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Asymptotics of Partial Waves in Regge Theory and Constraints on Partial-Wave Subtraction Constants*

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In a Regge theory of spinless elastic scattering, subtraction constants in physical partial waves are not always free parameters. In a partial wave where there is no Castillejo-Dalitz-Dyson (CDD) pole, the subtraction constant $A_l(s_0)$, $s_0<4m^2$, is uniquely determined when the left-cut term and the elasticity η_l are specified. If there are CDD poles, all at finite points, then the subtraction constant is again fixed uniquely, but its value depends on the CDD parameters. If there is a CDD pole at infinity, the subtraction constant is unconstrained. These results are proved assuming that high-energy behavior is determined by a moving Pomeranchuk pole, with or without associated branch points. The analysis, although model independent, has implications for a dynamical model based on Reggeon exchange—namely, a model in which the input to the inelastic N/D equation (left- and right-cut parts) is constructed from crossed-channel Regge terms. In such a model the N/D equation is a regular Fredholm equation without high-energy truncation. In general, the phase shift obtained as output from the N/D equation has the high-energy behavior required by Regge theory if, and only if, the subtraction-constant constraint is satisfied. It is argued that new calculations are needed to test the Reggeon-exchange model. Earlier calculations in the Chew-Jones scheme are not conclusive for the formulation given here.

I. INTRODUCTION

IN this paper we are interested in certain relations between the low- and the high-energy behavior of physical partial waves. For simplicity, we treat elastic scattering of spinless mesons. The high-energy scattering is determined, we suppose, by a moving Pomeranchuk pole with $\alpha(0)=1$. There may be branch points which accompany the pole. The high-energy behavior of partial waves may be deduced in such a Regge theory, under reasonable technical assumptions about the behavior of the amplitude at large momentum transfers.[1] One finds the following asymptotes[2] for the real phase shift δ and the elasticity η:

$$\delta(s)/\pi \sim n-\tfrac{1}{2}\gamma \ln^{-2}s\,, \tag{1.1}$$

$$\eta(s)\sim 1-2\gamma \ln^{-1}s\,,\quad s\to\infty\,. \tag{1.2}$$

Here n is an integer, and γ is a positive constant.

We shall use the N/D method to show that Eqs. (1.1) and (1.2) have direct implications for the low-energy behavior of partial waves. We emphasize that the results will be true in any Regge-Pomeranchuk theory, whether or not the N/D method is used in construction of the theory.

The high-energy behavior of the force function $B(s)$ of the N/D equation may be derived from Eqs. (1.1) and (1.2). In fact, $B(s)$ may be expressed in terms of η and δ as follows:

$$B(s)=B_L(s)+\frac{s-s_0}{2\pi}P\int_{s_I}^{\infty}\frac{1-\eta(s')}{\rho(s')(s'-s_0)(s'-s)}ds' \tag{1.3}$$

$$=\frac{\eta(s)\sin2\delta(s)}{2\rho(s)}-a-\frac{s-s_0}{\pi}P\int_4^{\infty}\frac{\eta(s')\sin^2\delta(s')ds'}{(s'-s_0)(s'-s)}\,, \tag{1.4}$$

where s_I is the threshold for inelastic processes. The contribution of the left-hand cut is $B_L(s)$, and $a=A_l(s_0)$ is the value of the partial-wave amplitude at $s_0<4$. The N/D equation employed is that of Refs. 1 and 3, in which $B_L(s)$ and $\eta(s)$ are regarded as given data. With appropriate bounds on the derivatives of δ and η, it follows from (1.1), (1.2), and (1.4) that B has the asymptotic form

$$B(s)=B(\infty)-\tfrac{1}{2}\pi\gamma \ln^{-2}s+O(\ln^{-3}s)\,. \tag{1.5}$$

The purpose of this analysis is to determine the value of the constant a, supposing that functions B and η, which satisfy (1.2) and (1.5), are given. We find that a is uniquely fixed by a linear equation [Eq. (3.11)] when there is no Castillejo-Dalitz-Dyson (CDD) pole. When there are CDD poles at finite points only, we

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[1] R. L. Warnock, in *Lectures in Theoretical High-Energy Physics*, edited by H. H. Aly (Interscience, New York, 1968), Chap. 10.

[2] We use the symbols $\sim$, O, o in the standard way. When x tends to some limit, $f(x)\sim g(x)$ means that $f(x)/g(x)\to 1$, $f(x)=O(g(x))$ means that $|f(x)|\le M|g(x)|$, and $f(x)=o(g(x))$ means that $f(x)/g(x)\to 0$.

[3] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

again find a linear equation for a [Eq. (3.35)], but the equation depends on the residues and locations of the CDD poles. Finally, if there is a CDD pole at infinity, the value of a is undetermined.

These results imply that the total number of free parameters (CDD parameters and subtraction constants) is always twice the number of CDD poles. This is a new result of some academic interest. There is also a more practical aspect of our analysis, however, since the determination of a is an issue in dynamical models, as well as in a "correct Regge theory." In the simplest model to which our analysis applies, the left-cut term $B_L(s)$ would be constructed from the left-hand singularities of a small number of crossed-channel Regge terms, including the Pomeranchuk term, projected onto s-channel partial waves. At high energy, the elasticity η would also be obtained from the partial-wave projection of the Regge terms. In such a model, B has the behavior (1.5), and there are also certain bounds on the derivatives B' and B''. The bounds imply that the N/D equation is a regular Fredholm equation in L^2, so that we obtain a convergent Reggeon exchange model without high-energy truncations. [We note in passing that our equation[1,3] is more convenient in this respect than the N/D equation of Chew and Mandelstam,[4] in which inelasticity is parametrized by a function $R(s)$, the quotient of total and elastic partial-wave cross sections. The latter equation is marginally singular[5] under conditions (1.1) and (1.2).] The value of the constant $B(\infty)$ given by the model is not likely to be correct, however, since $B(\infty)$ represents short-range forces depending on effects more complicated than Reggeon exchange. We should then regard $B(\infty)$ as a parameter which is yet to be determined. Our constraint equations [(3.11) and (3.35)] actually determine the sum $B(\infty)+a$, when there is no CDD pole at infinity.

The Reggeon-exchange model described is, of course, similar to the old N/D models based on single-particle exchange. It is quite different in detail, however, since it is convergent without cutoffs, and it includes Pomeranchukon exchange. Could it possibly overcome some of the defects of the old models? It is difficult to answer this question from information currently available in the literature. Some new calculations will be required. Among previous attempts, the closest to our proposal is that of Chew and Jones.[6] In a scheme called the "new strip approximation," Chew and Jones suggested unitarizing Regge terms by means of an N/D equation referring to a finite interval of energy. The Chew-Jones procedure for π-π scattering was carried out by Collins and Teplitz.[7,8] They encountered several difficulties, a principal one being that the $I=0$ (Pomeranchuk) trajectory in the crossed channels produced an overwhelming repulsive force in the direct channel. The attractive effect of the $I=1$ exchange was insufficient to overcome this repulsion, so that no ρ resonance was produced in the $I=1$ direct channel. It is not certain that the situation will be the same in the model we have described. When account is taken of our constraint, or of CDD poles, the outcome might be quite different. Even the difference between the truncated N/D equation of Chew and Jones and our untruncated equation could be quite significant. In fact, high-energy contributions are demonstrably important in our equation. The norm of the kernel exists only by virtue of logarithmic factors, and if it were not for a delicate cancellation between the left- and right-cut parts of the force function at high energy, the equation would be singular, and the Regge behavior (1.1) and (1.2) would probably not be obtained.

In any case, we believe that CDD poles will be needed in a realistic model with a manageable number of channels. In π-π scattering, for instance, a CDD pole in the ρ channel is indicated by the outcome of several attempts to make bootstrap theories of the ρ in which all channels contain two pseudoscalar mesons.[9–12] In such models the width of the ρ is habitually too large by a factor of 2 or more, whether the whole pseudoscalar octet with physical masses is included,[12] or just the π mesons. Additional channels, such as P-V and V-V (P=pseudoscalar meson, V=vector meson) seem capable of reducing the width.[9,13] It is known that a CDD pole can reduce the width, and that coupled channels can induce CDD poles in the single-channel description.[14] Thus, the attitude toward CDD poles should be positive rather than negative as it has been in the past. In appropriate circumstances, they offer a remarkably economical way of accounting for many-channel effects in a one-channel or few-channel formalism. At the present stage of theory, we shall probably have to make a semiempirical determination of the CDD parameters. In principle, however, these parameters may be sharply restricted by the powerful condition that CDD poles not generate Kronecker δ singularities in the l plane.[15,16] This is certainly a subtle

[4] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[5] D. Atkinson and A. P. Contogouris, J. Math. Phys. **9**, 1489 (1968).

[6] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

[7] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

[8] P. D. B. Collins, Phys. Rev. **142**, 1163 (1966).

[9] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[10] Chan Hong-Mo, P. C. Decelles, and J. E. Paton, Nuovo Cimento **33**, 70 (1964).

[11] J. Fulco, G. Shaw, and D. Wong, Phys. Rev. **137**, B1242 (1965).

[12] The effects of mass difference in the pseudoscalar octet on the vector-meson bootstrap were studied by S. K. Gupta, Ph.D. Dissertation, Illinois Institute of Technology, 1967 (unpublished).

[13] Chan Hong-Mo and C. Wilkin, Ann. Phys. (N. Y.) **39**, 300 (1966).

[14] D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966); J. B. Hartle and C. E. Jones, *ibid.* **38**, 348 (1966).

[15] S. Mandelstam, Phys. Rev. **137**, B949 (1965).

[16] In the situation of small coupling constants and small CDD pole residues, one necessarily gets Kronecker δ's from CDD poles; see D. Atkinson and R. L. Warnock, Phys. Rev. **188**, 2098 (1969). At larger couplings there could be "inelastic" CDD poles with parameters depending on l in such a way that Kronecker δ's are not implied; see Hartle and Jones (Ref. 14).

condition, and suitable means to enforce it have not been invented. At a more practical level, we can at least require that the CDD poles do not induce ghost zeros of the D function.

The matter of ghosts, by the way, is another area in which Reggeon-exchange models should be reexamined. It is not clear to us that adequate checks for ghosts were carried out in earlier work. Sometimes a CDD pole may be needed to *avoid* a ghost.

An apparent exception to the rule that elastic models give too large a value for the ρ width is to be found in the work of Collins and Johnson.[17] Collins[8] argued that the Regge terms alone could not make a suitable N/D force function. He suggested that a better force function could be obtained by including a part of the elastic double-spectral function. Collins and Johnson calculated this double-spectral function in the strip approximation by the Mandelstam iteration, beginning with Regge terms as the zeroth iterate. The results of the Collins-Johnson calculation appeared to be relatively satisfactory. They claimed to have a bootstrap solution with reasonable P and ρ trajectories, good values of the ρ width and s-wave phase shifts, etc. Recently, Webber[18] has attempted to reproduce the Collins-Johnson calculation. Webber uses only the Mandelstam iteration in the strip approximation, rather than the combination of Mandelstam iteration and N/D equations employed by Collins and Johnson. Webber's technique is better in principle (if more difficult numerically) since, if he finds a solution, it is bound to be crossing symmetric and free of ghosts. In the Collins-Johnson approach, exact crossing symmetry is not ensured, and there could be ghosts. Webber is unable to find a solution of the type claimed by Collins and Johnson. In fact, in his nearest approach to a solution, the ρ width is again much too large, and there are other unsatisfactory features. It might be that the narrow ρ width in the Collins-Johnson calculation is an artifact arising from a ghost.[18,19]

It is probably true, as Collins argues, that Regge terms alone will not make an entirely adequate input for the N/D equation. There must be some violation of crossing symmetry, at least, in such a scheme. Perhaps the Mandelstam iteration offers the best hope of improving the model, but it should be a Mandelstam iteration with CDD poles, similar to that formulated by Atkinson and Warnock.[16] This implies a rather complicated program, but it does offer a way of phrasing the requirement of no Kronecker δ's in the l plane. This requirement appears as the vanishing of single spectral functions, which are expressed as the difference between N/D absorptive parts and the corresponding partial-wave absorptive parts obtained from the double-spectral terms of the total amplitude.

We have noted that Regge branch points (at least those of the class considered in Ref. 1) do not affect the results of this paper. The solution of the N/D equation in the Reggeon exchange model almost certainly entails Regge branch points, however. The effects of branch points appear first in terms of order $\ln^{-3}s$ in the phase shift of the direct channel. Since our asymptotic analysis is concerned only with the leading terms of order $\ln^{-2}s$, we do not meet the branch-point terms face to face. The N/D equation with Reggeon input could be regarded as a means of generating Regge branch points, analogous to but different from the eikonal or multiperipheral models. We have not yet obtained any clear idea of the character of the branch points, but it may be possible to do so by carrying the asymptotic analysis to higher terms.

The background for this paper is to be found in Ref. 1. In Sec. II we define notation and summarize results of Ref. 1 which are needed in the following. Section III contains the proof of the constraint equations. The calculational method of imposing the constraints is explained at the end of Sec. III. A useful theorem on asymptotic behavior of principal-value integrals is proved in Appendix A.

II. PRELIMINARIES

Let $A_l(z)$ be the partial-wave amplitude for elastic scattering of spinless mesons of unit mass. Henceforth, the subscript l is dropped. We assume that A is analytic in some neighborhood Ω of the physical cut $4\leq s<\infty$,[20] where s is the square of the energy in the c.m. frame. This neighborhood can be an arbitrarily thin sliver, with a width tending to zero at $s=+\infty$. The physical amplitude is the boundary value

$$A_+(s)=A(s+i0)=[\eta(s)e^{2i\delta(s)}-1]/2i\rho(s)\,,\quad s\geq 4 \quad (2.1)$$

$$\rho(s)=\left(\frac{s-4}{s}\right)^{1/2}.$$

The elasticity η and the real phase shift δ are assumed to be Hölder-continuous on any finite interval, and η is assumed to have no zeros; therefore, $0<\eta\leq 1$.[21] By Cauchy's integral theorem, the amplitude may be represented in Ω as follows:

$$A(z)=a+(z-s_0)\left[A_U(z)+\frac{1}{\pi}\int_4^\infty \frac{\operatorname{Im}A_+(s)ds}{(s-s_0)(s-z)}\right]. \quad (2.2)$$

Here $s_0<4$ is a real point at which $A(s_0)=a$ is defined.

[17] P. D. B. Collins and R. C. Johnson, Phys. Rev. **177**, 2472 (1969); **182**, 1755 (1969); R. C. Johnson and P. D. B. Collins, *ibid.* **185**, 2020 (1969).

[18] B. Webber, LRL Report No. UCRL-20134 (unpublished).

[19] This explanation is advanced tentatively by G. F. Chew (private communication).

[20] Such analyticity has been proved in Lehmann-Symanzik-Zimmermann field theory by J. Bros, H. Epstein, and V. Glaser, Nuovo Cimento **31**, 1265 (1964).

[21] If η has a zero, the N/D equation becomes a so-called "Fredholm equation of the third kind," which we intend to discuss elsewhere. Zeros of η do not, in fact, spoil the results of the present paper.

The "unphysical" term A_U consists of the stable-particle poles (if there are any) plus an integral around some path in Ω which closes the Cauchy contour.

For application of the N/D method it is not necessary to have a partial-wave dispersion relation in the customary sense, where A_U is an integral over a left-hand cut, and the contribution of the Cauchy contour at infinity is zero. The representation (2.2) in Ω is sufficient as far as analyticity is concerned. It is important to require that the phase shift be bounded, however. Otherwise, the standard N/D theory does not go through, since the function $\mathfrak{D}(z)$ (notation of Ref. 1) acquires an essential singularity. The phase shift appears to be bounded, fortunately, in all current models of high-energy behavior.

We define the force function $B(s)$ as

$$B(s)=(s-s_0)\times\left[A_U(s)+\frac{1}{2\pi}P\int_{s_I}^{\infty}\frac{[1-\eta(s')]ds'}{\rho(s')(s'-s_0)(s'-s)}\right], \quad (2.3)$$

where s_I is the threshold for inelastic processes; $s_I=16$ in π-π scattering. Any amplitude of the class we have described may be represented as[1]

$$A(z)=N(z)/D(z), \quad D_+(s)=|D_+(s)|e^{-i\delta(s)}, \quad (2.4)$$

$$D(z)=1+(z-s_0)\times\left[c-\sum_{i=1}^{n}\frac{c_i}{s_i-z}-\frac{1}{\pi}\int_4^{\infty}\frac{\rho(s)\varphi(s)ds}{s-z}\right], \quad (2.5)$$

$$\varphi(s)=-\mathrm{Im}D_+(s)/[\rho(s)(s-s_0)].$$

The numerator function $N(z)$ is analytic in a neighborhood of the line segment $4\leq s\leq s_I$. In general, the D function has a finite number n of CDD poles at the distinct finite energies $s_i>4$, as well as a CDD pole at infinity [the term cz in (2.5)]. The real constants c, c_i, and s_i depend on the particular amplitude A, and they are not always uniquely specified for a given A. A *necessary* condition on φ, which follows without further assumptions on asymptotic behavior, etc., is the following N/D integral equation, derived in Ref. 1:

$$\eta(s)\varphi(s)=\frac{a+B(s)}{s-s_0}+cB(s)+\Lambda(\infty)+\sum_i c_i\frac{B(s_i)-B(s)}{s_i-s}+\frac{1}{\pi}\int_4^{\infty}\frac{B(s)-\alpha(s,s')B(s')}{s-s'}\rho(s')\varphi(s')ds'. \quad (2.6)$$

The constant $\Lambda(\infty)$ is defined in Ref. 1, and $\alpha(s,s')\equiv 1$ unless the following integral diverges, in which case $\alpha(s,s')=(s-s_0)/(s'-s_0)$:

$$\frac{1}{\pi}\int_4^{\infty}\frac{\varphi(s)B(s)\rho(s)ds}{s-z}. \quad (2.7)$$

Conversely, if φ is a Hölder-continuous solution of (2.6) such that D as defined in (2.5) exists and is free of zeros in the cut plane, then we can construct from φ a solution of (2.1) and (2.2) by means of the following formula:

$$A(z)=B(z)+\frac{z-s_0}{D(z)}\left[\frac{a}{z-s_0}+\Lambda(\infty)+\sum_i c_i\frac{B(s_i)}{s_i-z}+\frac{1}{\pi}\int_4^{\infty}\frac{\alpha(z,s)\rho(s)\varphi(s)B(s)ds}{s-z}\right]. \quad (2.8)$$

Notice that with the assumptions made up to now, we do not know that (2.6) is a regular Fredholm equation (i.e., that it has a completely continuous kernel in some Banach space, with the inhomogeneous term in that space). We do know that (2.6) has a Hölder-continuous solution, however, for any amplitude A and some choice of the constants $\Lambda(\infty)$, c, c_i, and s_i.

We now consider the implications of the Regge model for the asymptotic behavior of $B(s)$. Equation (1.4), which follows directly from (2.1)–(2.3), shows that the asymptotic behavior of B may be deduced from that of η and δ. To derive the behavior of η and δ, we consider the contribution of the Pomeranchuk pole to the amplitude $A(s,t)$ at large s, viz.,

$$\mathcal{B}(t)\frac{1+e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}s^{\alpha(t)}+\bar{\mathcal{B}}(u)\frac{1+e^{-i\pi\alpha(u)}}{\sin\pi\alpha(u)}s^{\alpha(u)}. \quad (2.9)$$

The partial-wave amplitude is

$$A_l(s)=\frac{1}{s-4}\int_{4-s}^{0}A(s,t)P_l\left(1+\frac{2t}{s-4}\right)dt. \quad (2.10)$$

The region of integration expands as s increases, so the contribution of a Regge-pole term cannot be ascertained without an assumption on the behavior of trajectories and residues at large negative t. Even if we make such an assumption, there is still the question of uniformity with respect to t of the approximation of the amplitude by Regge poles at a given s. We sweep these questions under the rug by simply assuming that the leading contribution to the partial wave is obtained by integration over fixed but arbitrary regions of the two momentum transfers: $-T\leq t\leq 0$, $-U\leq u\leq 0$. This restriction to forward and backward peaks seems plausible, since it is suggested by experiment, as well as by our intuitive notions of peripheralism. The problem is then to find the large-s behavior of the following integral:

$$\frac{1}{s-4}\int_{-T}^{0}\mathcal{B}(t)\frac{1+e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}s^{\alpha(t)}P_l\left(1+\frac{2t}{s-4}\right)dt+\frac{(-1)^l}{s-4}\int_{-U}^{0}\bar{\mathcal{B}}(u)\frac{1+e^{-i\pi\alpha(u)}}{\sin\pi\alpha(u)}s^{\alpha(u)}\times P_l\left(1+\frac{2u}{s-4}\right)du. \quad (2.11)$$

Following most authors, we suppose that

$$\alpha'(t)\geq 0\,,\quad -T\leq t\leq 0 \tag{2.12}$$

$$\alpha'(0)>0\,, \tag{2.13}$$

$$\alpha(0)=1\,. \tag{2.14}$$

We suppose also that $\alpha(t)$, $\mathcal{B}(t)$, and $\bar{\mathcal{B}}(t)$ have continuous second derivatives for $-T\leq t\leq 0$. If $\alpha(\tau)=0$, -2, -4, ... for $-T<\tau<0$, we naturally demand that $\mathcal{B}(\tau)=0$, since there can be no singularity of the scattering amplitude in the physical region. Now it is a tedious exercise, spelled out in Ref. 1, to show that (2.11) is asymptotic to the following function:

$$\frac{1}{\alpha'(0)}[\mathcal{B}(0)+(-1)^l\bar{\mathcal{B}}(0)]\left(\frac{\pi}{2\ln^2 s}-\frac{i}{\ln s}\right). \tag{2.15}$$

Secondary trajectories $\alpha_i(t)$ of either signature, which satisfy (2.12) and (2.13), contribute a term to the partial wave which is $O(s^{\alpha_i(0)-1}\ln^{-2}s)$. This does not count in comparison with (2.17), since $\alpha_i(0)<1$ by definition. Also, we estimate that Regge branch points associated with the Pomeranchuk pole give a term which is down by at least a factor $\ln^{-1}s$ in comparison to (2.15); see Ref. 1. From (2.15) we then have

$$\delta_l(s)\sim\pi n_l-\pi\gamma_l/2\ln^2 s\,, \tag{2.16}$$

$$\eta_l(s)\sim 1-2\gamma_l/\ln s\,, \tag{2.17}$$

$$n_l=\text{integer}, \tag{2.18}$$

$$\gamma_l=-\frac{1}{\alpha'(0)}[\mathcal{B}(0)+(-1)^l\bar{\mathcal{B}}(0)]\geq 0\,. \tag{2.19}$$

The non-negative property of γ_l comes from unitarity: $\eta_l\leq 1$. We assume that the leading terms in the derivatives of A_l are obtained correctly from the derivatives of (2.11). One then finds that differentiation of (2.16) and (2.17) leads to valid asymptotic relations. Hence,

$$\frac{d^n}{ds^n}\delta(s)=O(s^{-n}\ln^{-3}s)\,,$$
$$\frac{d^n}{ds^n}\eta(s)=O(s^{-n}\ln^{-2}s)\,, \tag{2.20}$$
$$n=1,2,3\,.$$

We can now employ (2.16), (2.17), and (2.20) with formula (1.4) to find the high-energy behavior of B. For that we need the following theorem on asymptotic behavior of principal-value integrals, which is proved in Appendix A.

Theorem A: Let

$$g(s)=P\int_{s_0}^{\infty}\frac{f(t)dt}{s-t}\,,\quad s_0>1 \tag{2.21}$$

where f is a differentiable function such that $f(s)=O(s^{-\alpha}\ln^{-\beta}s)$, $f'(s)=O(s^{-1-\alpha}\ln^{-\beta}s)$, $s\geq s_0$. Then for $s>s_0$, we have

$$\begin{aligned} g(s)&=\frac{1}{s}\int_{s_0}^{s}f(t)dt+O(s^{-1}\ln^{-\beta}s)\,, && \alpha=1\,,\ \beta\geq 0\\ g(s)&=O(\ln^{1-\beta}s)\,, && \alpha=0\,,\ \beta>1\,. \end{aligned} \tag{2.22}$$

With the help of this result, one proves that the limit $B(\infty)$ exists, and that[1]

$$B(s)=B(\infty)-\pi\gamma/2\ln^2 s+O(\ln^{-3}s)\,. \tag{2.23}$$

Furthermore, B' and B'' exist at large s (say, $s>S$) and[22]

$$B'(s)=O(s^{-1}\ln^{-3}s)\,,\ B''(s)=O(s^{-2}\ln^{-3}s)\,,\ s>S\,. \tag{2.24}$$

We expect that in most problems of interest B' and B'' will exist and be continuous for all $s\geq 4$, which means that the bounds (2.24) will hold for all $s\geq 4$. We shall, in fact, assume (2.24) for $s\geq 4$. In a contrary case where B'', say, fails to be bounded at a low threshold, the proofs can be modified easily by treating the low-energy parts of integrals separately. These low-energy parts inevitably have whatever behavior is required, irrespective of the bounds (2.24) which are needed essentially at high energy only.

The quantities $\Lambda(\infty)$ and α in (2.6) may be evaluated using (2.23) and (2.24). We find[1]

$$\Lambda(\infty)=-cB(\infty)\,,\quad \alpha(s,s')\equiv 1\,. \tag{2.25}$$

One may now demonstrate,[1] under conditions (2.23), (2.24), and (2.17), that the N/D integral equation (2.6) is a regular Fredholm equation in $L^2[4,\infty)$. If $c=0$, the unknown L^2 function is φ, while if $c\neq 0$, one must multiply through by $s^{-1/2}\ln(s-s_0)$ so that the unknown in L^2 is $\psi(s)=s^{-1/2}\ln(s-s_0)\varphi(s)$. The latter redefinition of the unknown is to make the inhomogeneous term a member of L^2.

III. ASYMPTOTIC ANALYSIS OF N/D EQUATION, AND CONSTRAINT ON $B(\infty)+a$

We now make an asymptotic analysis of the N/D equation under conditions (2.23) and (2.24) on the force function, and condition (2.17) on the elasticity. We take first the case where no CDD poles are present, so that the integral equation is

$$\eta(s)\varphi(s)=\frac{a+B(\infty)+C(s)}{s-s_0}+\frac{1}{\pi}\int_4^{\infty}\frac{C(s)-C(s')}{s-s'}\rho(s')\varphi(s')ds'\,, \tag{3.1}$$

[22] G. R. Bart, Ph.D. Dissertation, Illinois Institute of Technology, 1970 (unpublished); see Eqs. (2.45) ff.

where

$$C(s)=B(s)-B(\infty)$$
$$=-(\pi/2)\gamma\ \ln^{-2}s+O(\ln^{-3}s)\,. \qquad (3.2)$$

We suppose that (3.1) has a unique solution in L^2, i.e., that the corresponding homogeneous equation has no nontrivial solution in L^2. There is no physical reason to expect the contrary. We are interested in the phase shift δ constructed from the solution φ of (3.1) according to the formula

$$\tan\delta=-\frac{\mathrm{Im}D_+}{\mathrm{Re}D_+}=\frac{(s-s_0)\rho(s)\varphi}{\mathrm{Re}D_+}\,, \qquad (3.3)$$

where D is obtained from (2.5), with $c=c_i=0$. We are looking for the conditions under which δ satisfies (2.16).

To get a start, let us suppose that φ' exists and that

$$\varphi(s)=O(s^{-1})\,,\quad \varphi'(s)=O(s^{-2})\,. \qquad (3.4)$$

(Later, we shall indeed establish these conditions.) Then the following integral exists:

$$I_1=I_1[\varphi]=-\frac{1}{\pi}\int_4^\infty C(s)\rho(s)\varphi(s)ds\,. \qquad (3.5)$$

Moreover,

$$K\varphi(s)=\frac{1}{\pi}\int_4^\infty\frac{C(s)-C(s')}{s-s'}\rho(s')\varphi(s')ds'$$
$$=I_1/s+O(s^{-1}\ln^{-1}s)\,, \qquad (3.6)$$

since

$$\bar{K}\varphi(s)=K\varphi(s)-I_1/s$$
$$=\frac{1}{\pi}\int_4^\infty\frac{sC(s)-s'C(s')}{s(s-s')}\rho(s')\varphi(s')ds'\,, \qquad (3.7)$$

and (3.7) is seen to be $O(s^{-1}\ln^{-1}s)$ by Theorem A. By (3.1) we then have

$$\varphi(s)\sim[a+B(\infty)+I_1]/s\,, \qquad (3.8)$$

$$\mathrm{Im}D_+(s)\sim\mathrm{Im}D_+(\infty)=-[a+B(\infty)+I_1]\,. \qquad (3.9)$$

If $\mathrm{Im}D_+(\infty)\neq0$, then Theorem A yields $\mathrm{Re}D_+(s)\sim(\mathrm{const})\ln s$, or $\tan\delta\sim(\mathrm{const})\ln^{-1}s$, contrary to the desired behavior

$$\tan\delta\sim-\tfrac{1}{2}\pi\,\gamma/\ln^2 s\,. \qquad (3.10)$$

Thus, we have the following necessary condition for the asymptotic form (3.10):

$$a+B(\infty)+I_1=0\,. \qquad (3.11)$$

This condition, our principal result, will prove to be also sufficient. Notice that φ, and hence I_1, is a linear function of $a+B(\infty)$; i.e., the resolvent of (3.1) is a linear operator. Equation (3.11) is a linear equation for $a+B(\infty)$.

Toward the goal of proving the bounds (3.4), we now consider the Fredholm equation

$$y=f+Ky\,. \qquad (3.12)$$

Suppose that 1 is not an eigenvalue of K, and let H be the resolvent, $1+H=(1-K)^{-1}$. Then

$$y=f+Hf$$
$$=f+Kf+KHf\,. \qquad (3.13)$$

By the Schwarz inequality,

$$|y|<|f|+|Kf|+k\|Hf\|\,, \qquad (3.14)$$

where

$$k^2(s)=\int_4^\infty|K(s,s')|^2ds'\,, \qquad (3.15)$$

and the double bars denote the L^2 norm.

To make use of the bound (3.14), we put

$$y(s)=\eta(s)s^{1/2}\varphi(s)/\ln(s-s_0)\,, \qquad (3.16)$$

$$f(s)=\frac{s^{1/2}}{\ln(s-s_0)}\left[\frac{a+B(\infty)+C(s)}{s-s_0}\right]\,, \qquad (3.17)$$

$$K(s,s')=\frac{1}{\pi}\left[\frac{C(s)-C(s')}{s-s'}\right]\left(\frac{s}{s'}\right)^{1/2}\frac{\ln(s'-s_0)}{\ln(s-s_0)}\frac{\rho(s')}{\eta(s')}\,, \qquad (3.18)$$

so that (3.12) is identical to (3.1). With the aid of (2.23), (2.24), and Theorem A, one easily finds that the first two terms on the right-hand side of (3.14) are $O(s^{-1/2}\ln^{-1}s)$. Appendix B contains a proof that $k(s)=O(s^{-1/2}\ln^{-1}s)$. [This shows, incidentally, that (3.1) is a regular Fredholm equation in L^2, as we have claimed.] Thus, any L^2 solution φ of (3.1) is automatically $O(s^{-1})$, since $y(s)=O(s^{-1/2}\ln^{-1}s)$.

To investigate φ', we first notice that φ certainly has a continuous derivative. This follows from the fact that the right-hand side of (3.1) has a continuous derivative, which may be evaluated by differentiating under the integral sign. The derivative of the first term on the right of (3.1) is $O(z^{-2})$, while the derivative of the integral term has the absolute value

$$\left|\int_4^\infty\rho(s')\varphi(s')ds'\int_0^1 uC''(s'+u[s-s'])du\right|$$

$$\leq M\int_4^\infty\frac{ds'}{s'}\int_0^1\frac{udu}{[s'+u(s-s')]^2\ln^3[s'+u(s-s')]}$$

$$=\frac{M}{s^2}\int_{4/s}^\infty\frac{dt}{t(1-t)^2}\int_t^1\frac{x-t}{x^2\ln^3(sx)}dx$$

$$\leq\frac{M}{s^2}\int_{4/s}^\infty\frac{dt}{t(1-t)}\int_t^1\frac{dx}{x\ln^3 sx}\,. \qquad (3.19)$$

The integral over t is bounded as s increases. This is obvious except, perhaps, for the part of the integral near $t=0$, which is handled as follows:

$$\int_{4/s}^{\epsilon}\frac{dt}{t(1-t)}\int_t^1\frac{dx}{x\ln^3 sx}$$

$$<\frac{1}{2(1-\epsilon)}\int_{4/s}^{\epsilon}\frac{dt}{t}\left(\frac{1}{\ln^2 st}-\frac{1}{\ln^2 s}\right)=O(1). \quad (3.20)$$

It follows that $\varphi'(s)=O(s^{-2})$, $s\to\infty$.

We have used the bounds (3.4) to show that Eq. (3.11) is a necessary condition for the Regge behavior of the phase shift (3.10). We are now concerned with showing that (3.11) is also sufficient for (3.10). We assume that (3.11) holds, then, and write the integral equation in the form

$$\eta\varphi=\frac{a+B(\infty)+C(s)}{s-s_0}+\frac{I_1}{s}+\bar{K}\varphi(s), \quad (3.21)$$

where $\bar{K}$ is defined in (3.7). We have observed above that $\bar{K}\varphi(s)=O(s^{-1}\ln^{-1}s)$, so it follows from (3.21) that $\varphi(s)=O(s^{-1}\ln^{-1}s)$. This improved bound on φ enables us to get an improved bound on $\bar{K}\varphi$, which will lead to $\varphi(s)=O(s^{-1}\ln^{-2}s)$. We have

$$|\bar{K}\varphi(s)|$$

$$=\left|\frac{1}{s}\int_4^{\infty}ds\,\rho(s')\varphi(s')\int_0^1 du[xC(x)]'|_{x=s'+u(s-s')}\right|$$

$$\le\frac{M}{s}\int_4^{\infty}\frac{ds'}{s'\ln s'}\int_0^1\frac{du}{\ln^2[s'+u(s-s')]}$$

$$=\frac{M}{s}\int_{4/s}^{\infty}\frac{dt}{t(1-t)\ln st}\frac{1}{s}\int_{st}^{s}\frac{du}{\ln^2 u}. \quad (3.22)$$

To bound the u integral, we need the following lemma, in which n is a positive integer:

$$\int_{s_0}^{s}\frac{du}{\ln^n u}=\frac{s}{\ln^n s}+O\left(\frac{s}{\ln^{n+1}s}\right), \quad s_0>1,\ s\to\infty. \quad (3.23)$$

To prove (3.23), note that

$$\frac{s}{\ln^n s}=\int_{s_0}^{s}\left(\frac{u}{\ln^n u}\right)'du+\frac{s_0}{\ln^n s_0}$$

$$=\int_{s_0}^{s}\left(\frac{1}{\ln^n u}-\frac{n}{\ln^{n+1}u}\right)du+\frac{s_0}{\ln^n s_0}. \quad (3.24)$$

Now apply l'Hospital's rule as follows:

$$\lim_{s\to\infty}\frac{\displaystyle\int_{s_0}^{s}\frac{du}{\ln^n u}-\frac{s}{\ln^n s}}{\displaystyle\frac{ns}{\ln^{n+1}s}}=\lim_{s\to\infty}\frac{\displaystyle n\int_{s_0}^{s}\frac{du}{\ln^{n+1}u}-\frac{s_0}{\ln^n s_0}}{\displaystyle\frac{ns}{\ln^{n+1}s}}$$

$$=\lim_{s\to\infty}\frac{\displaystyle\frac{n}{\ln^{n+1}s}}{\displaystyle\frac{n}{\ln^{n+1}s}-\frac{n(n+1)}{\ln^{n+2}s}}=1. \quad (3.25)$$

Result (3.23) follows; in fact, the second term on the right-hand side of (3.23) is actually asymptotic to $ns\ln^{-n-1}s$. To majorize the integral on the right-hand side of (3.22), we first look at the part near $t=0$, using (3.23). With $4/s<\epsilon<1$, we have

$$J_1=\int_{4/s}^{\epsilon}\frac{dt}{t(1-t)\ln st}\frac{1}{s}\int_{st}^{s}\frac{du}{\ln^2 u}$$

$$\le\frac{1}{s}\int_4^{s}\frac{du}{\ln^2 u}\int_{4/s}^{\epsilon}\frac{dt}{(1-\epsilon)t\ln st}$$

$$\le\left[\frac{1}{\ln^2 s}+O\left(\frac{1}{\ln^3 s}\right)\right]\frac{1}{1-\epsilon}[\ln\ln s\epsilon-\ln\ln 4]$$

$$=O(\ln^{-2}s\ln\ln s). \quad (3.26)$$

The part of the integral near $t=1$ is

$$J_2=\int_{\epsilon}^{1+\epsilon}\frac{dt}{t(1-t)\ln st}\int_t^1\frac{dx}{\ln^2(sx)}$$

$$\le\frac{1}{\ln^2 s\epsilon}\int_{\epsilon}^{1+\epsilon}\frac{dt}{t\ln st}=O(\ln^{-3}s). \quad (3.27)$$

The piece with t large is

$$J_3=\int_{1+\epsilon}^{\infty}\frac{dt}{t(1-t)\ln st}\frac{1}{s}\int_{st}^{s}\frac{du}{\ln^2 u}$$

$$=\int_{1+\epsilon}^{\infty}\frac{dt}{t(1-t)\ln st}\left[\frac{1}{\ln^2 s}-\frac{t}{\ln^2 st}+O\left(\frac{t}{\ln^3 st}\right)+O\left(\frac{1}{\ln^3 s}\right)\right]$$

$$\le\int_{1+\epsilon}^{\infty}\frac{dt}{t\ln^3 st}+O\left(\frac{1}{\ln^3 s}\right)=O(\ln^{-2}s). \quad (3.28)$$

Thus, $\bar{K}\varphi(s)=O(s^{-1}\ln^{-2}s\ln\ln s)$, and it follows from (3.21) that also $\varphi(s)=O(s^{-1}\ln^{-2}s\ln\ln s)$. Now if this bound on φ is used in a repetition of the above estimates of $\bar{K}\varphi$, we find that $\bar{K}\varphi(s)=O(s^{-1}\ln^{-2}s)$. Hence, $\varphi(s)=O(s^{-1}\ln^{-2}s)$, which was to be proved.

We also need $\varphi'(s)=O(s^{-2}\ln^{-2}s)$ which is proved in much the same way. On differentiating (3.21), it is seen that all terms on the right-hand side are clearly $O(s^{-2}\ln^{-2}s)$, except possibly $(\partial/\partial s)\bar{K}\varphi$, which is treated in the manner of Eq. (3.19). We find

$$\left|\frac{\partial}{\partial s}\bar{K}\varphi\right|\leq\frac{M}{s^2}\int_{4/s}^{\infty}\frac{dt}{t(1-t)\ln^2 st}\frac{1}{s}\int_{st}^{s}\frac{du}{\ln^3 u}+O(s^{-2}\ln^{-2}s). \quad (3.29)$$

After an analysis like that following (3.22), it turns out that $(\partial/\partial s)\bar{K}\varphi=O(s^{-2}\ln^{-2}s)$; hence, $\varphi'(s)=O(s^{-2}\ln^{-2}s)$.

Knowing that $\varphi=O(s^{-1}\ln^{-2}s)$, we may define

$$I_2=\frac{1}{\pi}\int_4^{\infty}\rho(s)\varphi(s)ds. \quad (3.30)$$

From the bounds on φ and φ' and Theorem A, it is easy to deduce from Eq. (3.21) that

$$\varphi(s)=\frac{-\pi\gamma}{2}\frac{1+I_2}{s\ln^2 s}+O(s^{-1}\ln^{-3}s). \quad (3.31)$$

Introducing this result in (2.5), we use Theorem A again to obtain

$$\mathrm{Re}D_+(s)=1-\frac{s-s_0}{\pi}P\int_4^{\infty}\frac{\rho(s')\varphi(s')ds'}{s'-s}=1+I_2+O(\ln^{-1}s). \quad (3.32)$$

According to (3.3), (3.31), and (3.32), our theorem is now proved, provided $1+I_2\neq O$:

$$\tan\delta(s)=\frac{(s-s_0)\rho(s)\varphi(s)}{\mathrm{Re}D_+(s)}=\frac{-\pi\gamma}{2\ln^2 s}+O(\ln^{-3}s). \quad (3.33)$$

In the singular case $1+I_2=0$, we could still have

$$\tan\delta(s)=O(\ln^{-2}s), \quad (3.34)$$

provided that $\mathrm{Re}D_+(s)$ is asymptotic to const $\ln^{-1}s$.

When n CDD poles are included at finite energies s_i in (2.5), keeping $c=0$, the preceding analysis goes through with little change. The reason is that the CDD terms appear only in the inhomogeneous term of the integral equation, so that the various estimates of integrals are not affected. The condition (3.11) takes the modified form

$$a+B(\infty)+\sum_{i=1}^{n}c_i[B(s_i)-B(\infty)]+I_1=0. \quad (3.35)$$

If Eq. (3.35) holds, then $1+I_2$ is replaced by $1+I_2+\sum c_i$ in (3.31) and (3.32) and, provided the latter sum is not zero, result (3.33) follows.

In (2.16) and (2.19), it is seen that the phase shift in Regge theory always approaches $n\pi$ from below. As is explained in Ref. 1 and Ref. 3, this means that when CDD poles are needed, one of them may be put at infinity. We shall show presently that whenever there is a CDD pole at infinity [$c\neq 0$ in (2.5)] then the Regge phase-shift asymptote (3.10) is obtained from the solution of the N/D equation *automatically*, without the constraint (3.35). In this connection, it is interesting to count the number of parameters. The number n_c of CDD poles (counting the possible pole at infinity) is determined by Levinson's relation, as follows[1]:

$$\delta(\infty)/\pi=n_c-n_b. \quad (3.36)$$

The number of stable-particle poles is n_b, and $\delta(4)=0$. The behavior of the phase shift may be such that it is possible[1] to take all CDD poles at finite energies s_i; [i.e., at points s_i such that $\sin\delta(s_i)=0$]. In that case, we have $2n_c+1$ parameters, viz., the n_c positions s_i, the n_c residues c_i, and the subtraction constant a. We must impose Eq. (3.35), however, which reduces the number of parameters to $2n_c$. On the other hand, it may happen that one of the CDD poles may be or *must* be placed at infinity. It must be there in the case when the number of finite zeros of $\sin\delta$ is less than n_c, as in the instance where the phase rises monotonically from zero to approach π from below at infinity. With one pole at infinity, we have $2(n_c-1)$ residues and finite positions, the residue c of the infinite pole, and the subtraction constant. Since Eq. (3.35) is now irrelevant, there are again $2n_c$ parameters. This clears up an old puzzle which was noticed when Ref. 3 was written: Why are there two parameters associated with finite CDD poles, while only one with a pole at infinity? This seemed strange since, with certain phase-shift behaviors, one has the option of placing a pole at infinity or not; i.e., if the number of finite zeros of $\sin\delta$ is at least equal to n_c, and the phase tends to $n\pi$ from below, then one has this option. We now know (in Regge theory, at least) that the subtraction constant is determined when all poles are at finite positions, but undetermined when one is at infinity, so that the total number of parameters is always $2n_c$.

In order to analyze the case $c\neq 0$, we must redefine the unknown function in the integral equation, as was remarked following Eq. (2.25). The integral equation then reads

$$\psi(s)=\frac{\ln(s-s_0)}{s^{1/2}\eta(s)}\left[\frac{a+B(\infty)+C(s)}{s-s_0}+cC(s)+\sum_i c_i\frac{C(s)-C(s_i)}{s-s_i}\right]+\frac{1}{\pi}\int_4^{\infty}\frac{\rho(s')}{\eta(s)}\left(\frac{s'}{s}\right)^{1/2}\frac{\ln(s-s_0)}{\ln(s'-s_0)}\times\left[\frac{C(s)-C(s')}{s-s'}\right]\psi(s')ds', \quad (3.37)$$

where

$$\psi(s)=\frac{\ln(s-s_0)\varphi(s)}{s^{1/2}}. \tag{3.38}$$

It is known[1] that if $M(s,s')$ is the kernelin (3.37), then

$$m^2(s)=\int_4^\infty M^2(s,s')ds'=O(s^{-1}\ln^{-2}s). \tag{3.39}$$

By the methods following (3.13) one finds that $\psi(s)=O(s^{-1/2}\ln^{-1}s)$. One can also verify, by techniques similar to those already used, that $M\psi=O(s^{-1/2}\ln^{-2}s)$, $(M\psi)'=O(s^{-3/2}\ln^{-2}s)$. Consequently, the inhomogeneous term in (3.37) determines the high-energy behavior of both ψ and ψ'. We find then

$$\varphi(s)=cC(s)+O(\ln^{-3}s), \tag{3.40}$$

$$\varphi'(s)=O(s^{-1}\ln^{-3}s), \tag{3.41}$$

from which we deduce via (2.5) and Theorem A that

$$\mathrm{Re}D_+(s)=cs+O(s\ln^{-1}s). \tag{3.42}$$

Thus,

$$\tan\delta(s)=\frac{(s-s_0)\rho(s)\varphi(s)}{\mathrm{Re}D_+(s)}\sim C(s)$$

$$=\frac{-\pi\gamma}{2\ln^2 s}+O(\ln^{-3}s), \tag{3.43}$$

which was to be proved.

The determination of the constant $a+B(\infty)$ from Eq. (3.11) or Eq. (3.35) requires very little computation beyond that needed to solve the N/D equation. We denote the integral operator in (3.1) by K, so that (3.1) reads

$$\eta(s)\varphi(s)=\frac{a+B(\infty)+C(s)}{s-s_0}+K\varphi(s). \tag{3.44}$$

Instead of solving (3.44), we solve two auxiliary equations, namely,

$$\eta(s)\varphi_1(s)=\frac{1}{s-s_0}+K\varphi_1(s), \tag{3.45}$$

$$\eta(s)\varphi_2(s)=\frac{C(s)}{s-s_0}+K\varphi_2(s). \tag{3.46}$$

We can then make the solution of (3.44) as an explicit linear function of $a+B(\infty)$:

$$\varphi(s)=[a+B(\infty)]\varphi_1(s)+\varphi_2(s). \tag{3.47}$$

When this is substituted in (3.5), the constraint equation (3.11) yields

$$a+B(\infty)=\frac{\frac{1}{\pi}\int_4^\infty C(s)\rho(s)\varphi_2(s)ds}{1-\frac{1}{\pi}\int_4^\infty C(s)\rho(s)\varphi_1(s)ds}. \tag{3.48}$$

A vanishing of the denominator in (3.48) would mean that the input functions $B(s)$ and $\eta(s)$ were inconsistent with the Regge behavior (2.16) of the output phase shift. That would be the only way out of a contradiction, since the left-hand side of (3.48) is finite: $a=A_l(s_0)$ is finite by definition, and so is $B(\infty)$ as a convergent integral.

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APPENDIX A

To prove Theorem A, as stated at the end of Sec. II, we break up the integral as follows:

$$g(s)=P\int_{s_0}^\infty\frac{f(t)dt}{s-t}=\sum_{i=1}^4 I_i,$$

$$I_1=\frac{1}{s}\int_{s_0}^{s(1+\epsilon)}f(t)dt,\quad I_2=\frac{1}{s}\int_{s_0}^{s(1-\epsilon)}\frac{tf(t)dt}{s-t}, \tag{A1}$$

$$I_3=\frac{P}{s}\int_{s(1-\epsilon)}^{s(1+\epsilon)}\frac{tf(t)dt}{s-t},\quad I_4=\int_{s(1+\epsilon)}^\infty\frac{f(t)dt}{s-t}.$$

We take $0<\epsilon<1$ and $s_0<s(1-\epsilon)$. Since

$$|I_2|\le Ms^{-2}\int_{s_0}^{s(1-\epsilon)}t^{1-\alpha}\ln^{-\beta}t\,dt=Ms^{-2}J_2, \tag{A2}$$

we may apply l'Hospital's rule to $J_2/(s^{2-\alpha}\ln^{-\beta}s)$ to show that $I_2=O(s^{-\alpha}\ln^{-\beta}s)$. For I_4 we have

$$\alpha=1:\quad |I_4|\le M\ln^{-\beta}[s(1+\epsilon)]\times\int_{s(1+\epsilon)}^\infty t^{-1}(t-s)^{-1}dt=O(s^{-1}\ln^{-\beta}s), \tag{A3}$$

$$\alpha=0,\beta>1:\quad |I_4|\le M\int_{s(1+\epsilon)}^\infty\ln^{-\beta}t\,(t-s)^{-1}dt<M\int_{s\epsilon}^\infty\ln^{-\beta}u\,u^{-1}du=O(\ln^{1-\beta}s). \tag{A4}$$

Now I_3 may be written as

$$I_3=\frac{1}{s}\int_{s(1-\epsilon)}^{s(1+\epsilon)}\frac{tf(t)-sf(s)}{s-t}dt, \tag{A5}$$

since

$$P\int_{s(1-\epsilon)}^{s(1+\epsilon)}\frac{dt}{s-t}=0. \tag{A6}$$

The mean-value theorem may be used to bound the integrand of (A5). For some x such that $s(1-\epsilon)\leq x\leq s(1+\epsilon)$, we have

$$\left|\frac{tf(t)-sf(s)}{t-s}\right|=|(xf(x))'|\leq Mx^{-\alpha}\ln^{-\beta}x$$
$$\leq M[s(1-\epsilon)]^{-\alpha}\ln^{-\beta}[s(1-\epsilon)]=O(s^{-\alpha}\ln^{-\beta}s). \tag{A7}$$

Hence

$$I_3=O(s^{-\alpha}\ln^{-\beta}s). \tag{A8}$$

Finally, we note that

$$\begin{aligned}I_1&=\frac{1}{s}\int_{s_0}^{s}f(t)dt+\frac{1}{s}\int_{s}^{s(1+\epsilon)}f(t)dt\\&=\frac{1}{s}\int_{s_0}^{s}f(t)dt+O(s^{-\alpha}\ln^{-\beta}s),\end{aligned} \tag{A9}$$

which completes the proof.

APPENDIX B

Our intention here is to majorize the function $k^2(s)$, defined in Eqs. (3.15) and (3.18). We prove that

$$k^2(s)=O(s^{-1}\ln^{-2}s),\quad s\to\infty. \tag{B1}$$

For this purpose it is convenient to bound the difference quotient of $C(s)$ as follows, using (2.24):

$$\left|\frac{C(s)-C(s')}{s-s'}\right|=\left|\int_0^1 C'(s'+u[s-s'])du\right|$$
$$\leq M\int_0^1\frac{du}{[s'+u(s-s')]\ln^3[s'+u(s-s')]}. \tag{B2}$$

This gives

$$k^2(s)\leq M_1\int_4^\infty\frac{s}{s'}\frac{\ln^2(s'-s_0)}{\ln^2(s-s_0)}$$
$$\times\left[\int_0^1\frac{du}{[s'+u(s-s')]\ln^3[s'+u(s-s')]}\right]^2 ds'$$
$$=\frac{M_1}{s\ln^2(s-s_0)}\int_{4/s}^\infty\frac{dt\ln^2(st-s_0)}{t(1-t)^2}\left(\int_t^1\frac{dx}{x\ln^3 sx}\right)^2. \tag{B3}$$

To show that the integral over t in (B3) is bounded, we consider three separate intervals of t: $4/s\leq t\leq\epsilon<1$, $\epsilon\leq t\leq 1+\epsilon$, and $1+\epsilon\leq t<\infty$. For the first interval, we have

$$\int_{4/s}^{\epsilon}\frac{dt\ln^2(st-s_0)}{t(1-t)^2}\left(\int_t^1\frac{dx}{x\ln^3 sx}\right)^2$$
$$<\frac{1}{4(1-\epsilon)^2}\int_{4/s}^{\epsilon}\frac{dt\ln^2 st}{t}\left(\frac{-1}{\ln^2 s}+\frac{1}{\ln^2 st}\right)^2=O(1). \tag{B4}$$

To show that (B4) is bounded, the integration may be performed explicitly. The interval near $t=1$ contributes

$$\int_{\epsilon}^{1+\epsilon}\frac{dt\ln^2(st-s_0)}{t(1-t)^2}\left(\int_t^1\frac{dx}{x\ln^3 sx}\right)^2$$
$$<\frac{\ln^2(s[1+\epsilon]-s_0)}{\epsilon^3\ln^6 s\epsilon}=O(\ln^{-4}s), \tag{B5}$$

while the tail of the integral yields

$$\int_{1+\epsilon}^{\infty}\frac{dt\ln^2(st-s_0)}{t(1-t)^2}\left(\int_t^1\frac{dx}{x\ln^3 sx}\right)^2$$
$$<\frac{1}{\ln^4 s}\int_{1+\epsilon}^{\infty}\frac{dt[1+\ln(t-s_0/s)]^2\ln^2 t}{t(1-t)^2}=O(\ln^{-4}s). \tag{B6}$$