the following expression for $D_1(s,t)$:

$$D_1(s,t)=\frac{2D}{\sqrt{\pi}}\sum_{0\leq n\leq\alpha_c(t)}(-1)^n\binom{-\alpha(s)-1}{n}\times[\alpha_c(t)-n]^{1/2}e^{n-\alpha_c(t)}\,. \tag{2.17}$$

Using this expression, one explicitly finds

$$\int_{t_0}^{\infty} D_1(s,t)dt$$
$$=\frac{2D}{a\sqrt{\pi}}\sum_{n=0}^{\infty}\binom{n+\alpha(s)}{\alpha(s)}\int_0^{\infty} e^{-(\lambda-n)}(\lambda-n)^{1/2}\theta(\lambda-n)d\lambda$$
$$=\frac{2D}{a\sqrt{\pi}}\sum_{n=0}^{\infty}\binom{n+\alpha(s)}{\alpha(s)}\frac{\sqrt{\pi}}{2}=0\quad\text{for }\alpha(s)+1<0\,, \tag{2.18}$$

where we have used the known result

$$\sum_{n=0}^{N}\binom{n+\alpha}{\alpha}=\binom{\alpha+1+N}{\alpha+1}\sim\frac{N^{\alpha+1}}{\Gamma(\alpha+2)}\quad\text{as }N\to\infty\,.$$

Equation (2.18) is a direct verification of the superconvergence relation (2.10) for $n=0$. For the FESR version, we obtain from (2.17)

$$\int_{t_0}^{R} D_1(s,t)dt=\frac{2D}{\sqrt{\pi}}\sum_{n=0}^{\infty}\binom{n+\alpha(s)}{\alpha(s)}I_n(R)\,, \tag{2.19}$$

where

$$I_n(R)=\int_{t_0}^{R} e^{-(\alpha_c(t)-n)}(\alpha_c(t)-n)^{1/2}\theta(\alpha_c(t)-n)dt\,. \tag{2.20}$$

This integral may be evaluated to give

$$I_n(R)=(1/a)\gamma(\tfrac{3}{2},\,\alpha_c(R)-n)\theta(\alpha_c(R)-n)\,, \tag{2.21}$$

where $\gamma(a,x)$ is the incomplete gamma function.[8]

[7] H. Bateman, *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. I.

[8] H. Bateman, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I.

Equation (2.19) therefore becomes

$$\int_{t_0}^{R} D_1(s,t)dt$$
$$=\frac{2D}{a\sqrt{\pi}}\sum_{0\leq n\leq\alpha_c(R)}\binom{n+\alpha(s)}{\alpha(s)}\gamma(\tfrac{3}{2},\,\alpha_c(R)-n)\,. \tag{2.22}$$

It is interesting to compare results (2.18) and (2.22) to the corresponding ones that obtain in the Veneziano model. The superconvergence condition in (2.18), namely, the identity

$$\sum_{n=0}^{\infty}\binom{n+\alpha}{\alpha}=0\quad\text{for }\alpha+1<0\,,$$

is exactly the same in both cases.[3] Further, if in (2.22) R is so large that

$$\gamma(\tfrac{3}{2},\,\alpha_c(R)-n)\simeq\Gamma(\tfrac{3}{2})=\tfrac{1}{2}\sqrt{\pi}\,,$$

then the sum on the right-hand side of (2.22) looks like the sum over the t-channel pole residues of the Veneziano amplitude. In the present model, $D_1(s,t)$ is the absorptive part over the two-particle cut. This is not surprising, however, since large R takes one into the asymptotic region where both give Regge-pole behavior.

From the asymptotic behavior (2.8), we see that the zero-moment FESR for $D_1(s,t)$ takes the form

$$\int_{t_0}^{R} D_1(s,t)dt\simeq\frac{D[\alpha_c(R)]^{\alpha(s)+1}}{a\Gamma(\alpha(s)+2)}\,. \tag{2.23}$$

This sum rule is then satisfied to the extent within which the approximation

$$\sum_{0\leq n\leq\alpha_c(R)}\binom{n+\alpha(s)}{\alpha(s)}\frac{\gamma(\tfrac{3}{2},\,\alpha_c(R)-n)}{\Gamma(\tfrac{3}{2})}\simeq\frac{[\alpha_c(R)]^{\alpha(s)+1}}{\Gamma(\alpha(s)+2)} \tag{2.24}$$

is maintained. For large R the asymptotic expansion of the incomplete gamma function,[8]

$$\frac{\gamma(\tfrac{3}{2},\,\alpha_c(R)-n)}{\Gamma(\tfrac{3}{2})}=1-\frac{[\alpha_c(R)-n]^{1/2}e^{-\alpha_c(R)-n}}{\Gamma(\tfrac{3}{2})}\times\left[1+\frac{1}{2[\alpha_c(R)-n]}-\frac{1}{4[\alpha_c(R)-n]^2}+\cdots\right],$$

shows that (2.24) is essentially the relation

$$\sum_{0\leq n\leq\alpha_c(R)}\binom{n+\alpha(s)}{\alpha(s)}\simeq\frac{[\alpha_c(R)]^{\alpha(s)+1}}{\Gamma(\alpha(s)+2)}\,,$$

i.e.,

$$\frac{\Gamma(\alpha(s)+\alpha_c(R)+2)}{\Gamma(\alpha_c(R)+1)}\simeq[\alpha_c(R)]^{\alpha(s)+1}\,. \tag{2.25}$$

This relation is asymptotically satisfied, as one expects.

We proceed to discuss the superconvergence sum rule on $D_2(s,t)$.

B. Superconvergence of $T(t,s)$

The procedure for the calculation of the discontinuity $D_2(s,t)$ of $T(t,s)$ follows the same steps as those leading to $D_1(s,t)$ in Sec. II A. The result is

$$D_2(s,t)=\frac{2D}{b\sqrt{\pi}}\sum_{0\leq n\leq b(t-t_0)} a_n(s)\delta\left(t-t_R-\frac{n}{b}\right), \quad (2.26)$$

where $a_n(s)$, the polynomial residues at the t-channel poles, are given by

$$a_n(s)=\int_0^\infty \binom{n+\alpha_c(s)-x}{\alpha_c(s)-x} e^{-x}x^{1/2}dx. \quad (2.27)$$

From Eq. (2.26) one then obtains

$$\int_{t_0}^R D_2(s,t)dt=\frac{2D}{b\sqrt{\pi}}\sum_{0\leq n\leq \alpha(R)} a_n(s). \quad (2.28)$$

Let $N(R)$ be the integer such that $\alpha(R)-1<N(R)\leq\alpha(R)$. Then

$$\int_{t_0}^R D_2(s,t)dt =\frac{2D}{b\sqrt{\pi}}\int_0^\infty \binom{N(R)+\alpha_c(s)+1-x}{\alpha_c(s)+1-x} e^{-x}x^{1/2}dx. \quad (2.29)$$

Now if the superconvergence relation (2.11) for $n=0$ is written in the form of a FESR using (2.9), one has

$$\int_{t_0}^R D_2(s,t)dt\simeq -\frac{D}{\Gamma(\alpha_c(s)+1)}\int_R^\infty \frac{[\alpha(t)]^{\alpha_c(b)}}{(\ln t)^{3/2}}dt, \quad \alpha_c(s)+1<0. \quad (2.30)$$

Using the identity

$$(\ln\lambda)^{-3/2}=\frac{1}{\Gamma(\frac{3}{2})}\int_0^\infty x^{1/2}\lambda^{-x}dx, \quad \lambda>1$$

one can see that for large R

$$\int_R^\infty \frac{[\alpha(t)]^{\alpha_c(s)}}{(\ln t)^{3/2}}dt=-\frac{1}{b\Gamma(\frac{3}{2})}\int_0^\infty \frac{[\alpha(R)]^{\alpha_c(s)+1-x}}{\alpha_c(s)+1-x}x^{1/2}dx.$$

The asymptotic contribution comes from the neighborhood of $x=0$ and is immediately obtained on expanding the denominator in the integrand about $x=0$. When this is substituted in (2.30), one has for the FESR

$$\int_{t_0}^R D_2(s,t)dt \simeq\frac{D}{b}\frac{[\alpha(R)]^{\alpha_c(s)+1}}{\Gamma(\alpha_c(s)+2)[\ln\alpha(R)]^{3/2}}\left[1+O\left(\frac{1}{\ln R}\right)\right]. \quad (2.31)$$

To show that (2.29) satisfies the sum rule (2.31), we again observe that the asymptotic contribution comes from the neighborhood of $x=0$, so that in the binomial coefficient

$$\binom{N(R)+\alpha_c(s)+1-x}{\alpha_c(s)+1-x}=\frac{\Gamma(N(R)+\alpha_c(s)+2-x)}{\Gamma(N(R)+1)\Gamma(\alpha_c(s)+2-x)}$$

we may use the expansions

$$\frac{\Gamma(N(R)+\alpha_c(s)+2-x)}{\Gamma(N(R)+1)}=[N(R)]^{\alpha_c(s)+1-x}\times\left\{1+\frac{[\alpha_c(s)+1-x][\alpha_c(s)+2-x]}{2N(R)}+\cdots\right\},$$

$$\frac{1}{\Gamma(\alpha_c(s)+2-x)}=\frac{[\alpha_c(s)+2]^x}{\Gamma(\alpha_c(s)+2)}\times\left\{1-\frac{x(x+1)}{2[\alpha_c(s)+2]}+\cdots\right\},$$

valid for large R and small x, respectively. One then obtains for the right-hand side of (2.29)

$$\frac{2D}{b\sqrt{\pi}}\frac{[N(R)]^{\alpha_c(s)+1}}{\Gamma(\alpha_c(s)+2)}\int_0^\infty [N(R)]^{-x}e^{-x}x^{1/2}[1+O(x)]dx =\frac{2D}{b\sqrt{\pi}}\frac{[N(R)]^{\alpha_c(s)+1}}{\Gamma(\alpha_c(s)+2)}\Gamma(\tfrac{3}{2})[\ln N(R)]^{-3/2}\times\left[1+O\left(\frac{1}{\ln R}\right)\right].$$

Replacing $N(R)$ by $\alpha(R)$, we see that the FESR (2.31) is satisfied.

The superconvergence condition (2.11) for $n=0$ gives

$$\lim_{R\to\infty}\int_0^\infty \binom{\alpha(R)+\alpha_c(s)+1-x}{\alpha_c+1-x} e^{-x}x^{1/2}dx=0, \quad \alpha_c(s)+1<0 \quad (2.32)$$

which is, of course, identically satisfied.

C. Discussion

(a) We first note that the sum rules on D_1 and D_2 add to give the general FESR:

$$\int_0^R \mathrm{Im}A\,dt\simeq\frac{\pi D[\alpha_c(R)]^{\alpha(s)+1}}{a\Gamma(\alpha(s)+1)}+\frac{\pi D}{b}\frac{[\alpha(R)]^{\alpha_c(s)+1}}{\Gamma(\alpha_c(s)+2)[\ln\alpha(R)]^{3/2}}. \quad (2.33)$$

It is thus clear that this FESR cannot, by itself, imply any type of duality; it is satisfied by all models exhibiting the required asymptotic behavior. What the cross-duality model specifically excludes is the approximation of the left-hand side of (2.33) entirely by direct-channel resonances.

One may further observe that complete resonance saturation cannot, alone, lead to direct duality of resonances and Regge poles. To see this, consider the resonance-saturated FESR

$$\int_0^R \mathrm{Im}A_{\mathrm{res}}dt \simeq \sum_i \beta_i(s)R^{\alpha_i(s)+1} + \sum_i \beta_i^c(s)\frac{R^{\alpha_i^c(s)+1}}{(\ln R)^{\lambda_i}}, \quad (2.34)$$

where the second sum on the right-hand side is over the leading cut contributions. For simplicity take the pole trajectories to be parallel. Since the branch-point trajectories are of smaller slope, they will overtake the pole trajectories at some $s=s_1$. We may then write for (2.34)

$$\int_0^R \mathrm{Im}A_{\mathrm{res}}dt \simeq \sum_i \beta_i(s)R^{\alpha_i(s)+1}, \quad s\gg s_1 \quad (2.35)$$

and

$$\int_0^R \mathrm{Im}A_{\mathrm{res}}dt \simeq \sum_i \frac{\beta_i^c(s)R^{\alpha_i^c(s)+1}}{(\ln R)^{\lambda_i}}, \quad s\ll s_1. \quad (2.36)$$

When (2.36) holds, resonance–Regge-pole duality cannot be maintained. This argument makes plausible an expectation that the direct- and cross-duality constructions are alternatively good approximations to the amplitude $A(s,t)$ in different regions of the st plane.

(b) It is evident that there is no question of double counting, since the resonant and nonresonant parts of the amplitude are separately superconvergent. Further, the sum rule

$$\int_0^R \mathrm{Im}A_{\mathrm{res}}dt=0,$$

ascribed by Schmid[9] to generalized interference models, does not hold in this case, and there is, consequently, no immediate conflict with empirical results.

(c) Since different parameters appear on the left- and right-hand sides of the sum rules (2.23) and (2.31), one does not have bootstrap conditions from these FESR. Whether or not effective bootstrap restrictions obtain on supplementing the model with assumptions on the relation of cuts to Regge poles and direct-channel resonances is a matter that requires further investigation.

(d) Equation (2.15), with $F(s,x)$ given by (2.13), may be considered as a continuous-parameter sum rule of the exponential type.[10] For small x it gives

$$\int_{t_0}^{\infty} e^{-xt}D_1(s,t)dt \simeq \frac{D}{a}\left(\frac{x}{a}\right)^{-\alpha(s)-1}, \quad x \text{ small.} \quad (2.37)$$

When the high-energy contribution is approximated by (2.8), one obtains

$$\int_{t_0}^{R} e^{-xt}D_1(s,t)dt + \frac{Da^{\alpha(s)}}{\Gamma(\alpha(s)+1)}\int_R^{\infty} e^{-xt}t^{\alpha(s)}dt \simeq \frac{D}{a}\left(\frac{x}{a}\right)^{-\alpha(s)-1}.$$

The second integral on the left-hand side gives the incomplete gamma function γ, plus a contribution that cancels with the term on the right-hand side leaving

$$\int_{t_0}^{R} e^{-xt}D_1(s,t)dt \simeq \frac{D(x/a)^{-\alpha(s)-1}}{a\Gamma(\alpha(s)+1)}\gamma(\alpha(s)+1, xR), \quad x\to+0,\ R\to\infty. \quad (2.38)$$

Writing $xR=\eta$, where η is an arbitrary positive parameter, we obtain for large R the FESR

$$\int_0^R e^{-\eta t/R}D_1(s,t)dt \simeq \frac{D}{a}\gamma^*(\alpha(s)+1,\eta)[\alpha_c(R)]^{\alpha(s)+1}, \quad (2.39)$$

where $\gamma^*(a,x)$ is the single-valued analytic function related to the incomplete gamma function.[8] For $\eta=0$, $\gamma^*(\alpha+1,0)=1/\Gamma(\alpha+2)$ and (2.39) reduces to the sum rule (2.23).

III. EXCHANGE DEGENERACY

In this section we give a general discussion, i.e., without using the specific form of the amplitude given in (2.1)–(2.3), of some consequences of cross-duality in the presence of an exotic channel. We consider a spinless process in which the u channel is exotic. Since there are no direct-channel resonances in u, the discontinuity of the amplitude $M^{(u)}(s,t,u)$ consists of a term corresponding to D_1, which we denote by $\Delta_u M^{(u)}(s,t,u)$, while D_2 is absent. If the asymptotic behavior of the amplitude satisfies cross-duality, i.e., if Regge-pole exchange dualizes the direct-channel unitarity cut contribution while Regge-cut exchange is dual to the direct-channel resonances, then one has for large u at fixed t

$$\Delta_u M^{(u)} \simeq \sum_i \sigma_i\beta_i^{(u)}(t)\sin\pi\alpha_i(t)u^{\alpha_i(t)}, \quad (3.1)$$

$$0 \simeq \sum_i \sigma_i^c\gamma_i^{(u)}(t)\sin\pi\alpha_i^c(t)u^{\alpha_i^c(t)}, \quad (3.2)$$

[9] C. Schmid, CERN Report No. TH. 1128, 1969 (unpublished).

[10] M. O. Taha, Nucl. Phys. B10, 656 (1969).

where σ_i and $\sigma_i{}^c$ are the signatures of the contributing poles and cuts; the usual gamma and logarithmic factors are absorbed in the definition of the residue functions $\beta_i{}^{(u)}$ and $\gamma_i{}^{(u)}$.

From Eq. (3.2) we are immediately led to *exchange-degeneracy relations among the trajectories of the leading Regge branch points*. Within this scheme one therefore expects the leading branch-point trajectories to occur in degenerate pairs of opposite signature. Thus if the dominant contributions to the right-hand side of (3.2) come from only two branch points $\alpha_{12}{}^c(t)$ and $\alpha_{13}{}^c(i)$, then the equation reads

$$\sigma_{12}{}^c\gamma_{12}{}^{(u)}(t)\,\sin\pi\alpha_{12}{}^c(t)u^{\alpha_{12}{}^c(t)} +\sigma_{13}{}^c\gamma_{13}{}^{(u)}(t)\,\sin\pi\alpha_{13}{}^c(t)u^{\alpha_{13}{}^c(t)}\simeq 0\,,$$

so that one has the degeneracy relations

$$\alpha_{12}{}^c(t)=\alpha_{13}{}^c(t)\,, \tag{3.3}$$

$$\sigma_{12}{}^c=-\sigma_{13}{}^c\,, \tag{3.4}$$

$$\gamma_{12}{}^{(u)}(t)=\gamma_{13}{}^{(u)}(t)\,, \tag{3.5}$$

where (3.5) is reaction dependent.

Under the general assumption that Regge cuts are generated by the exchange of Regge poles, one further expects such relations to impose constraints on the exchanged Regge poles. This is not difficult to show. Suppose that the branch points $\alpha_{12}{}^c$ and $\alpha_{13}{}^c$ are generated by the exchange of the Regge-pole trajectories α_1, α_2 and α_1, α_3, respectively. The signatures and trajectories are then related by

$$\sigma_{12}{}^c=\sigma_1\sigma_2\,, \tag{3.6}$$

$$\alpha_{12}{}^c(t)=\max_{(t_1,t_2)}\,[\alpha_1(t_1)+\alpha_2(t_2)-1]\big|_{\sqrt{(-t_1)}+\sqrt{(-t_2)}=\sqrt{(-t)}}\,, \tag{3.7}$$

with similar equations for $\sigma_{13}{}^c$ and $\alpha_{13}{}^c$. Equations (3.4) and (3.6) then give

$$\sigma_2=-\sigma_3\,, \tag{3.8}$$

so that the pole trajectories α_2 and α_3 are of opposite signature. To see the constraint on the trajectories themselves, let us write Eq. (3.7) in the form

$$\alpha_{12}{}^c(t)=\max_{(x)}\,[\alpha_1(x)+\alpha_2(Z(x,t))-1]\,, \tag{3.9}$$

where

$$Z(x,t)=-[\sqrt{(-t)}-\sqrt{(-x)}]^2. \tag{3.10}$$

If the point of maximum value is $x=x_m(t)$, Eq. (3.9) gives

$$\alpha_{12}{}^c(t)=\alpha_1(x_m(t))+\alpha_2(Z(x_m,t))-1\,. \tag{3.11}$$

When this is substituted in (3.3), one obtains

$$\alpha_2(Z(x_m,t))=\alpha_3(Z(x_m,t))\,. \tag{3.12}$$

But $Z(x_m(t),t)$ is a general variable so that (3.12) implies

$$\alpha_2(t)=\alpha_3(t)\,. \tag{3.13}$$

From (3.8) and (3.13) it is seen that we have obtained the *exchange degeneracy of the Regge-pole trajectories* α_2 *and* α_3 collaborating with the trajectory α_1 to produce the exchange-degenerate branch points $\alpha_{12}{}^c$ and $\alpha_{13}{}^c$. It should be noted that we have not used the assumption of the linearity of the trajectories, in which case (3.13) immediately follows from (3.3). We also note that this discussion applies to the case when the branch points are generated by two poles only, i.e., for branch points $\alpha_{11}{}^c$, $\alpha_{12}{}^c$ generated by α_1-α_1 and α_1-α_2 exchanges, respectively. This is clearly seen on replacing the index 3 by 1 in the above argument.

The emergence of Regge-pole exchange degeneracy, in the absence of exotic resonances, from a scheme in which resonances are dual to Regge cuts and not to Regge poles is a most interesting feature for at least two reasons:

(i) Direct duality of the Veneziano type has been closely associated with Regge-pole exchange degeneracy and it is generally believed that such a connection is absent in the case of interference-type models.

(ii) Exchange degeneracy of Regge poles is rather well established experimentally and it is therefore not satisfactory merely to replace it by the degeneracy of branch-point trajectories about which almost nothing is known.

Now that exchange degeneracy of branch-point trajectories is suggested by cross duality in the presence of exotic channels, we observe that such degeneracy should in fact have been expected quite generally and *independently of cross duality*. For, the converse of the above argument is also valid, i.e., exchange degeneracy of pole trajectories imposes the same on the branch points which they generate.[11] It is clear that (3.8) and (3.13) imply (3.3) and (3.4). Further, if $\beta_2{}^{(u)}(t)=\beta_3{}^{(u)}(t)$, then it follows—see (4.3) below—that $\gamma_{12}{}^{(u)}(t)=\gamma_{13}{}^{(u)}(t)$; hence, exchange degeneracy among branch-point trajectories without assuming cross duality. This observation that the experimental degeneracy of pole trajectories impose branch-point degeneracy may be used to reduce the number of unknown parameters in phenomenological fits with cuts.

IV. POMERANCHUK SINGULARITY

So far we have not mentioned the Pomeranchuk singularity in this scheme. An interesting possibility, however, arises from our discussion on branch-point degeneracy in Sec. III, which we now present.

Suppose that the trajectory α_1 which collaborates with α_2 and α_3 to generate the leading branch points is the Pomeranchukon-pole trajectory α_P. Let us further

[11] V. Barger and R. J. N. Phillips, Phys. Letters **29B**, 676 (1969), make a related observation on the "exchange degeneracy" of the leading Pomeranchukon-induced cuts, in the sense that they are continua of exchange-degenerate poles. We are specifically concerned with the exchange degeneracy of the branch-point trajectories.

assume that α_P, α_2, and α_3 are the only dominant contributions to the right-hand side of Eq. (3.1), on the left-hand side of which we have the discontinuity over the direct-channel unitarity cut, i.e., the imaginary part of the "background." Equation (3.1) may then be written

$$\Delta_u M^{(u)} \simeq \beta_P{}^{(u)}(t) \sin\pi\alpha_P(t) u^{\alpha_P(t)} + \sigma[\beta_2{}^{(u)}(t) - \beta_3{}^{(u)}(t)] \sin\pi\alpha(t) u^{\alpha(t)}, \quad (4.1)$$

where $\sigma_2 = -\sigma_3 = \sigma$ and $\alpha_2 = \alpha_3 = \alpha$ from (3.8) and (3.13).

Now in Sec. III we did not deduce the residue condition $\beta_2{}^{(u)} = \beta_3{}^{(u)}$ from the degeneracy of branch-point trajectories. We shall now argue that this relation may also obtain. The contribution to the scattering amplitude of a cut generated by two Regge poles is generally given by a double-integral transform over the pole contributions with a model-dependent kernel.[12–16] The equation relating the residues γ_{12}, β_1, and β_2 is obtained by keeping only the asymptotically dominant contribution of the integral. In the case of the Glauber rescattering formulation,[15] for example, the contribution $f_{12}{}^{(u)}(u,t)$ of the cut generated by α_1 and α_2 is given by

$$f_{12}{}^{(u)}(u,t) = \int \beta_1{}^{(u)}(-|\mathbf{q}_1|^2) \beta_2{}^{(u)}(-|\mathbf{q}-\mathbf{q}_1|^2) \times u^{\alpha_1(-|\mathbf{q}_1|^2)+\alpha_2(-|\mathbf{q}-\mathbf{q}_1|^2)-1} d^2q_1, \quad (4.2)$$

where $\mathbf{q}$ and $\mathbf{q}_1$ are two-dimensional momentum vectors with $|\mathbf{q}|^2 = -t$. The dominant contribution to the integral comes from the region where $|\mathbf{q}_1| + |\mathbf{q}-\mathbf{q}_1| = |\mathbf{q}|$, so that from (4.2) one obtains for the residue $\gamma_{12}{}^{(u)}$,

$$\gamma_{12}{}^{(u)}(t) = \int \beta_1{}^{(u)}(-|\mathbf{q}_1|^2) \times \beta_2{}^{(u)}(-(|\mathbf{q}| - |\mathbf{q}_1|)^2) d^2q_1. \quad (4.3)$$

Equation (3.5) then implies

$$\int \beta_1{}^{(u)}(-|\mathbf{q}_1|^2)[\beta_2{}^{(u)}(-(|\mathbf{q}| - |\mathbf{q}_1|)^2) - \beta_3{}^{(u)}(-(|\mathbf{q}| - |\mathbf{q}_1|)^2)] d^2q_1 = 0. \quad (4.4)$$

This equation will then hold for all values of $|\mathbf{q}|$, and it is not therefore unreasonable to expect it to be simply satisfied by

$$\beta_2{}^{(u)}(t) = \beta_3{}^{(u)}(t). \quad (4.5)$$

We shall, in any case, take (4.5) to be the solution to (4.4).

When Eq. (4.5) is substituted into (4.1), one obtains

$$\Delta_u M^{(u)} \simeq \beta_P{}^{(u)}(t) \sin\pi\alpha_P(t) u^{\alpha_P(t)}, \quad (4.6)$$

i.e., we are left with the contribution of the Pomeranchuk pole to the background term. This discussion may be summarized as follows:

(a) In an exotic channel, cross-duality gives exchange degeneracy among branch-point trajectories of opposite signature. If a degenerate pair of branch points is generated by exchange of the poles α_1, α_2 and α_1, α_3, then α_2 and α_3 are of opposite signature and are exchange degenerate.

(b) If α_1, α_2, and α_3 are the dominant poles in Eq. (3.1), then the contributions of α_2 and α_3 cancel leaving only that of α_1.

(c) It is tempting to identify α_1 with the Pomeranchuk pole as is often done in phenomenological generation of cuts. We then have Eq. (4.6), which amounts to the part of the Freund-Harari[6] conjecture that associates the background contribution to FESR with that of the Pomeranchuk trajectory. In our case this holds in the presence of an exotic channel.

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[12] The exact form of this double-integral transform depends on the model used for the generation of the cut. We refer to the absorptive, multiperipheral and rescattering models in Refs. 13–15, respectively. General reviews are given in Ref. 16.

[13] J. D. Jackson, in *High Energy Physics*, edited by C. DeWitt and M. Jacob (Gordon and Breach, New York, 1966).

[14] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962); D. Amati, M. Stanghellini, and S. Fubini, *ibid.* **26**, 6 (1962).

[15] R. J. Glauber, *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland, Amsterdam, 1967).

[16] G. E. Mite, Rev. Mod. Phys. **41**, 669 (1969); J. D. Jackson, *ibid.* **42**, 12 (1970); W. Drechsler, ICTP, Trieste, Report No. IC/69/39, 1969 (unpublished).