Direct-Channel Reggeization of Strong-Interaction Scattering Amplitudes. III

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A class of representations is discussed which describe the direct-channel S-matrix partial-wave projection in terms of direct-channel Regge poles $\{\alpha_n{}^{\rm I}(s)\}$, Regge zeros $\{\alpha_n{}^{\rm II}(s)\}$, and a convergence factor $\phi(\lambda)$, as follows:

$$\ln S_l=\left[\sum_n\int_{\alpha_n{}^{\rm I}}^{\alpha_n{}^{\rm II}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right]\Big/\phi(l).$$

Mathematical methods, introduced in Paper II of this series, are expanded here to take advantage of the particular representation of the function $\phi(\lambda)$ which is shown to be required by proper behavior of the amplitude near the closest cross-channel singularities. This paper discusses the ways in which more detailed crossing-symmetry requirements specify $\phi(\lambda)$, given an approximate set of direct-channel poles and zeros. It is shown that the representation for $\phi(\lambda)$ produces large families of infinitely rising crossed-channel trajectories in a simple way. It is shown that these amplitudes automatically have the proper direct-channel phase-shift threshold behavior, provided that the functions $\{\alpha_n{}^{\rm I}(s),\alpha_n{}^{\rm II}(s)\}$ have the threshold behavior usually required. In addition, it is shown that this trajectory threshold behavior follows, for our amplitudes, from imposition of proper behavior in the neighborhood of the nearest crossed-channel singularity; i.e., no recourse to results derived outside the context of this class of representations is necessary. A simple bootstrap calculation is performed, for the purposes of illustration. It is shown that the zeros of the function $\phi(\lambda)$ are closely related to the background integral in the conventional Sommerfeld-Watson summation scheme.

I. INTRODUCTION

In two previous papers[1] this author indicated that there exist a large class of direct-channel Reggeized representations for $\ln S_l$ which have the proper asymptotic behavior as $l\to\infty$, $\text{Re}\,l>-\frac{1}{2}$. In this paper, the specification of these representations is reduced to the choice of an entire function $\phi(\lambda)$ with the following asymptotic properties:

$$\phi(\lambda)\sim\frac{e^{\lambda\xi}}{\sqrt{\lambda}}\times\text{const},\quad \text{as }\lambda\to\infty,\ \text{Re}\lambda>-\tfrac{1}{2},\quad \cosh\xi=1+M_x{}^2/2q_s{}^2 \qquad (1)$$

$$\phi(\lambda)=o(\lambda^{-1}),\qquad \text{as }\lambda\to\infty,\ \text{Re}\lambda<-\tfrac{1}{2}$$

where $\phi(\lambda)$ is used to represent the scattering amplitude in terms of direct-channel Regge trajectories, as follows:

$$\ln S_l=\left[\sum_n\int_{\alpha_n{}^{\rm I}}^{\alpha_n{}^{\rm II}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right]\Big/\phi(l). \qquad (2)$$

It will then be shown that some of these representations have a greater number of desirable properties than one might naively expect to follow from the imposition of proper asymptotic behavior for large l. One such representation is examined in fair detail, and a rough "bootstrap" calculation is performed for illustrative purposes. Some mathematical techniques are discussed which should be useful in performing calculations with such representations, and which shed some light on the general problem of constructing approximate scattering amplitudes which have a large number of desirable properties. In order to concentrate on the essential features of these techniques, we will suppress explicit references to Regge cuts, regard our scattering amplitudes as having only one crossed channel with a pole being the nearest singularity, and consider only elastic scattering.

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[1] S. P. Creekmore, preceding papers, Phys. Rev. D **3**, 1400 (1971); **3**, 1407 (1971).

II. GENERALIZED CHENG-LIKE REPRESENTATIONS: CHOICE AND REPRESENTATION OF FUNCTION

Consider the integral $\oint_c\phi(\lambda)(\lambda-l)^{-1}\ln S_\lambda d\lambda$ as the contour is taken to infinity. Earlier authors[2] derived Reggeized representations by choosing $\phi(\lambda)=e^{\lambda\xi}$ and postulating that $e^{\lambda\xi}\ln S_\lambda\to 0$ as $|\lambda|\to\infty$ sufficiently fast in all directions away from $\lambda=-\frac{1}{2}$ to allow the contour integral to approach zero as the contour is expanded to infinity, avoiding the singularities of $\ln S_\lambda$. In fact, this requirement can be modified somewhat since it is known that

$$\ln S_l\sim\text{const}\,e^{-l\xi}/\sqrt{l},\quad \text{as}\quad l\to\infty,\quad \text{Re}\,l\geq-\tfrac{1}{2} \qquad (3a)$$

$$\ln S_l\sim 2\pi il,\qquad \text{as}\quad l\to\infty,\quad \text{Re}\,l<-\tfrac{1}{2}. \qquad (3b)$$

In potential theory, it has been shown[3] that $\ln S_l$ satisfies Eq. (3b) for a large class of potentials. Equation (3a) follows from the well-known asymptotic behavior of a_l in that limit:

$$a_l\sim O(e^{-l\xi}/\sqrt{l}), \qquad (4a)$$

$$S_l=1+a_lf(s). \qquad (4b)$$

[2] Hung Cheng, Phys. Rev. **144**, 1237 (1966); W. J. Abbe, P. Kaus, P. Nath, and Y. N. Strivastava, *ibid.* **140**, B1595 (1965), henceforth referred to as AKNS. A specific computational scheme was investigated in W. J. Abbe *et al.*, Phys. Rev. **141**, 1513 (1966), and W. J. Abbe and G. A. Gary, *ibid.* **160**, 1510 (1967).

[3] Hung Cheng and Tai Tsun Wu, Phys. Rev. **144**, 1232 (1966).

An immediate consequence of the behavior in the right half-plane is the correct behavior of the whole amplitude (defined in terms of the partial-wave series) in the neighborhood of the nearest crossed-channel singularity. As we remarked in an earlier paper,[4] the Cheng representation $\phi(\lambda)=e^{\lambda\xi}$ does not have this behavior if the sum over Regge poles is approximated by a finite number of terms. In order to avoid that difficulty, we will require $\phi(\lambda)\sim O(e^{\lambda\xi}/\sqrt{\lambda})$ in the right half-plane.

The essential conclusions of this section are that, with quite unrestrictive further assumptions about the function ϕ, one obtains the following representation:

$$\phi(\lambda)=\exp[(1-a)(\lambda+\tfrac{1}{2})\xi]\psi(\lambda)\,, \tag{5}$$

where

$$\psi(\lambda)=\int_{-a\xi}^{a\xi}\frac{\exp[(\lambda+\frac{1}{2})w]\,k(w,a\xi)}{\pi\sqrt{2}(\cosh w-\cosh a\xi)^{1/2}}dw\,. \tag{6}$$

Furthermore, ϕ has an infinite number of zeros lying close to the line $\mathrm{Re}\lambda=-\frac{1}{2}$. The line segment $(-a\xi, a\xi)$ is called the indicator diagram of ψ, and the coefficient of the exponential in Eq. (6) is the Borel transform of ψ, which for mathematical reasons is required to have singularities at the endpoints of the indicator diagram. Equation (6) itself is called a Polya representation.

The precise meaning of these statements is made more clear in the following discussion and in the references cited therein, but it is possible for a reader uninterested in these details to proceed directly to Sec. III.

The asymptotic behavior of $\phi(\lambda)$ is less well specified in the left half-plane than in the right. We will restrict our consideration to functions having the following asymptotic behavior in the left half-plane:

$$\limsup_{|\lambda|\to\infty;\ \mathrm{Re}\lambda<-\frac{1}{2}}\frac{\ln|\phi(\lambda)|}{|\lambda|}=\xi'<\infty\,, \tag{7}$$

where ξ' does not necessarily equal ξ. This behavior allows flexibility in the behavior of $\ln S_\lambda$ in the left-half plane, should this be required by relativistic generalizations of the potential-scattering result, Eq. (3b).

We also require

$$\limsup_{|\lambda|\to\infty}\frac{\ln\ln M_{|\lambda|}(\phi)}{\ln|\lambda|}=1\,, \tag{8}$$

where

$$M_{|\lambda|}(\phi)=\max_{0\leq\theta\leq2\pi}|\phi(|\lambda|e^{i\theta})|\,.$$

It is useful to replace $\phi(\lambda)$ by a function whose asymptotic behavior is symmetric, as follows:

$$\phi(\lambda)=\exp[(1-a)(\lambda+\tfrac{1}{2})\xi]\psi(\lambda)\,, \tag{9}$$

$$\limsup_{|\lambda|\to\infty}\frac{\ln|\psi(\lambda)|}{|\lambda|}=a\lambda\xi\,,\qquad \mathrm{Re}\lambda\geq-\tfrac{1}{2}$$
$$=-a\lambda\xi\,,\quad \mathrm{Re}\lambda<-\tfrac{1}{2}\,.$$

[4] S. P. Creekmore, preceding paper, Phys. Rev. D 3, 1407 (1971).

a will appear as a parameter in our later formulas. We could also obtain representations by integrating over the variable a to obtain $\psi(\lambda)$.

It does not appear that these asymptotic conditions are overly restrictive on the class of allowed $\phi(\lambda)$. In mathematical terms, we have restricted our consideration to those $\phi(\lambda)$ which are entire functions of exponential growth of order 1.[5] We require that $|\psi(\lambda)|$ be bounded along the line $\mathrm{Re}\lambda=-\frac{1}{2}$.

There is a theorem to the effect that any entire function of exponential type which is bounded on the line $\mathrm{Re}\lambda=-\frac{1}{2}$ is a function of class A, of completely regular growth, and its indicator diagram is an interval on the real axis. In this case, being of class A means that

$$\sum_k|\mathrm{Re}(a_k+\tfrac{1}{2})^{-1}|<\infty\,,$$

where the $\{a_k\}$ are the zeros of $\psi(\lambda)$. All the zeros of such a function, except possibly those of a set of zero density, lie inside rays passing through $\lambda=-\frac{1}{2}$ making arbitrarily small angles with the line $\mathrm{Re}\lambda=-\frac{1}{2}$. The zeros are spaced to allow the limit

$$\lim_{r\to\infty}\sum_{|a_k|<r}a_k^{-1}$$

to exist.[6]

We compute the indicator function of $\psi(\lambda)$ as follows:

$$h_\psi(\theta)=\limsup_{r\to\infty}\frac{\ln|\psi(re^{i\theta}-\frac{1}{2})|}{r}=a\xi|\cos\theta|\,. \tag{10}$$

The indicator diagram, the convex set supported by the function $h_\psi(\theta)$, is the line segment $(-a\xi, a\xi)$ which is also the conjugate indicator diagram.[7] Recalling another mathematical theorem,[8] we use the fact that any entire function of exponential type can be written in a Polya representation:

$$\psi(\lambda)=\frac{1}{2\pi i}\int_C e^{(\lambda+\frac{1}{2})w}F(w)dw\,, \tag{11}$$

where C is a contour surrounding the conjugate indicator diagram and $F(w)$ is the Borel transform of $\psi(\lambda)$. $F(w)$ must have singularities at the points $w=\pm a\xi$.[9] We will assume, without further discussion,[10] that C can be shrunk to the boundary segment $(-a\xi, a\xi)$:

$$\psi(\lambda)=\frac{1}{\pi}\int_{-a\xi}^{a\xi}e^{(\lambda+\frac{1}{2})w}f(w,a\xi)dw\,. \tag{12}$$

[5] A very clear exposition of the theory of the functions we will be using is contained in B. Ja. Levin, *Distribution of Zeros of Entire Functions* (American Mathematical Society, Providence, R. I., 1964); see also R. P. Boas, *Entire Functions* (Academic, New York, 1954).

[6] B. Ja. Levin (Ref. 5), Theorem 11, p. 251.

[7] R. P. Boas (Ref. 5), Sec. 5.3, pp. 73–75.

[8] R. P. Boas (Ref. 5), Theorem 5.3.5, p. 74.

[9] R. P. Boas (Ref. 5), Theorem 5.3.12, p. 75.

[10] Actually, all we need is the conclusion of R. P. Boas (Ref. 5), Theorem 6.8.14, p. 107.

By an examination of the asymptotic properties of this representation in the right half-plane, assuming f to have reasonable behavior, it is seen that $f(w)$ must have a singularity of the following type near $w=a\xi$:

$$f(w)\cong\text{const}/(a\xi-\omega)^{1/2}. \quad (13)$$

Similarly, the behavior of f near $w=-a\xi$ determines the precise asymptotic behavior in the left half-plane. Since we want to retain some flexibility in the behavior for $\text{Re}l<-\frac{1}{2}$, we will not make requirements for $w\cong-a\xi$, except that the representation converge.

We make the following replacement (our choice for the argument of the square-root function will turn out to have simplifying consequences in the discussion of threshold behavior of these representations):

$$f(w,a\xi)=\frac{1}{\sqrt{2}}\frac{k(w,a\xi)}{(\cosh a\xi-\cosh w)^{1/2}}, \quad (14)$$

where the asymptotic l-plane behavior requires that k be finite at the point $w=a\xi$, although its behavior for $w=-a\xi$ is less rigidly defined, and it could still have singularities within the interval $(-a\xi,\ a\xi)$. Evidently, we have reduced the problem of specifying our representation to the specification of the function $k(w,a\xi)$ on the finite segment $(-a\xi,\ a\xi)$.

III. CONSTRAINTS ON BEHAVIOR OF $k(w,a\xi)$: DIRECT-CHANNEL THRESHOLD BEHAVIOR

The asymptotic behavior of $\phi(\lambda)$ in the right half l plane is

$$\phi(l)\sim\frac{\exp[(l+\frac{1}{2})\xi]\,k(a\xi,a\xi)}{[2\pi(l+\frac{1}{2})\sinh a\xi]^{1/2}}. \quad (15)$$

Suppose that the nearest crossed-channel singularity is a pole at $t=-2q_s^2(1-\cosh\xi)$. The residue of the crossed-channel pole is then

$$\text{Residue}=-\lim_{l\to\infty}\left[\left(\frac{1}{2iq_s}\ln S_l\right)\bigg/\left(\frac{1}{2q_s^2}Q_l(\cosh\xi)\right)\right], \quad (16)$$

where the limit is taken along positive real values of l. Inserting our representation, and keeping only one Regge pole, we obtain

$$\text{Residue}\cong 4q_s(\sinh\xi\,\sinh a\xi)^{1/2}k^{-1}(a\xi,a\xi)\times\frac{1}{2i}\int_{\alpha^{\text{I}}}^{\alpha^{\text{II}}}\phi(\lambda)d\lambda, \quad (17)$$

where we have used the asymptotic behavior[11] of $Q_l(\cosh\xi)$ to obtain this result.

On the other hand, quite unrestrictive assumptions about $k(w,a\xi)$ result in the following asymptotic behavior for large $a\xi$ and fixed l:

$$\phi(l)\sim\frac{\exp[(l+\frac{1}{2})\xi]}{(\sinh a\xi)^{1/2}}k(a\xi,a\xi)f(l), \quad (18)$$

where $f(l)$ is a function of l only.

Since $\phi(\lambda)$ only appears in ratios of its values at different values of λ, we will lose no generality in restricting our consideration to k for which $k(a\xi,a\xi)=1$.[12] Therefore, we will suppress the factors $k(a\xi,a\xi)$ appearing in the asymptotic formulas.

Now let us investigate the threshold behavior of amplitudes parametrized in terms of our functions $\phi(\lambda)$; this behavior is closely related to the asymptotic behavior of $\phi(\lambda)$ in λ and $a\xi$.

The expression for the s-channel phase shifts is as follows:

$$\delta_l=\frac{1}{2i}\int_{\alpha^{\text{I}}}^{\alpha^{\text{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda/\phi(l), \quad (19)$$

where for simplicity we consider a single trajectory approximation. As $q_s\to 0$, $e^{\xi}\sim M_x^2/q_s^2$, where M_x is the mass of the nearest crossed-channel pole. Consequently using the behavior of $\phi(l)$ for large ξ and fixed l, we obtain

$$\delta_l\cong\frac{1}{2i}\int_{\alpha^{\text{I}}}^{\alpha^{\text{II}}}q_s^{2(l-\lambda)}f(\lambda)M_x^{2(\lambda-l)}d\lambda/f(l). \quad (20)$$

As $q_s\to 0$, $\alpha^{\text{I}}\to\alpha^{\text{II}}$, so that a mean-value approximation to the integral approaches the correct value:

$$\delta_l\cong\text{const}\ \text{Im}\alpha\ q_s^{2(l-\text{Re}\alpha)}. \quad (21)$$

This result is independent of a in the range $0<a<\frac{1}{2}$ and it is also true for the original Cheng representation and the AKNS modification. All of these representations have the property that, *provided* $\text{Im}\alpha$ has the correct threshold behavior,

$$\text{Im}\alpha\cong\text{const}\ q_s^{2\text{Re}\alpha+1}, \quad (22)$$

the phase shift has the correct threshold behavior:

$$\delta_l\cong\text{const}\ q_s^{2l+1}. \quad (23)$$

We should not be satisfied merely with this result, however; it would be preferable to choose $\phi(\lambda)$ so that, when we perform a practical calculation, in almost any approximation we will obtain the threshold constraint on $\text{Im}\alpha(s)$ as an automatic feature of the solution. In a practical calculation, the trajectory near threshold should be determined by the constraints placed on the amplitude by its required behavior near the closest crossed-channel singularities. In potential theory, for example, this statement reduces to some simple assumptions about the potential.

[11] E. W. Hobson, *Spherical and Ellipsoidal Harmonics* (Chelsea, New York, 1965), Eq. (22), p. 305.

[12] Clearly, we are restricted to $\phi(\lambda)$ corresponding to amplitudes whose nearest crossed-channel singularity is a pole, and therefore whose asymptotic behavior in λ is strictly $e^{-\lambda\xi}/\sqrt{\lambda}$. The generalization to crossed-channel cuts, however, is straightforward.

We have already obtained an expression for the one-trajectory approximation to the crossed-channel residue. Inserting in Eq. (17) the calculated asymptotic behavior of $\phi(\lambda)$ as $q_s \to 0$, and using the same kind of mean-value argument as before, we obtain the s-channel threshold behavior of the (approximate) t-channel pole residue function:

$$\text{Residue} \cong \text{const Im}\alpha \quad q_s^{-(2\text{Re}\alpha+1)}. \tag{24}$$

Now, even in the crudest approximation, the crossed-channel residue should be required to remain finite as $q_s \to 0$. This condition is equivalent (for this class of representations) to the desired result:

$$\text{Im}\alpha(s) \cong \text{const } q_s^{2\text{Re}\alpha+1}. \tag{25}$$

Again, we have obtained a threshold condition independent of a in the range $0<a<\frac{1}{2}$. Only a loose sort of constraint on the behavior of the function $k(w,a\xi)$, which was required to obtain our asymptotic formulas in l and ξ, is necessary to ensure the connection between the crossed-channel singularity and the direct-channel Regge trajectory threshold behavior, which we know from potential theory must exist.[13]

In this respect our representation is a distinct improvement over those of Cheng and AKNS. Such a constraint on the threshold behavior of $\alpha(s)$ cannot automatically hold for these older formulations, since the first of these representations does not allow a reasonable crossed-channel singularity at all, and the second specifies the residue of the Born term *a priori*, as a parameter independent of the direct-channel Regge trajectories.

IV. MATHEMATICAL TECHNIQUES RELATING k TO CROSSED-CHANNEL DISCONTINUITY: MECHANISM FOR IMPOSING CROSSING SYMMETRY

We have developed a parametrization of the scattering amplitude in terms of direct-channel Regge trajectories and an auxiliary function $\phi(\lambda)$, which we have so far avoided specifying precisely, other than to state asymptotic conditions which lead to a representation in terms of the Borel transform

$$k(w,a\xi)\pi^{-1}[(\cosh a\xi-\cosh\omega)\times 2]^{-1/2}.$$

Presumably, the disposition of the direct-channel Regge trajectories provides most of the salient behavior in the s channel, at least for low energies, since the sequence of resonances will be correctly specified, in a way that is fairly insensitive to the choice of ϕ. On the other hand, ϕ is closely related to the behavior of the amplitude in the crossed channel. In Paper II we discussed methods of relating the analytic continuation of the amplitude to the continuation of a simpler function, in terms of an integrodifferential equation. Briefly, if $A(s,z)$ is the scattering amplitude in our representation,

$$A(s,z)=\frac{1}{2iq_s}\sum_{l=0}^{\infty}(2l+1)\times\left\{\exp\left[\left(\sum_n\int_{\alpha_n^{\text{I}}}^{\alpha_n^{\text{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right)\Big/\phi(l)\right]-1\right\}P_l(z) \tag{26}$$

and $g(s,z)$ is a related function defined as follows:

$$g(s,z)=\sum_{l=0}^{\infty}(2l+1)\times\left[\left(\sum_n\int_{\alpha_n^{\text{I}}}^{\alpha_n^{\text{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right)\Big/\phi(l)\right]P_l(z), \tag{27}$$

then the solution of the following equation,

$$\frac{\partial\Delta f(x,s,z)}{\partial x}=\Delta g(s,z)+\frac{1}{\pi}\iint_{K>0}\frac{\Delta g(s,z'')\Delta f(x,s,z')}{K^{1/2}(z,z',z'')}dz'dz'',$$

$$K(z,z',z'')=[(z'z''-z)^2-(z'^2-1)(z''^2-1)], \tag{28}$$

with the boundary condition

$$\Delta f(0,s,z)=0, \tag{29}$$

gives the discontinuity of the amplitude:

$$\Delta A(s,z)=\frac{1}{2iq_s}\Delta f(1,s,z), \tag{30}$$

where $\Delta g(z)$ denotes the crossed-channel discontinuity of the function $g(z)$, defined in terms of a Cauchy-integral representation:

$$g(z)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{\Delta g(z')dz'}{z'-z}.$$

The sum for $g(s,z)$ still appears quite formidable, so we will develop some methods for carrying out the summation.

First, we decompose $g(s,z)$ into a sum over the various Regge-pole contributions:

$$g(s,z)=\sum_n\int_{\alpha_n^{\text{I}}}^{\alpha_n^{\text{II}}}\tilde{g}(s,z,\lambda)\phi(\lambda)d\lambda, \tag{31}$$

where

$$\tilde{g}(s,z,\lambda)=\sum_{l=0}^{\infty}(2l+1)(\lambda-l)^{-1}\phi^{-1}(l)P_l(z). \tag{32}$$

[13] See, e.g., R. G. Newton, *The Complex j-Plane* (Benjamin, New York, 1964), Chap. 9.

Then we notice that, since $\phi(l)\sim O(e^{l\xi}/\sqrt{l})$ the series for $\tilde{g}(s,z,\lambda)$ has the same ellipse of convergence as the following series:

$$h(s,z)=\sum_{l=0}^{\infty} e^{-2l\xi}\phi(l)P_l(z)\,. \tag{33}$$

The leading singularity of $h(z)$ is weaker, however.

Following a procedure used in Paper II, we observe that

$$\frac{1}{4\pi}\int \tilde{g}(s,z',\lambda)h(z'')\,|_{z''=zz'+(z^2-1)^{1/2}(z'^2-1)^{1/2}\cos\phi'}d\Omega' =\sum_{l=0}^{\infty} e^{-2l\xi}(\lambda-l)^{-1}P_l(z)\,. \tag{34}$$

The analytic properties of the right-hand side of this equation are straightforwardly related to the properties of the Legendre function, since[14]

$$\sum_{l=0}^{\infty} e^{-2l\xi}(\lambda-l)^{-1}P_l(z) =e^{-2\lambda\xi}\left[\frac{\pi}{\sin\lambda\pi}P_\lambda(-z)+\int_{-\infty}^{2\xi}\frac{e^{(\lambda+\frac{1}{2})x}dx}{[2(\cosh x-z)]^{1/2}}\right]. \tag{35}$$

Combining the parts of $g(z)$ associated with the various Regge poles, we obtain an integral equation relating $g(z)$ to $h(z)$:

$$\frac{1}{4\pi}\int g(s,z')h(z'')d\Omega'=\sum_n \int_{\alpha_n{}^{\rm I}}^{\alpha_n{}^{\rm II}} \phi(\lambda)I(\lambda,z)d\lambda \tag{36}$$

(as well as the corresponding equation for the discontinuities), where by $I(\lambda,z)$ we mean the right-hand side of Eq. (34), which is a known function.

We now observe that $h(z)$ can be expressed in terms of the Borel transform of $\phi(\lambda)$ in a closed-form expression[15];

$$h(z)=\frac{1}{\pi}\int_{-a\xi}^{a\xi}\frac{k(w,a\xi)e^{\xi}dw}{[2(\cosh a\xi-\cosh w)]^{1/2}\{2[\cosh(\xi(1+a)-w)-z]\}^{1/2}}. \tag{37}$$

This expression then allows us to write the equation for the input equation, $g(z)$, directly in terms of the Regge trajectories and the function specifying the representation, $k(w,a,\xi)$, thereby avoiding explicit reference to summations over l:

$$\frac{1}{\pi}\int_{-a\xi}^{a\xi}\frac{k(w,a\xi)e^{\xi}}{[2(\cosh a\xi-\cosh w)]^{1/2}} \times\frac{1}{4\pi}\int\frac{g(s,z')d\Omega'}{\{2[\cosh(\xi(1+a)-w)-z'']\}^{1/2}}dw =\sum_n\int_{\alpha_n{}^{\rm I}}^{\alpha_n{}^{\rm II}}\phi(\lambda)I(\lambda,z)d\lambda\,. \tag{38}$$

This equation is the vehicle for imposing crossing symmetry in a calculation involving these representations. In order to close the system of equations, we add the equation, derived in Paper II, which determines the input function $g(z)$ in terms of the whole amplitude. Given

$$B(s,z)=-2iq_sA(s,z)=-\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(z)\,, \tag{39}$$

then

$$g(s,z)=\sum_{l=0}^{\infty}(2l+1)\ln S_l\,P_l(z) \tag{40}$$

is determined by

$$g(s,z)=-\int_0^1 f(s,z,t)dt\,, \tag{41}$$

where

$$f(s,z,t)=B(s,z)+\frac{t}{4\pi}\int f(s,z',t)B(s,z'')d\Omega' \tag{42}$$

and the corresponding equations hold for the discontinuities.[16] Finally, we impose crossing:

$$A_t(s,t)=A_s(s,t)\,|_{s\leftrightarrow t}\,, \tag{43}$$

i.e., requiring the t-channel discontinuity to be equal to the s-channel discontinuity (with a suitable change of variable) which itself is determined, in our approximation, by the s-channel Regge trajectories and the function $k(w,a\xi)$. Then one would cycle through the integral equations to obtain a consistency check, adjusting the trajectories and function(s) k to achieve optimum agreement.

It is instructive to compare the present scheme with other proposed Reggeized scattering calculations. As an example, we consider the method of Abbe *et al.*[2] In their scheme, the scattering amplitude was parametrized according to a modified Cheng representation and the analytic continuation necessary to impose crossing was carried out using the Sommerfeld-Watson transformation, applied to the amplitude in complex angular momentum space. In the present method, we avoid explicit continuation in angular momentum space since all the

[14] This function is simply related to the Khuri representation [N. N. Khuri, Phys. Rev. **130**, 429 (1963)]. In a sense, thus, our representation is a "unitarization" of the Khuri representation.

[15] We have used the generating function for $P_l(z)$ to perform this sum. See Ref. 11, p. 15.

[16] ΔB would have a δ-function part corresponding to the pole term, but we will suppress explicit reference to this complication. The modifications introduced in the formulas are quite trivial.

partial-wave series involved in the integral equations are exactly summable, before approximations are made. (In Paper II an alternative method is detailed for continuation of the partial-wave series, using only the numerical values of the partial-wave amplitudes at integer values of l. This method could be applied in much the same way as the collocation procedure suggested in Abbe *et al.*, except that a strictly numerical computation would be possible involving no unphysical amplitudes.) In our case, we obtain equations in terms of the total scattering amplitude defined along the unitarity cuts. Direct-channel Regge trajectories are explicit, as in the Abbe scheme, but in contrast to the older method, we have retained a great deal of generality in the choice of representation. Thus, when approximations are made, for example, the neglecting of distant trajectories, the functions $k(w,a\xi)$ are to be adjusted, according to the integral equations, in such a way as to minimize the damage done by the approximations. In the following sections we explain more precisely what is meant by this statement, how Regge trajectories arise in the crossed channel, and how the function $k(w,a\xi)$ relates to the background integral in the Sommerfeld-Watson continuation method.

Now we outline a technique useful for obtaining approximate solutions to Eq. (38), by describing a method for obtaining $g(z)$ "directly," in terms of a convergent infinite series. A reader uninterested in the computational details involved with these equations can proceed directly to Sec. V.

The significance of this development is that an approximation to such a series would be useful as a starting point in a numerical scheme for solving Eq. (38), proceeding by successive approximations.

We consider the problem of continuing the sum in Eq. (32) to all complex z. Consider the following contour integral:

$$\oint \frac{d\lambda'(-l_0)}{Q_{\lambda'}(\cosh\xi)(\lambda'-l)\phi(\lambda')(\lambda'-\lambda)(\lambda'-l_0)}. \tag{44}$$

First taking the contour to infinity in a sequence of finite contours in the λ plane symmetric about the real axis and passing between zeros of the function $\phi(l)$ as well as of zeros of the function $Q_l(\cosh\xi)$, one obtains an equation for the quantity

$$[\phi(l)Q_l(\cosh\xi)(\lambda-l)]^{-1}.$$

The zeros of the denominator of the integrand occur at $\lambda'=\lambda$, l, l_0, at the zeros of the $Q_{\lambda'}(\cosh\xi)$ function, which lie along the negative real axis between integers and half-integers[17]: $-n-\frac{1}{2}<\Lambda_n{}^Q(\xi)<-n$, $n=1,2,3,\ldots$, and at the zeros of the $\phi(\lambda')$ function, which, as we remarked previously, lie close to the line $\mathrm{Re}\lambda'=-\frac{1}{2}$. For any finite l_0 the contribution of the infinite contour is zero, as can be demonstrated using the asymptotic expansions of the functions $\phi(l)$ and $Q_l(\cosh\xi)$.

Finally, one obtains an expansion for $[\phi(l)(\lambda-l)]^{-1}$. The summation over l required upon insertion of this expression into Eq. (32) can be carried out explicitly.[18] Summing these various terms, we arrive at the analytic continuation of the input function $g(z)$ in terms of a convergent infinite series, which is also the solution to the integral equation (38). The solution is given in terms of the zeros of $\phi(\lambda)$ and the function values and derivatives at a sequence of points. In practical problems, it may be more efficient to solve the integral equation directly, but a few terms of the infinite-series solution would be useful as a first step.

V. INTERPRETATION OF FUNCTION $\phi(l)$: LEFT HALF l PLANE AND BACKGROUND INTEGRAL

If the positions of all the Regge zeros and poles were known precisely, then the choice of any of a wide class of possible functions $\phi(\lambda)$ would produce the same result, upon summation over all trajectories. Approximating the sum by a finite number of terms, or in some other way misrepresenting some of the terms, however, we obtain an expression whose usefulness is sensitive to the choice of $\phi(\lambda)$. Since in general we do not expect to be able to pinpoint the positions of trajectories below the first few leading Reggeons, either from experiment or from some kind of crossing calculation, we hope to use information derived from crossing to specify the amplitude, through the influence of crossing on the function $\phi(\lambda)$ in our approximation.

For the sake of illustration, suppose that the nature of our approximation consisted of the neglecting of all trajectories to the left of the line $\mathrm{Re}l=-\frac{1}{2}$. Then our representation, which began as an identity, would become

$$\begin{aligned}\ln S_l &= \left[\sum_{n=1}^{\infty}\int_{\alpha_n{}^{\mathrm{I}}}^{\alpha_n{}^{\mathrm{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right]\Big/\phi(l)\\ &\cong \left[\sum_{\mathrm{Re}\alpha_n\geq-\frac{1}{2}}\int_{\alpha_n{}^{\mathrm{I}}}^{\alpha_n{}^{\mathrm{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right]\Big/\phi(l),\end{aligned} \tag{45}$$

where the validity of the approximation is determined by the extent to which one can neglect the terms

$$\left[\sum_{\mathrm{Re}\alpha_n<-\frac{1}{2}}\int_{\alpha_n{}^{\mathrm{I}}}^{\alpha_n{}^{\mathrm{II}}}\phi(\lambda)(\lambda-l)^{-1}d\lambda\right]\Big/\phi(l) \tag{46}$$

in the partial-wave sum. The choice of $\phi(\lambda)$ will presumably be dictated by the behavior of the approximate amplitude continued to the crossed channel. As we have seen, the behavior of the function in the neighborhood

[17] E. Hille, Arkiv Mat. Astron. Fysik **13**, No. 17 (1918–1919); **17**, No. 22 (1922–1923).

[18] S. P. Creekmore, Ph.D. thesis, California Institute of Technology, 1969, pp. 107–110 (unpublished).

of the nearest crossed-channel singularity is related to coefficients in the asymptotic expansion of the partial-wave amplitude

$$a_l \sim \frac{e^{-l\xi}}{\sqrt{l}} \sum_n b_n l^{-n}, \tag{47}$$

with b_0 proportional to the residue of the crossed-channel pole, and b_n proportional to coefficients of singularities which become weaker as n increases. Insisting that the coefficients of the first N singularities be correctly specified by the sum over $\mathrm{Re}\alpha_n > -\frac{1}{2}$ would require

$$\sum_{\mathrm{Re}\alpha_n < -\frac{1}{2}} \int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}} \lambda^n \phi(\lambda) d\lambda = 0 \qquad (n=0, 1, 2, \ldots, N-1), \tag{48}$$

while the error term would be

$$\epsilon_{N,l} = \left[\sum_{\mathrm{Re}\alpha_n < -\frac{1}{2}} \int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}} (1-\lambda^N l^{-N-1}) \times (\lambda - l)^{-1} \phi(\lambda) d\lambda \right] \Big/ \phi(l). \tag{49}$$

With a simple model for the distribution of the α_n, we could obtain bounds for the error term. There is no reason to assume it to diminish indefinitely as $N \to \infty$, and in this sense our approximations are asymptotic. More precise statements of the nature and behavior of the error terms will have to await some experience in estimating the bounds on the errors. It should be clear, however, that a successful calculation will eventually have to consider, in some fashion, the behavior of the trajectories in the left half-plane in order to obtain agreement with crossing to a high accuracy.

If one correctly specifies the positions of all the Regge poles and zeros, then the sum

$$\sum_n \int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}} \phi(\lambda)(\lambda - l)^{-1} d\lambda \tag{50}$$

must have zeros in l at the same positions as the zeros of the function $\phi(l)$. If one truncates or otherwise approximates the sum over Regge poles, the approximate expression for $\ln S_l$ has an infinite number of poles lying close to the line $\mathrm{Re}l = -\frac{1}{2}$ which is the location of the background integral of the conventional Sommerfeld-Watson representation. The existence and positions of these poles is a consequence of the required asymptotic behavior of $\phi(l)$, which in turn was imposed by the requirement that the representation converge and produce the correct asymptotic behavior as $l \to \infty$, $\mathrm{Re}l > -\frac{1}{2}$, regardless of the number of trajectories retained in an approximation. Consequently, the poles along $\mathrm{Re}l \cong -\frac{1}{2}$ represent the effect of trajectories, in the left half l plane for example, which have been neglected or treated incorrectly, and in this way are related to the background integral of the Sommerfeld-Watson transformation.

VI. EXAMPLE OF ALLOWABLE k: ROUGH "BOOTSTRAP" CALCULATION

For the purpose of illustration, we exhibit a possible function $\phi(\lambda)$. Recalling the Polya representation,

$$\phi(\lambda) = \exp[(1-a)(\lambda+\tfrac{1}{2})\xi]\pi^{-1} \times \int_{-a\xi}^{a\xi} \frac{k(w,a\xi) e^{(\lambda+\frac{1}{2})w} dw}{[2(\cosh a\xi - \cosh w)]^{1/2}},$$

we set the integration density $k(w,a\xi)=1$. In some sense, this is the "simplest" possible form for $\phi(\lambda)$. In fact, it reduces to the well-known Legendre function, since[19]

$$P_\lambda(\cosh a\xi) = \frac{1}{\pi} \int_{-a\xi}^{a\xi} \frac{e^{(\lambda+\frac{1}{2})w} dw}{[2(\cosh a\xi - \cosh w)]^{1/2}}, \tag{51}$$

so that $\phi(\lambda) = \exp[(1-a)(\lambda+\frac{1}{2})\xi] P_\lambda(\cosh a\xi)$. The behaviors of $P_\lambda(\cosh a\xi)$ for large values of λ and ξ imply[20]

$$\phi(\lambda) \underset{\lambda\to\infty;\, \mathrm{Re}\lambda > -\frac{1}{2}}{\sim} e^{(\lambda+\frac{1}{2})\xi} [2\lambda\pi \sinh a\xi]^{-1/2} \tag{52a}$$

and

$$\phi(\lambda) \underset{\xi\to\infty}{\sim} e^{(\lambda+\frac{1}{2})\xi} \frac{e^{-\frac{1}{2}a\xi}\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)\Gamma(\frac{1}{2})}, \tag{52b}$$

respectively, which conform to our earlier estimates. As a consequence, we recover the same behavior for the residue of the crossed-channel pole:

$$\mathrm{Residue} = 4q_s(\sinh\xi \sinh a\xi)^{1/2} \times \sum_n \frac{1}{2i} \int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}} \exp[(1-a)(\lambda+\tfrac{1}{2})\xi] P_\lambda(\cosh a\xi) d\lambda, \tag{53}$$

where, for $t \cong M_x^2$,

$$A(s,t) \cong \mathrm{Residue}/(t-M_x^2) + (\text{regular function}).$$

The residue of a pole in the direct channel is

$$\rho_s = -\frac{(2\alpha+1)\mathrm{Im}\alpha}{q_s(d\alpha/ds)} \exp\left\{ \int_{\alpha^{\mathrm{I}}}^{\alpha^{\mathrm{II}}} \left[\frac{\phi(\lambda)}{\phi(\alpha^{\mathrm{I}})} - 1\right] \frac{d\lambda}{\lambda - \alpha^{\mathrm{I}}} + \sum_{\alpha_n^{\mathrm{I}} \neq \alpha^{\mathrm{I}}} \int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}} \frac{\phi(\lambda) d\lambda}{\phi(l)(\lambda - \alpha^{\mathrm{I}})} \right\}. \tag{54}$$

All zeros of the function $P_\lambda(\cosh a\xi)$ as a function of λ lie on the line $\mathrm{Re}\lambda = -\frac{1}{2}$, and are symmetrically distributed on each side of the real axis.[17]

Since this representation allows a specification of both direct-channel and crossed-channel behavior, one can

[19] Reference 11, Eq. (141), p. 270.

[20] The behavior for large l is given by Ref. 11, Eq. (24), p. 306. The behavior for large ξ is given by Ref. 11, Eq. (70), p. 234. Note that for $a=0$, one recovers the Cheng representation.

attempt a bootstrap calculation. The representation has the same difficulties as the Cheng and AKNS representations for large s (see Paper I); moreover, the treatment of the crossed-channel cut will have to be modified at high energies, since one expects difficulties for high-spin exchanged systems unless the crossed-channel intermediate states are Reggeized. As a consequence, consideration is restricted for the moment to a low-energy calculation.

We have in mind $\pi\pi$ scattering with a ρ resonance in both direct channel and crossed channel. We ignore the effects of isospin and inelastic thresholds. We do *not* intend this calculation to develop a reliable model of the ρ meson; instead, we are simply investigating the behavior of our representation, and we will be satisfied if we obtain dynamical quantities which are of the order of magnitude characteristic of the strong interactions.

Recalling that we have formulas for determining the residues of poles in the direct channel and crossed channel, we obtain a pair of simplified equations by dropping all but the smallest number of Regge trajectories needed to approximate the quantities. The highest Regge trajectory lies at $\mathrm{Re}\alpha=1$ at the energy of the resonance:

direct channel:

$$\rho_s \cong \frac{1}{2iq_s} \frac{(\alpha^{\mathrm{I}}-\alpha^{\mathrm{II}})(2\alpha^{\mathrm{I}}+1)}{(-d\alpha^{\mathrm{I}}/ds)}\bigg|_{s=m_\rho^2} \cong -\frac{3\,\mathrm{Im}\alpha}{q_s(d\alpha/ds)}\bigg|_{s=m_\rho^2};$$

crossed channel:

$$\rho_t P_1(\cos\theta_t)\,|_{t=m_\rho^2;\,s=m_\rho^2} \cong -4q_s\,\mathrm{Im}\alpha P_{\mathrm{Re}\alpha}(\cosh a\xi) \times\exp[\xi(1-a)(\mathrm{Re}\alpha+\tfrac{1}{2})] \times(\sinh\xi\,\sinh a\xi)^{1/2}\,|_{s=m_\rho^2;\,\mathrm{Re}\alpha=1},$$

where

$$A(s,t)\cong\rho_s\frac{P_1(\cos\theta_s)}{s-m_\rho^2}+\rho_t\frac{P_1(\cos\theta_t)}{t-m_\rho^2}.$$

We would obtain another equation if we were to write a dispersion relation for $\alpha(s)$:

$$\alpha(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{Im}\alpha(s')ds'}{(s'-s)}.$$

In a more elaborate calculation, we would assume such a relationship between $\mathrm{Re}\alpha(s)$ and $\mathrm{Im}\alpha(s)$, and investigate our equations as functions of s. At this level, however, we fix s at $s=m_\rho^2$; $d\alpha/ds$ and $\mathrm{Im}\alpha$ are undetermined. Experimentally,[21] $\alpha(0)=0.57$, so we can set $d\alpha/ds \approx(65m_\pi^2)^{-1}$. By requiring equality between the direct-channel and crossed-channel residues, we will obtain a relation which determines m_ρ^2 in terms of $d\alpha/ds$. $\mathrm{Im}\alpha$ factors out of the equation, so its value is irrelevant at this level of approximation.

We still have a free parameter: a. In our approximation, this parameter has a straightforward physical interpretation. It appears to be a measure of the fundamental strength of the interactions. As $a\to 0$ with $\mathrm{Im}\alpha$ and $d\alpha/ds$ held fixed, it can be shown that the value of m_ρ^2 goes to infinity while the residue goes to zero. The minimum value of m_ρ^2 and maximum value of residue occur as a approaches $\frac{1}{2}$. In accordance with an intuitive feeling that the strong interactions should be "as strong as possible" we set $a=\frac{1}{2}$, although it should be pointed out[22] that the value of m_ρ^2 that we obtain is still a function of, indeed perhaps very sensitive to, the approximations we are making regarding the neglecting of the lower Regge trajectories; the choice of the value of a may have a bearing on this problem, and in principle, should we possess an exact knowledge of the disposition of all Regge poles and zeros, the value of a should be irrelevant, as long as it is in the range $0<a<\frac{1}{2}$. In this regard, we should recall once again the spirit in which we have constructed this representation. Specifically, we have arranged the mathematics so that it is possible to compensate for our ignorance of direct-channel trajectories to the left of the background integral by introducing information about the crossed-channel cut. In general, in order to obtain more precise agreement between the behavior of our approximate amplitude and the requirements of crossing and unitarity, we will have to introduce an integration over a and a nonconstant $k(w,a\xi)$.

Our equation is

$$q_s^2=\frac{3}{4}\frac{\cosh\xi(d\alpha/ds)^{-1}}{e^{3\xi/4}\cosh\frac{1}{2}\xi(\sinh\xi\,\sinh\frac{1}{2}\xi)^{1/2}}, \tag{55}$$

where

$$q_s^2=\tfrac{1}{4}(m_\rho^2-4).$$

This equation can be solved by hand, using the method of successive approximations, starting with

$$m_\rho^2\cong(d\alpha/ds)^{-1}.$$

The solution is $m_\rho\cong 8.2m_\pi$ or $m_\rho\cong 1100$ MeV.

Reviewing our approximations, we might have more confidence in the direct-channel residue equation than in the crossed-channel equations. Other investigations[23] have indicated that the approximation $\mathrm{Im}\alpha=q_s(d\alpha/ds)\beta$ may be quite good, in spite of the corrections one might expect from lower trajectories. Inclusion of the lower trajectories in the crossed-channel equation will lower the value of m_ρ obtained in our "model." To obtain a

[21] This value was taken from R. J. Eden, *High-Energy Collisions of Elementary Particles* (Cambridge U. P., New York, 1967), p. 245. For the purposes of this discussion, the precise value of this constant is immaterial.

[22] Evidently, for the case $k=1$, a may approach, but not actually reach, the value $a=\frac{1}{2}$, because of the difficulties with convergence in the left half-plane. This difficulty is easily removed by modification of k in the neighborhood of $w=-a\xi$, without affecting the essential features of our argument.

[23] See, e.g., S.-Y. Chu, G. Epstein, P. Kaus, R. C. Slansky, and F. Zachariasen, Phys. Rev. 175, 2098 (1968).

quantitative evaluation of this correction, we would require an estimate of α_n and $\text{Im}\alpha_n/\text{Im}\alpha$ for the lower trajectories in question.

All known Regge particle trajectories have approximately the same slope at the origin. Taking the ρ trajectory as representative, we have $d\alpha/ds \cong (65m_\pi^2)^{-1}$. The quantity $(d\alpha/ds)^{-1/2}$ appears to fix a mass scale for strong interactions in our model, at least in regions where one of the Mandelstam variables assume a small value, since we have $m_\rho \cong \text{const}\ (d\alpha/ds)^{-1/2}$.

Even in this crude level of approximation, our bootstrap theory possess an important feature in common with observed scattering processes. As we increase the mass of the external particles, holding $d\alpha/ds$ fixed, the size of the self-consistent bound-state mass *decreases* relative to the external mass. Thus, for example, a two-"pion" system can bind a particle of mass $8.2m_\pi$, but if we allow the external particles to have the mass of the K, the system binds a particle whose mass satisfies the equation

$$q_s^2 = \frac{3}{4}\frac{65\cosh\xi}{e^{3\xi/4}\cosh\frac{1}{2}\xi(\sinh\xi\sinh\frac{1}{2}\xi)^{1/2}}\left(\frac{m_\pi^2}{m_K^2}\right)$$

for which we obtain the solution "m_ρ"$\cong 2.8m_K$.

VII. REGGEIZATION OF CROSSED CHANNEL: SINGULARITIES IN $k(w,a\xi)$

At this point, we see that it is fairly simple to arrive at a scattering-amplitude approximation which, in some sense, reproduces many of the qualitative features of scattering for low energies, where one can represent the crossed channel by the exchange of a single particle of fixed spin. Simply setting $k\equiv 1$ in our basic formulas allowed us to perform a simple (albeit somewhat unrealistic) bootstrap calculation by hand. Allowing k to vary and enforcing stricter crossing conditions while including a few more trajectories would presumably produce even more interesting results. These approaches are limited, however, unless we can show that such amplitudes can simply be written in such a form that Regge recurrences are obtained in the crossed channel as well as the direct channel.

To this end, we consider a step which is one of the most simple extensions of the single-exchange model, $k=1$. Recalling briefly the function $F(w)$, the Borel transform of the convergence factor $\phi(\lambda)$, we note that there is in principle no reason for $F(w)$ not to have singularities at other points than $w=\pm a\xi$. Equivalently, we could investigate singularities of the function $k(w,a\xi)$. Suppose $F(w)$ were to have a square-root type of divergence at some point within the interval $(-a\xi, a\xi)$. Then we could represent $\phi(\lambda)$ by a sum of two pieces, each being an integral with no singularities of the integrand in the range of the integration:

$$\phi(\lambda) = \exp[(1-a)(\lambda+\tfrac{1}{2})\xi] \times\left\{\frac{1}{\pi}\int_{-a\xi}^{a\xi}\frac{k_1(w,a\xi)e^{(\lambda+\frac{1}{2})w}dw}{[2(\cosh a\xi-\cosh w)]^{1/2}} + \frac{g}{\pi}\int_{-b\xi}^{b\xi}\frac{k_2(w,b\xi)e^{(\lambda+\frac{1}{2})w}dw}{[2(\cosh b\xi-\cosh w)]^{1/2}}\right\}, \quad (56)$$

where

$$k_1(a\xi,a\xi) = k_2(b\xi,b\xi) = 1\,, \quad 0<b<a$$

and

$$k_2(-b\xi,\, b\xi) = 0\,.$$

The asymptotic behavior of this function as $\lambda\to\infty$ is obtained simply as

$$\phi(\lambda)\sim\frac{e^{(\lambda+\frac{1}{2})\xi}}{(2\pi\lambda\,\sinh a\xi)^{1/2}}[F_1(\lambda)+fe^{(b-a)(\lambda+\frac{1}{2})\xi}F_2(\lambda)]\,, \quad \text{Re}\lambda > -\tfrac{1}{2} \quad (57)$$

where

$$F_1(\lambda)\sim 1 \quad\text{and}\quad F_2(\lambda)\sim 1\,.$$

Then the asymptotic behavior of $\ln S_l$ is obtained as follows:

$$\ln S_l = \sum_n \int_{\alpha_n^{\text{I}}}^{\alpha_n^{\text{II}}} \frac{\phi(\lambda)d\lambda}{(\lambda-l)\phi(l)} \sim e^{-(l+\frac{1}{2})\xi}(2\pi\,\sinh a\xi)^{-1/2}l^{-1/2} \times\sum_{n=0}^{\infty} f^n R_{n,l}\exp[n(b-a)(l+\tfrac{1}{2})\xi]\,. \quad (58)$$

Now, we have learned to identify a term with asymptotic behavior $e^{-(l+\frac{1}{2})\xi}/\sqrt{l}$ with the exchange of a particle whose mass satisfies the equation $\cosh\xi = 1+M^2/2q^2$. We then identify the remaining terms with the exchange of particles with the "mass" relation

$$1+\frac{M_n^2}{2q_s^2} = \cosh\{\xi[1+n(a-b)]\}\,.$$

Evidently, the "masses" obtained in this way are a function of s. Since we are dealing with a t-channel process, it would appear that the exchange masses are a function of a channel angle. Presumably, this deficiency could be corrected by a more careful approximation. However, if we take $s\to\infty$, the limit in which we expect a description in terms of a t-channel Reggeon to be most valid, then we obtain, noticing that $\xi\sim M/q$ as $q\to\infty$,

$$\cosh\{[1+n(a-b)]\xi\}\sim 1+[1+n(a-b)]^2M^2/2q^2$$

or

$$M_n\sim[1+n(a-b)]M+O(1/q^2)\,. \quad (59)$$

That is, our sequence of peculiar singularities reduces, as $s\to\infty$, to a set of legitimate exchange terms in the t channel, with positions evenly spaced in energy. Now, if we make the simple substitution

$$f=A+Bs=A'+B'\cos\theta_t,$$

we see that the residues of these poles indicate that each is composed of a sequence of partial-wave states, of angular momentum values $0\leq l\leq n$, where M_n is the mass of the pole position. The behavior of our amplitude in the t channel is very much like a sequence of Regge recurrences and daughters, even in this extremely crude approximation. A more systematic development, using the integral equations, should arrive at an amplitude with Regge poles in all channels, provided we allow the Borel transform $F(w)$ to have singularities at points in the interior of the indicator diagram, $(-a\xi, +a\xi)$. Furthermore, such an amplitude would be unitary in all channels, to the extent that it satisfies crossing, since we begin with a class of amplitudes unitary in the direct channel.[24]

Finally, we note that the device of introducing additional singularities in the Borel transform

$$k(w,a\xi)\pi^{-1}[2(\cosh a\xi-\cosh w)]^{-1/2}$$

can be used to represent quite diverse types of singularity structures as long as they are characterized by the fairly regular spacing of an infinite number of singularities in the t channel. For example, an infinite sequence of crossed-channel production thresholds could be treated in this way, with a different choice of the "angular factor" f, which we chose as $f=A+Bs$ to obtain Regge recurrences. In this picture, therefore, an infinitely rising crossed-channel Reggeon and an infinite sequence of crossed-channel thresholds are seen on the same footing—or as different manifestations of the same mathematical structure, a fact which should not be too surprising.[25]

VIII. EXTENSIONS OF THESE PRELIMINARY RESULTS

In order to extend these results to a more interesting class of reactions, the formalism must be generalized to allow for unequal external masses, multiple trajectory exchanges, inelastic channels, external particles with spin, bound states lying below threshold, and the treatment of more than one crossed channel. It does not seem that these modifications would pose any problems in principle, but they lie outside the scope of this work. Unequal-mass kinematics would be straightforward; extra complications arise from distinct t-channel and u-channel states, in which case the different-signature amplitudes must be treated individually. Scattering of particles with spin requires the separation of scalar amplitudes which are then Reggeized.

In our parametrization, the higher inelastic s-channel thresholds manifest themselves in various ways. The integral

$$\int_{\alpha^{\rm I}}^{\alpha^{\rm II}}\frac{\phi(\lambda)}{\lambda-l}d\lambda$$

develops a new piece corresponding to each new open channel. Presumably, the integrands $\phi^i(\lambda)$ in these new pieces would involve functions of $q^i{}_{\rm rel}$, the relative momentum of the inelastically produced particles, just as the purely elastic pieces involve functions of the elastic c.m. momentum q_s. The functions α and ϕ themselves develop branch cuts on the various sheets. The behavior of the functions ϕ^i as $q^i{}_{\rm rel}\rightarrow 0$ would determine the inelastic threshold behavior of the α, just as in the elastic case.

Application of unitarity would involve detailed consideration of all the various channels involved in the problem. In an intuitive sense, however, we might expect the solution of the problem to follow certain patterns. Especially, we could speculate that quasi-two-body approximations could be made to represent the effects of inelastic channels on the scattering in the basic elastic channel. For example, a three-pion channel would be represented as $\rho\pi$ two-body scattering, with an "effective" Regge trajectory equal to the original trajectory displaced in the l plane according to the rules for the addition of angular momentum, by the value of the ρ trajectory at the relative c.m. energy of the two pions bound in the fictitious ρ. Evidently, the "effective" Regge trajectory would never rise much above $\alpha=1$, and, in this sense, the scattering would always resemble a low-energy process, with the threshold behavior and rate of rise of the effective trajectory, and hence the original trajectory, being governed by the kind of two-body threshold constraints we described earlier. Therefore, the opening of new thresholds would allow the trajectories to rise indefinitely[25] at the same time, the higher inelastic thresholds would influence the behavior of the Regge trajectory in different manners on different sheets, and the coupling of the Regge trajectory to the original elastic channel would decrease with rising energy, as $\alpha^{\rm II}(s)\rightarrow\alpha^{\rm I}(s)$.

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[24] Paper I discusses some of the modifications necessary to extend these results to amplitudes which satisfy inelastic unitarity.

[25] This idea is widespread, that the opening of infinitely many new thresholds is associated with infinitely rising trajectories, perhaps by simulating low-energy behavior in some sense. The author is not presenting it here as an idea that is very new, but is only indicating how it arises naturally out of this class of Reggerized representations.