for $L(s)$ ensures that dips and peaks in the amplitude occur at points of constant t for rising s, at least to the nearest power of s. There is evidently a certain similarity to the duality approach in this discussion, although there is a distinct difference in the mathematical details, as will be clear in the subsequent papers. Paper II describes calculations made with a product representation using a large number of trajectories, in which the aforementioned properties for nonforward angles are exhibited explicitly.

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Direct-Channel Reggeization of Strong-Interaction Scattering Amplitudes. II

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Direct-channel Reggeized strong-interaction scattering amplitudes are defined in terms of a set of direct-channel phase shifts and elasticity factors. These quantities are functions of the positions of direct-channel Regge poles and certain convergence factors, which are related to crossed-channel behavior. Two mathematical techniques are discussed which can be used to continue these partial-wave expansions outside their ordinary region of convergence. Unphysical values of angular momentum are not used in the continuation methods. The numerical method requires the partial-wave amplitudes specified at positive-integer values of l only. The analytic method can be used to calculate precisely the discontinuity across the crossed-channel cut, and thus would be valuable in further theoretical work using these representations. Applications of these methods are discussed, with reference to two earlier phase-shift Reggeization schemes.

I. INTRODUCTION

IN principle, all that is necessary for a complete specification of the total scattering amplitude is knowledge of the set of s-channel partial-wave amplitudes $\{a_l(s)\}$ for non-negative integer l along with a viable method of analytic continuation of the partial-wave series in $\cos\theta_s$. We seek an improvement of the Sommerfeld-Watson transformation as a practical continuation technique for this purpose: It is clumsy in dealing with direct-channel unitarity and in determining the behavior of the amplitude around nearby crossed-channel singularities, and there is no simple way to approximate the background integral.

We shall, however, utilize certain information abstracted from the Sommerfeld-Watson method. For example, we restrict our attention to a class of representations for the scattering amplitude for which some direct-channel Regge poles are manifest, thereby treating whole sequences of observed resonances in a unified way. We also incorporate non-Regge information, insisting on certain asymptotic behavior of our amplitudes, independent of the level of approximation.

In order to satisfy unitarity we have introduced a "Regge zero" as well as a Regge pole, and thereby have lost the possibility of independently specifying a Regge residue.[1] Near a Regge pole, in fact,

$$S_l(s)\cong\exp[\phi_n(\alpha_n{}^{\mathrm{I}},s)+G(\alpha_n{}^{\mathrm{I}},s)]\left(\frac{\alpha_n{}^{\mathrm{I}}-\alpha_n{}^{\mathrm{II}}}{l-\alpha_n{}^{\mathrm{I}}}\right)$$
$$\times\prod_{m\neq n}\left[\left(\frac{\alpha_n{}^{\mathrm{I}}-\alpha_m{}^{\mathrm{II}}}{\alpha_n{}^{\mathrm{I}}-\alpha_m{}^{\mathrm{I}}}\right)\exp[\phi_m(\alpha_n{}^{\mathrm{I}},s)]\right]\quad\text{for}\quad l\cong\alpha_n{}^{\mathrm{I}}.\quad(1)$$

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This paper is the second in a series of three papers dealing with direct-channel Reggeization of partial-wave amplitudes. It is concerned with mathematical techniques useful for calculations with such amplitudes. An introductory discussion and additional references are contained in Paper I.[1]

II. ANALYTIC METHOD FOR CONTINUATION OF PARTIAL-WAVE SERIES

We shall use product representations, in which $\ln S_l$ is specified in terms of physically relevant quantities. In this case, $\ln S_l$ is a simpler function than S_l, from the point of view of practical analytic methods. Nevertheless, we must continue the sum:

$$A(s,\cos\theta_s)=[2iK(s)]^{-1}\sum_{l=0}^{\infty}(2l+1)\times[\exp(\ln S_l(s))-1]P_l(\cos\theta_s).\quad(2)$$

Let us therefore consider a class of methods for the continuation of functions defined in terms of Legendre series. It will be obvious that these methods, with appropriate modifications, can be used for expansions in terms of any set of orthogonal functions.

A discussion of the significance of the continuation methods is given in Paper III, where they are used to develop a set of equations to impose crossing and Reggeized behavior.

Consider the analytic continuation of

$$f(z)=\sum_{l=0}^{\infty}(2l+1)[e^{bl}-1]P_l(z)\quad(3)$$

[1] S. P. Creekmore, preceding paper, Phys. Rev. D **3**, 1400 (1971).

and the related function

$$g(z)=\sum_{l=0}^{\infty}(2l+1)b_lP_l(z). \quad (4)$$

The latter will yield more readily to analysis. Writing Cauchy formulas for $f(z)$ and $g(z)$,

$$f(z)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{dz'\Delta f(z')}{z'-z}, \quad (5a)$$

$$g(z)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{dz'\Delta g(z')}{z'-z}, \quad (5b)$$

we want to determine $\Delta f(z)$ in terms of $\Delta g(z)$. We define a new function:

$$f_\lambda(z)=\sum_{l=0}^{\infty}(2l+1)(e^{\lambda b_l}-1)P_l(z), \quad (6)$$

which for $0<\lambda\leq 1$ converges within the same region as the sum for $f(z)$. Trivially, $f_0(z)=0$ and $f_1(z)=f(z)$. Differentiating the new function, we obtain

$$\frac{\partial f_\lambda(z)}{\partial\lambda}=g(z)+\sum_{l=0}^{\infty}(2l+1)b_l(e^{\lambda b_l}-1)P_l(z). \quad (7)$$

Now we have obtained a new Legendre series whose coefficients are products of coefficients belonging to the two other series. The addition theorem[2] and the orthogonality condition[3] for the Legendre polynomials imply that our new series can be expressed as follows:

$$\sum_{l=0}^{\infty}(2l+1)b_l(e^{\lambda b_l}-1)P_l(z)=\frac{1}{4\pi}\int_{-1}^{1}dz'\int_0^{2\pi}d\phi \times f_\lambda(z')g(z'')\big|_{z''=zz'+(z^2-1)^{1/2}(z'^2-1)^{1/2}\cos\phi}. \quad (8)$$

Consequently, we have reduced the problem of continuation of $f(z)$ to the continuation of an auxiliary function and the solution of an integrodifferential equation:

$$\frac{\partial f_\lambda(z)}{\partial\lambda}=g(z)+\frac{1}{4\pi}\int d\Omega' f_\lambda(z')g(z''), \quad (9)$$

with the boundary condition $f_0(z)=0$.

The quantity of real physical interest is the discontinuity of $f(z)$, since the cut in $f(z)$ is related to physical intermediate states in the crossed channel. In fact, to complete the dynamical specification of an amplitude in terms of direct-channel Regge poles, we would impose unitarity in the crossed channel, and consequently we would have to know the discontinuity across the crossed-channel cut.

The following identity[4] will be useful here:

$$\frac{1}{4\pi}\int d\Omega_p(\tau_1-\hat{p}_1\cdot\hat{p})^{-1}(\tau_2-\hat{p}_2\cdot\hat{p})^{-1}=\int_{\eta_0}^{\infty}\frac{d\eta}{\eta-\hat{p}_1\cdot\hat{p}_2}K^{-1/2}(\eta,\tau_1,\tau_2),$$

where

$$\eta_0=\tau_1\tau_2+(\tau_1^2-1)^{1/2}(\tau_2^2-1)^{1/2}$$

and

$$K(x,\tau_1,\tau_2)=[(\tau_1\tau_2-x)^2-(\tau_1^2-1)(\tau_2^2-1)].$$

Using this relation, we obtain an equation specifying the discontinuity of $f(z)$ in terms of the discontinuity of $g(z)$:

$$\frac{\partial}{\partial\lambda}\Delta f_\lambda(z)=\Delta g(z)+\frac{1}{\pi}\iint_{K>0}dz'dz'' \times\Delta f_\lambda(z')\Delta g(z'')K^{-1/2}(z,z',z''), \quad (10a)$$

$$\Delta f_0(z)=0, \quad (10b)$$

$$\Delta f(z)=\Delta f_1(z). \quad (10c)$$

This relation is a linear integral equation with the kernel being the form $K^{-1/2}$, familiar from the potential-theory Mandelstam iteration scheme, integrated over a known function Δg. This method should not be confused with the Mandelstam scheme, however, since the latter involves a nonlinear integral equation which determines the singularities of the scattering amplitude from a partial knowledge of those singularities. Such a determination is on a dynamical footing and is equivalent to imposing direct-channel unitarity. Our method involves a linear equation which determines the singularities of a function, hopefully an approximation to the scattering amplitude. In any case, the function is completely specified beforehand by the set $\{b_l\}$. This method is purely mathematical; it is correct independent of direct-channel unitarity. After we determine the analytic continuation of a partial-wave expansion, we could compare its behavior with the requirements of unitarity in the crossed channel and crossing. For example, if the direct and crossed channels were identical, then we could bootstrap by adjusting $\{b_l\}$ so that the crossed-channel amplitude has resonances which correspond to the Reggeized resonances in the direct channel.

It should be clear that our method and the Mandelstam scheme have one important feature in common: The cuts propagate in exactly the same way. If $g(z)$ has a singularity at $z=\cosh\xi$, then our equation requires $f(z)$ to have an infinite sequence of singularities at the points $z=\cosh n\xi$, $n=1,2,\ldots$. This fact makes iterative methods of solution possible. More important, it means that the nearest singularity of $f(z)$ is exactly the same as that of $g(z)$.

Of course, the success of this approach depends on our being able to sum the series in Eq. (4), representing

[2] E. W. Hobson, *Spherical and Ellipsoidal Harmonics* (Chelsea, New York, 1965), p. 143.
[3] Reference 2, p. 37.

[4] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 69 (1960).

it in some form that can be analytically continued to find $\Delta g(z)$. In Secs. IV and V we present some explicit examples of this sort of summation, using representations for $b_l = \ln S_l$ which have practical significance. Should it happen that our methods of analysis are not sufficiently powerful to continue the input function $g(z)$, however, we might still be able to find some function which relates the Legendre coefficients in our sum to the coefficients in a tractable sum. In the previous example, the function e^{b_l} performed this role. We introduced a spurious parameter λ, which "turned on" our unknown sum and its cut structure.

Another example is a rational function:

$$f(z,t)=\sum_{l=0}^{\infty}(2l+1)b_l(1-tb_l)^{-1}P_l(z)\,, \tag{11}$$

$$g(z)=\sum_{l=0}^{\infty}(2l+1)b_lP_l(z)\,. \tag{12}$$

It is quite straightforward to obtain an integral equation to determine f:

$$f(z,t)=g(z)+\frac{t}{4\pi}\int d\Omega' g(z')f(z'',t)\,. \tag{13}$$

Evidently,

$$\sum_{l=0}^{\infty}(2l+1)\ln(1-b_lt)P_l(z)=-\int_0^t f(z,t')dt'\,. \tag{14}$$

These methods of analytic continuation make no mention of unphysical values of angular momentum. The integral equations, however, involve unphysical partial-wave amplitudes in intermediate steps.

III. NUMERICAL METHOD FOR CONTINUATION OF PARTIAL-WAVE SERIES

In cases where there is insufficient knowledge of the analytic structure of the partial-wave amplitude to carry out the integral-equation scheme, it should still be possible to continue the Legendre series, since in principle all that is needed for complete specification of the scattering amplitude is the set of *numerical* values of the partial-wave amplitudes for integer l. We will illustrate this fact by investigating a particular method of numerical continuation of divergent Legendre series.

Consider the set of partial sums of the Legendre series:

$$\begin{aligned}\sigma_L(z)&=\sum_{l=0}^{L}(2l+1)a_lP_l(z),\quad L=1,2,\cdots\\ \sigma_0(z)&=0\,.\end{aligned} \tag{15}$$

If the sum converges, then we have $f(z)=\lim_{L\to\infty}\sigma_L(z)<\infty$. If the sum fails to converge, we still expect that the set $\{\sigma_L(z)\}$ contains enough information to specify the analytic continuation of $f(z)$, even though the sequence does not have a limit in the classical sense.

A well-known result is $a_l\sim KQ_l(\cosh\xi)$ as $l\to\infty$, $\mathrm{Re}\,l>-\frac{1}{2}$, where $\cosh\xi=1+M_x^2/2q_s^2$ and M_x is the mass of the lowest-mass exchanged system. If $\mathrm{Re}(\eta-\xi)>0$, the sequence of partial sums $\{\sigma_L(\cosh\eta)\}$ will not converge. In any case, for large enough L,[5]

$$\sigma_L\sim \text{const}+K\sum_{l=0}^{L}(2l+1)Q_l(\cosh\xi)P_l(\cosh\eta) \tag{16a}$$

$$\sim K(L+1)(\cosh\xi-\cosh\eta)^{-1}[Q_{L+1}(\cosh\xi)P_L(\cosh\eta)-Q_L(\cosh\xi)P_{L+1}(\cosh\eta)] \tag{16b}$$

$$\sim K'e^{L(\eta-\xi)} \tag{16c}$$

or

$$\lim_{L\to\infty}\frac{\ln\sigma_L}{L}=\text{const}<\infty\,.$$

Asymptotically, at least, σ_L resembles an exponential function.

Most methods of summing divergent series involve a linear transformation of the sequence $\{\sigma_L(z)\}$ to obtain a new sequence which also converges to $f(z)$. We refer the reader to a standard text[6] in this field for discussion of the linear methods.

Little attention seems to have been paid by elementary-particle physicists to *nonlinear* transformations as summation methods for partial-wave expansions. In a remarkable paper, Shanks[7] investigated a class of nonlinear transformations of nonconvergent and slowly convergent sequences.

If $\{A_n\}$ is a sequence, Shanks defines the kth-order transform of $\{A_n\}$:

$$B_{k,n}=\frac{\begin{vmatrix}A_{n-k} & \cdots & A_n\\ \Delta A_{n-k} & \cdots & \Delta A_n\\ \Delta A_{n-k+1} & \cdots & \Delta A_{n+1}\\ \vdots & & \vdots\\ \Delta A_{n-1} & \cdots & \Delta A_{n+k-1}\end{vmatrix}}{\begin{vmatrix}1 & \cdots & 1\\ \Delta A_{n-k} & \cdots & \Delta A_n\\ \vdots & & \vdots\\ \Delta A_{n-1} & \cdots & \Delta A_{n+k-1}\end{vmatrix}}\,, \tag{17}$$

where

$$\Delta A_n=A_{n+1}-A_n\,,$$
$$k=1,2,\ldots\,,$$
$$n=k,k+1,\ldots\,,$$
$$B_{0,n}=A_n\,.$$

Shanks showed that if

$$A_n=B+\sum_{i=1}^{k}a_iq_i^{\,n}\,,$$

[5] Reference 2, p. 60.

[6] G. H. Hardy, *Divergent Series* (Oxford U. P., Oxford, England, 1949).

[7] Daniel Shanks, J. Math. Phys. **34**, 1 (1955).

then $B_{k,n}=B$. If the sequence converges, $|q_i|<1$ for all i, and B is the limit of the sequence. If the sequence diverges, $|q_i|\geq 1$ for at least one i, and Shanks calls B the *anti-limit* of the sequence.

Thus Shank's transform removes exponential behavior from a sequence. Application of a transform will generate a new sequence, so that transforms can be compounded. Shanks defined a number of special transforms, but we are most interested in the "diagonal transform"

$$e_j(A_n)=B_{n,n}. \tag{18}$$

Shanks showed that if $\{A_n\}$ is the set of partial sums of a power series, then the diagonal transform obtains the diagonal of the Padé table for the power series. Roughly speaking, the diagonal-transform sequence is a set of attempts to fit the partial sums of the series with increasing numbers of exponential components. Asymptotically, we expect this to be a justifiable characterization for a Legendre series.

The simplest example of a Legendre series representing a function with a "Regge pole" is the partial-wave expansion of the Legendre function itself[8]:

$$P_\alpha(z)=\frac{\sin\alpha\pi}{\pi}\sum_{l=0}^{\infty}(-)^l\left(\frac{1}{\alpha-l}-\frac{1}{\alpha+l+1}\right)P_l(z). \tag{19}$$

Computing the diagonal transform of the partial-sum sequence for this series, we have compared the result with a contour-integral determination of $P_\alpha(z)$ for several values of complex α and z. These results are presented in Appendix A. Inside or outside the ordinary region of convergence, the agreement is excellent within the range of practical applicability of Shanks's diagonal transform, and one would expect improvement with the introduction of multiple-precision arithmetic.

One way to use Shanks's method in a bootstrap calculation would be to expand the total amplitude in partial-wave sums in two identical channels. Continuing to a region in complex s and t where the diagonal-transform sequence can be applied to both sums, we would then compare the results.

The application of Shanks's technique is limited by the numerical accuracy of the input sequence as well as the precision retained in the intermediate arithmetic steps. Errors will propagate, and in general the accuracy of the determination of the anti-limit is less than the accuracy of the defining sequence.

Wynn[9] simplified the computations involved in applications of Shanks's transforms, eliminating the need for calculating determinants in Shanks's formulas.

IV. CHENG REPRESENTATION: APPLICATION OF ANALYTIC METHOD

Cheng constructed the first reasonable product representation as an outgrowth of a study of the asymptotic behavior of the S-matrix partial-wave projections, in a certain class of potential scattering problems.[10] Cheng considered the integral

$$\oint \frac{d\lambda}{\lambda-l}e^{\lambda\xi}\ln S_\lambda. \tag{20}$$

The integral vanishes as the contour is expanded to infinity, assuming the asymptotic behavior of $\ln S_\lambda$ is uniform. The only singularities in λ (for potential scattering) are logarithmic branch points corresponding to the Regge poles and Regge zeros. The following representation is obtained:

$$\ln S_l=\sum_n\int_{\alpha_n{}^{\mathrm{I}}}^{\alpha_n{}^{\mathrm{II}}}\frac{d\lambda}{\lambda-l}e^{(\lambda-l)\xi} \tag{21a}$$

$$=\sum_n\left[E_1((l-\alpha_n{}^{\mathrm{I}})\xi)-E_1((l-\alpha_n{}^{\mathrm{II}})\xi)\right], \tag{21b}$$

where the sum is taken over the Regge poles of the S matrix and where

$$E_1(z)=\int_1^\infty e^{-zt}\frac{dt}{t},$$

where the integral converges, and elsewhere by analytic continuation.[11]

It is immediately apparent that, for any finite number of Regge poles, the asymptotic behavior of this representation is $\ln S_l\sim O(e^{-l\xi}/l)$ rather than $\ln S_l\sim O(e^{-l\xi}/\sqrt{l})$ as $l\to\infty$, $\mathrm{Re}\,l>-\frac{1}{2}$. This fact means that the nearest singularity of the total amplitude is not automatically in the form of a pole or cut with finite discontinuity. Consequently, the Cheng representation is not as powerful as we desire for a practical calculation. Its simple analytic form, however, makes it interesting for a study of the application of some of the techniques developed in Sec. III.

We have reduced the problem of summing

$$f(z)=\sum_{l=0}^{\infty}(2l+1)\left[\exp\left(\sum_n\left[E_1((l-\alpha_n{}^{\mathrm{I}})\xi)-E_1((l-\alpha_n{}^{\mathrm{II}})\xi)\right]\right)-1\right]P_l(z) \tag{22}$$

to the problem of summing

$$g(z)=\sum_{l=0}^{\infty}(2l+1)\left\{\sum_n\left[E_1((l-\alpha_n{}^{\mathrm{I}})\xi)-E_1((l-\alpha_n{}^{\mathrm{II}})\xi)\right]\right\}P_l(z) \tag{23}$$

and solving an integrodifferential equation. We have also seen that leading singularities of the two functions, $f(z)$ and $g(z)$, are exactly the same. In a straightforward manner, we can continue the Legendre expansion for $g(z)$ in the Cheng representation. We perform the summation by considering a function $F(\alpha,z)$ which is

[8] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 167.

[9] P. Wynn, Mathematical Tables and Other Aids to Computation **10**, 91 (1956). Note that Wynn's notation differs slightly from that of Shanks.

[10] Hung Cheng, Phys. Rev. **144**, 1237 (1966).

[11] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), pp. 228–254.

simply related to $g(z)$:

$$F(\alpha,z)=\sum_{l=0}^{\infty}(2l+1)E_1((l-\alpha)\xi)P_l(z). \qquad (24)$$

For $\mathrm{Re}\alpha<0$, we obtain

$$F(\alpha,z)=\int_\xi^\infty \frac{e^{\alpha x}}{x}[\sum_l (2l+1)e^{-lx}P_l(z)]dx. \qquad (25)$$

Using the generating function for Legendre polynomials,[12] it is trivial to transform Eq. (25) into the following:

$$F(\alpha,z)=\frac{1}{\sqrt{2}}\int_\xi^\infty e^{(\alpha+\frac{1}{2})x}\sinh x(\cosh x-z)^{-3/2}x^{-1}dx \qquad (26)$$

and thus

$$\frac{\partial F}{\partial\alpha}=\frac{1}{\sqrt{2}}\int_\xi^\infty e^{(\alpha+\frac{1}{2})x}\sinh x(\cosh x-z)^{-3/2}dx. \qquad (27)$$

Integrating by parts, we obtain

$$\frac{\partial F}{\partial\alpha}=e^{(\alpha+\frac{1}{2})\xi}\sqrt{2}(\cosh\xi-z)^{-1/2}+\sqrt{2}(\alpha+\tfrac{1}{2})\int_\xi^\infty e^{(\alpha+\frac{1}{2})x}(\cosh x-z)^{-1/2}dx. \qquad (28)$$

For $-1<\mathrm{Re}\alpha<0$ the following representation of the Legendre function is valid[13]:

$$P_\alpha(-z)=-\frac{\sin\pi\alpha}{\pi\sqrt{2}}\int_{-\infty}^{\infty}e^{(\alpha+\frac{1}{2})x}(\cosh x-z)^{-1/2}dx. \qquad (29)$$

For $-1<\mathrm{Re}\alpha<0$ we can therefore rewrite Eq. (28):

$$\frac{\partial F}{\partial\alpha}=\sqrt{2}e^{(\alpha+\frac{1}{2})\xi}(\cosh\xi-z)^{-1/2}+(2\alpha+1)\pi\left[-\frac{P_\alpha(-z)}{\sin\pi\alpha}-\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{\xi}e^{(\alpha+\frac{1}{2})x}(\cosh x-z)^{-1/2}dx\right], \qquad (30)$$

which can be continued to all $\mathrm{Re}\alpha>-1$. Now

$$F(\alpha,z)=\frac{1}{\sqrt{2}}\int_\xi^\infty \sinh x(\cosh x-z)^{-3/2}x^{-1}dx+\int_{-1/2}^{\alpha}\frac{\partial F(\alpha',z)}{\partial\alpha'}d\alpha'. \qquad (31)$$

So putting in the conjugate trajectories $\alpha_n{}^{\mathrm{I}}$ and $\alpha_n{}^{\mathrm{II}}$ and summing over n, we obtain the analytic continuation of our auxiliary function, $g(z)$, defined in terms of an arbitrary set of Regge poles and zeros:

$$g(z)=\frac{1}{\sqrt{2}}\int_\xi^\infty \{\sum_{\mathrm{Re}\alpha_n<-\frac{1}{2}}[e^{\alpha_n{}^{\mathrm{I}}x}-e^{\alpha_n{}^{\mathrm{II}}x}]\}\frac{e^{\frac{1}{2}x}\sinh x\,dx}{x(\cosh x-z)^{3/2}}+[(\text{No. of }\mathrm{Re}\alpha_n{}^{\mathrm{I}}>-\tfrac{1}{2})-(\text{No. of }\mathrm{Re}\alpha_n{}^{\mathrm{II}}>-\tfrac{1}{2})]$$
$$\times\frac{1}{\sqrt{2}}\int_\xi^\infty\frac{\sinh x\,dx}{x(\cosh x-z)^{3/2}}+\sqrt{2}(\cosh\xi-z)^{-1/2}\xi^{-1}\sum_{\mathrm{Re}\alpha_n>-\frac{1}{2}}[e^{(\alpha_n{}^{\mathrm{I}}+\frac{1}{2})\xi}-e^{(\alpha_n{}^{\mathrm{II}}+\frac{1}{2})\xi}]$$
$$+\pi\sum_{\mathrm{Re}\alpha_n>-\frac{1}{2}}\int_{\alpha_n{}^{\mathrm{II}}}^{\alpha_n{}^{\mathrm{I}}}(2\alpha'+1)\left[-\frac{P_{\alpha'}(-z)}{\sin\pi\alpha'}-\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{(\alpha'+\frac{1}{2})x}dx}{(\cosh x-z)^{1/2}}\right]d\alpha'. \qquad (32)$$

This formula explicitly exhibits the singularity structure of $g(z)$. The contours of integration from $\alpha_n{}^{\mathrm{I}}$ to $\alpha_n{}^{\mathrm{II}}$ are immaterial for the purpose of finding the discontinuity, a reflection of the fact that the replacement $\ln S_l\to\ln S_l+2\pi i$ produces no change in the value of the Legendre series for the whole amplitude.

V. REPRESENTATION OF ABBE, KAUS, NATH, AND SRIVASTAVA: ANOTHER EXAMPLE OF ANALYTIC METHOD

Although the Cheng representation places the crossed-channel cut in the correct position, the amplitude fails to have the proper analytic structure near this singularity. Consequently, it is difficult to define a legitimate crossed-channel amplitude as an analytic continuation of the direct-channel Cheng amplitude. A bootstrap calculation would therefore not be sensible in the Cheng representation.

Abbe, Kaus, Nath, and Srivastava[14] devised a modification to the Cheng representation which, in a superficial sense, repairs the analytic structure of the Cheng amplitude, but at the expense of divorcing the complete specification of the crossed-channel singularity structure from the specification of the sequence of direct-channel Regge trajectories.

AKNS considered the contour integral

$$\oint\frac{d\lambda}{\lambda-l}[\ln S_\lambda-ig^2q_s{}^{-1}Q_\lambda(\cosh\bar{\xi})]e^{\lambda\xi}, \qquad (33)$$

where

$$\cosh\xi=1+4M^2/2q_s{}^2,$$
$$\cosh\bar{\xi}=1+M_x{}^2/2q_s{}^2,$$

and $-g^2$ is the t-channel residue. This integral tends to zero as the contour is expanded to infinity, assuming

[12] Reference 2, Eq. (10), p. 15.

[13] This formula can be obtained from Ref. 2, Eq. (122), p. 262.

[14] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. 140, B1595 (1965), referred to as AKNS. See also W. J. Abbe *et al.*, *ibid.* 141, 1513 (1966), as well as W. J. Abbe and G. A. Gary, *ibid.* 160, 1510 (1967).

that $\ln S_l$ tends uniformly to the asymptotic behavior derived by Cheng. This equality implies the following representation:

$$S_l=\exp[ig^2q_s^{-1}Q_l(\cosh\xi)]\prod_n S_n(l,s)\,, \qquad (34)$$

where

$$S_n(l,s)=\exp\left[\int_{\alpha_n^{\mathrm{I}}}^{\alpha_n^{\mathrm{II}}}\frac{e^{(\lambda-l)\xi}d\lambda}{\lambda-l} -\frac{ig^2e^{-(l+n)\xi}}{q_s(l+n)}P_{n-1}(\cosh\xi)\right]. \qquad (35)$$

In this case, the scattering amplitude will have a crossed-channel pole regardless of the disposition of the direct-channel Regge trajectories. The crossed-channel residue is completely independent of the direct-channel trajectories. Therefore, as far as a bootstrap calculation is concerned, this representation is no improvement over the Cheng formula, although it may produce a better fit to angular distribution in the direct channel. AKNS noted that the modified Cheng representation is more rapidly convergent in determining the residues of the direct-channel Regge poles in potential theory.

In the AKNS representation, it is straightforward to calculate the input function $g(z)$ for our analytic continuation scheme. If the model includes an infinite number of trajectories, as for example in an evenly spaced trajectory model, then the effect of the AKNS modification is to add the term

$$\frac{ig^2}{q_s}\left[Q_l(\cosh\xi)-\sum_{n=1}^{\infty}\frac{e^{-(l+n)\xi}}{l+n}P_{n-1}(\cosh\xi)\right] \qquad (36)$$

to the expression for $\ln S_l$.

Using the generating function for the Legendre polynomials, it is trivial to show that

$$\sum_{n=1}^{\infty}\frac{e^{-(l+n)\xi}}{l+n}P_{n-1}(\cosh\xi) =\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx. \qquad (37)$$

For $\mathrm{Re}\,l>-\frac{1}{2}$, $Q_l(\cosh\xi)$ can be represented as follows[15]:

$$Q_l(\cosh\xi)=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx. \qquad (38)$$

Subtracting these two equations, we obtain

$$Q_l(\cosh\xi)-\sum_{n=1}^{\infty}\frac{e^{-(l+n)\xi}}{l+n}P_{n-1}(\cosh\xi) =\frac{1}{\sqrt{2}}\int_{\bar{\xi}}^{\xi}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx. \qquad (39)$$

The integral vanishes as $\xi\to\bar{\xi}$ and the equation reduces to a representation[16] for the Legendre function of the second kind:

$$Q_l(\cosh\xi)=\sum_{n=0}^{\infty}\frac{e^{-(l+n+1)\xi}}{l+n+1}P_n(\cosh\xi)\,, \qquad (40)$$

convergent for all complex l not equal to a negative integer.

We could have written down the summation formula from intuitive arguments. The presence of the exponential factor in the contour-integral equation

$$\oint\left[\ln S_\lambda-\frac{ig^2}{q_s}Q_\lambda(\cosh\xi)\right]\frac{e^{\lambda\xi}}{\lambda-l}d\lambda=0 \qquad (41)$$

has the effect of suppressing exponential components $e^{-\mu\lambda}$ in a Laplace-transform representation of the function Q_λ for $\mu>\xi$. Accordingly, we write a convergent decomposition of Q_λ and simply drop the unwanted terms:

$$Q_l(\cosh\xi)=\frac{1}{\sqrt{2}}\int_{\bar{\xi}}^{\infty}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx\to \frac{1}{\sqrt{2}}\int_{\bar{\xi}}^{\xi}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx. \qquad (42)$$

Since this is a finite integral, it is convergent for all l. Approximating the sum

$$\sum_{n=1}^{\infty}\frac{e^{-(l+n)\xi}}{l+n}P_{n-1}(\cosh\xi)$$

by any finite number of terms, however, introduces spurious poles in the left half l plane.

In order to calculate the added contribution of the AKNS modification to the input function $g(z)$ obtained in the Cheng representation, we need to perform the following sum:

$$\sum_{l=0}^{\infty}(2l+1)\frac{1}{\sqrt{2}}\int_{\bar{\xi}}^{\xi}\frac{e^{-(l+\frac{1}{2})x}}{(\cosh x-\cosh\xi)^{1/2}}dx\,P_l(z). \qquad (43)$$

Using the generating function for $P_l(z)$, we can show that the AKNS contribution to the input function is

$$\frac{ig^2}{q_s}\left[(\cosh\xi-z)^{-1}-\frac{1}{2}\int_{\xi}^{\infty}(\cosh x-\cosh\xi)^{-1/2}\times(\cosh x-z)^{-3/2}\sinh x\,dx\right] \qquad (44)$$

in the case of an infinite number of trajectories. If we truncate the sum in the AKNS representation, the

[15] Reference 2, Eq. (82), p. 239.

[16] E. Hille, Arkiv Mat. Astron. Fysik **13**, No. 17 (1918–1919), Eq. (26), p. 18.

contribution to the input function becomes

$$\frac{ig^2}{q_s}\left[(\cosh\xi-z)^{-1}-\sum_{n=1}^{N}P_{n-1}(\cosh\xi)\times\int_{\xi}^{\infty}\frac{e^{-(n-\frac{1}{2})x}}{\sqrt{2}}(\cosh x-z)^{-3/2}dx\right]. \quad (45)$$

At this point, it should be apparent that the effect of the AKNS modification is to give the scattering amplitude reasonable analytic structure at the leading singularity simply by introducing a pole in an *ad hoc* fashion; difficulty remains, however, at the more distant singularities.

VI. INFINITE SEQUENCES OF DAUGHTER TRAJECTORIES: TOY MODEL

Other investigations have indicated that analyticity of the amplitude requires every Regge trajectory at $s=0$ to be associated with an infinite sequence of "daughter" trajectories evenly spaced below it in angular momentum.[17] It is worth considering an extrapolation of this arrangement of trajectories to physical $s\geq 4m^2$. Making this extrapolation within the context of the Cheng representation, we arrive at a toy-model scattering amplitude. While this amplitude is not intended to represent a physically realistic situation, it still serves to illustrate some of the behavior of amplitudes with a large number of Regge trajectories. Thus, properties described in a heuristic or abstract way in Paper I can now be illustrated graphically.[1] Designate by α^{I} and α^{II}, the leading pole and zero of a family of daughters. By Δ^{I} and Δ^{II} we mean the spacing of the pole-trajectory daughters and the zero-trajectory daughters, respectively.

The Cheng representation for our amplitude is

$$\ln S_l=\sum_{n=0}^{\infty}\left[E_1((l-\alpha^{\text{I}}+n\Delta^{\text{I}})\xi)-E_1((l-\alpha^{\text{II}}+n\Delta^{\text{II}})\xi)\right]. \quad (46)$$

It is straightforward to obtain a closed expression for this sum. If $\text{Re}l>\max(\text{Re}\alpha^{\text{I}},\text{Re}\alpha^{\text{II}})$, then the defining integral for $E_1(z)$ converges for all the Regge poles. Evidently, the series under the integral sign is absolutely and uniformly convergent. Exchanging the summation and integration, we obtain

$$\ln S_l=\int_1^{\infty}\left[\frac{e^{-(l-\alpha^{\text{I}})\xi t}}{1-e^{-\Delta^{\text{I}}\xi t}}-\frac{e^{-(l-\alpha^{\text{II}})\xi t}}{1-e^{-\Delta^{\text{II}}\xi t}}\right]\frac{dt}{t}. \quad (47)$$

In Appendix A, we show that the analytic continuation of this formula, valid for all l, is

$$\ln S_l=\ln\left[\Gamma\left(\frac{l-\alpha^{\text{I}}}{\Delta^{\text{I}}}\right)\Big/\Gamma\left(\frac{l-\alpha^{\text{II}}}{\Delta^{\text{II}}}\right)\right]+\gamma\left(\frac{l-\alpha^{\text{I}}}{\Delta^{\text{I}}}-\frac{l-\alpha^{\text{II}}}{\Delta^{\text{II}}}\right)+\int_0^1\frac{1}{x}\left[\frac{e^{-(l-\alpha^{\text{II}})\xi x}-1+(l-\alpha^{\text{II}})\xi x}{(1-e^{-\Delta^{\text{II}}\xi x})}-\frac{e^{-(l-\alpha^{\text{I}})\xi x}-1+(l-\alpha^{\text{I}})\xi x}{(1-e^{-\Delta^{\text{I}}\xi x})}\right]dx+\int_{\Delta^{\text{I}}}^{\Delta^{\text{II}}}\frac{dx}{x(1-e^{-\xi x})}+\frac{(l-\alpha^{\text{I}})}{\Delta^{\text{I}}}\ln(1-e^{-\Delta^{\text{I}}\xi})-\frac{(l-\alpha^{\text{II}})}{\Delta^{\text{II}}}\ln(1-e^{-\Delta^{\text{II}}\xi})+(\alpha^{\text{II}}-\alpha^{\text{I}})\xi, \quad (48)$$

where γ is the Euler-Mascheroni constant.

In Appendix C, we plot typical angular cross sections defined by partial-wave sums derived from these approximate phase shifts. Shanks's method was applied to accelerate the convergence of the Legendre series and, in some cases, to allow numerical extrapolation of the amplitudes outside the region of convergence of the expansion. The reader may note the change of shape of the curves in Figs. 2–4, as the leading zero trajectory crosses the real axis, other parameters being held constant. The phase of the amplitude in the forward direction can be varied over a wide range with such adjustments.

When the number of daughter trajectories is increased, the Cheng representation converges fairly rapidly to the infinite-trajectory amplitude. Specifically, the representation is relatively insensitive to the exact distribution of the trajectories to the left of the imaginary l axis, insofar as the qualitative behavior of the amplitude for physical angles is concerned.

The graphs also evidence a phenomenon discussed in Paper I; for nonforward angles, the amplitudes have small oscillations at about the same rate as a Legendre polynomial of the order equal to the leading trajectory. In the forward direction, and for a fairly wide range of parameters, the angular behavior is well approximated by an exponential, as an expanded logarithmic plot demonstrates in Fig. 1. Continuing outside the physical angles, we found that the exponential behavior persists in the forward direction; in the backward direction, however, the differential cross section appears to have a power-law bound.[18] As we consider higher energies, the Cheng representation is subject to the restrictions developed in Paper I concerning the behavior of the low partial waves in the case of infinitely rising trajectories.

The rate of convergence of the Legendre series was improved if account was explicitly taken of the singu-

[17] See, e.g., D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **18**, 863 (1967).

[18] These amplitudes have only t-channel singularities. In order to obtain more realistic amplitudes, we would use the signature device and two sets of direct-channel trajectories corresponding to the two signatures. For our purposes, however, the present case will suffice.

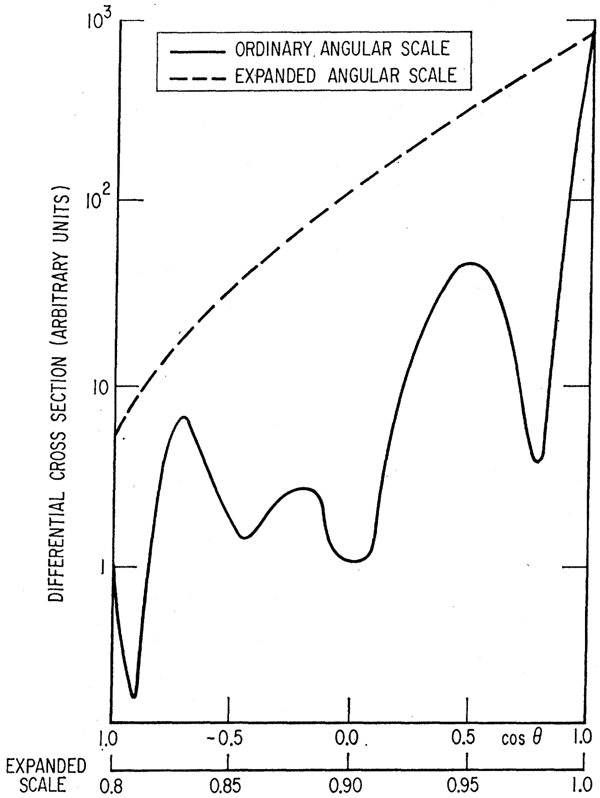

FIG. 1. Expanded graph of the forward peak in a typical Cheng-representation differential cross section with an infinite number of evenly spaced Regge poles.

larity structure of the function defined by the sum. For example, the function in the above discussion had a singularity at $z=\cosh\xi$. If

$$A(z)=\sum_{l=0}^{\infty}(2l+1)a_lP_l(z),$$

then rearranging the sum using one of the recursion relations for Legendre polynomials, we obtain

$$A(z)=(z-\cosh\xi)^{-1}\times\sum_{l=0}^{\infty}\left[la_{l-1}+(l+1)a_{l+1}-(2l+1)\cosh\xi\, a_l\right]P_l(z).$$

The new series and its diagonal transform converge faster than the old.

VII. CONCLUSIONS

We have exhibited tractable mathematical methods for precisely determining the crossed-channel discontinuity of an amplitude defined by a direct-channel partial-wave expansion. Our analytic method involves the solution of an integrodifferential equation, in terms of an input function which is simple to calculate exactly. We have given several examples of such computations. These methods are particularly appropriate for calculations involving amplitudes suggested in Paper I. In Paper III, we generalize the observations of Papers I and II to a class of amplitudes which exhibit Regge poles in both direct and crossed channels, while still having a great deal of flexibility.

APPENDIX A

$P_\alpha(z)$, the Legendre function of the first kind, was calculated for various values of α and z, in order to compare values obtained from three methods: first, the ordinary partial-wave expansion; second, the sequence

TABLE I. Comparison of values of $P_\alpha(z)$ obtained from three methods, for $\alpha=1.5+0.5i$, $z=2+2i$. From a contour-integral calculation, $P_\alpha(z)=-1.32+3.47i$.

Original partial-wave sequence	Diagonal-transform sequence
$0+0i$	$-0.0087+0.0104i$
$-0.17+0.098i$	$-0.06128+0.05643i$
$2.51-0.285i$	$-1.74-0.514i$
$-7.33+11.0i$	$-2.20+3.38i$
$24.9-5.74i$	$-1.44+3.45i$
$-93.6-33.2i$	$-1.34+3.46i$
$162.8+365.5i$	$-1.32+3.46i$
$604.2-1590.9i$	$-1.32+3.47i$
$-7030.2+3047.7i$	
$32788.2+13718.5i$	
$-60537.2-157778.9i$	
$-351862.0+739498.4i$	
$3831230.4-1247462.1i$	
a	
⋮	

[a] Absolute value of real or imaginary part greater than 10^7.

TABLE II. Comparison of values of $P_\alpha(z)$ obtained from three methods, for $\alpha=2.5+0.5i$, $z=2+2i$. From a contour-integral calculation, $P_\alpha(z)=-16.1+5.71i$.

Original partial-wave sequence	Diagonal-transform sequence
$0+0i$	
$0.0836-0.0295i$	$0.00433-0.00640i$
$-0.805-0.357i$	$0.00519-0.00638i$
$8.295+7.892i$	$0.0415-0.0536i$
$-60.51+8.024i$	$2.083+1.614i$
$98.56+68.27i$	$-13.0+14.7i$
$-183.86-438.72i$	$-16.3+6.60i$
$-773.28+1783.3i$	$-16.1+5.81i$
$7779.4-3124.8i$	$-16.1+5.73i$
$-34930.5-15399.6i$	$-16.1+5.73i$
$61582.4+167881.7i$	$-16.1+5.70i$
$377298.6-770005.4i$	
$-3992259.7+1268998.1i$	
a	
⋮	

[a] Absolute value of real or imaginary part greater than 10^7.

TABLE III. Comparison of values of $P_\alpha(z)$ obtained from three methods, for $\alpha=1.5+0.5i$, $z=1.5+i$. From a contour-integral calculation, $P_\alpha(z)=0.283+1.95i$.

Original partial-wave sequence	Diagonal-transform sequence
0+0*i*	−0.0159+0.0128*i*
−0.172+0.0983*i*	0.0983+0.0918*i*
1.46−0.477*i*	−1.23+2.45*i*
−0.710+4.98*i*	0.230+2.02*i*
4.33−2.30*i*	0.276+1.95*i*
−12.6+5.04*i*	0.282+1.95*i*
31.2+14.4*i*	0.283+1.95*i*
−46.4−74.5*i*	0.283+1.95*i*
−24.7+252*i*	
498−529*i*	
−2131+349*i*	
5735+3113*i*	
−8046−18319*i*	
−15192+60241*i*	
153678−119808*i*	
−616533+70.4*i*	
1549716+1208627*i*	

TABLE IV. Comparison of values of $P_\alpha(z)$ obtained from three methods, for $\alpha=1.5+0.5i$, $z=1.5+1.5i$. From a contour-integral calculation, $P_\alpha(z)=-0.568+2.30i$.

Original partial-wave sequence	Diagonal-transform sequence
0+0*i*	−0.0115+0.0134*i*
−0.172+0.0983*i*	−0.0692+0.0774*i*
1.84−0.189*i*	−1.855+0.734*i*
−3.91+6.00*i*	−0.782+2.29*i*
10.16−0.315*i*	−0.592+2.29*i*
−26.84−12.15*i*	−0.570+2.30*i*
24.27+89.66*i*	−0.568+2.30*i*
149.39−250.69*i*	−0.568+2.30*i*
−973.8+213.1*i*	
2843+2011*i*	
−1543−12368*i*	
−29646+34498*i*	
168409−4274*i*	
−435034−458025*i*	
−185603+2386023*i*	
7251866−5573949*i*	
a ⋮	

ᵃ Absolute value of real or imaginary part greater than 10^7.

TABLE V. Comparison of values of $P_\alpha(z)$ obtained from three methods, for $\alpha=1.5+0.5i$, $z=1.5i$. From a contour-integral calculation, $P_\alpha(z)=-1.06+0.334i$.

Original partial-wave sequence	Diagonal-transform sequence
0+0*i*	−0.000215+0.0257*i*
−0.172+0.0980*i*	−0.0323+0.103*i*
0.978+0.961*i*	−0.646+0.227*i*
−2.80−2.06*i*	−0.985+0.359*i*
−4.36+4.60*i*	−1.06+0.347*i*
8.93+6.21*i*	−1.06+0.335*i*
11.22−24.16*i*	−1.06+0.333*i*
−63.87−28.06*i*	−1.06+0.333*i*
−71.31+167.16*i*	
454.29+182.51*i*	
487.86−1269.49*i*	
−3603.7−1346.3*i*	
−3786.0+10366*i*	
30176.9+10811.8*i*	
31292.2−88726.1*i*	
−263077−91577*i*	
−270497+785741*i*	

obtained from the partial-wave expansion using Shanks's diagonal-transform method; third, a contour-integral representation. The results are shown in Tables I–V.

The partial-wave expansion of $P_\alpha(z)$ is

$$P_\alpha(z)=\lim_{n\to\infty} S_n(z,\alpha) \quad \text{(where the limit exists)},$$

where

$$S_n(z,\alpha)=\frac{\sin\pi\alpha}{\pi}\sum_{l=0}^{n}(-)^l\left(\frac{1}{\alpha-l}-\frac{1}{\alpha+l+1}\right)P_l(z).$$

This expansion converges for all z in the segment $-1\leq z\leq 1$ and for all nonintegral α.

APPENDIX B

Consider

$$\ln S_l=\sum_{n=0}^{\infty}\left[E_1((l-\alpha^{\rm I}+n\Delta^{\rm I})\xi)-E_1((l-\alpha^{\rm II}+n\Delta^{\rm II})\xi)\right].$$

In order to find an analytic continuation of this formula valid for all l, we use a representation of the E_1 function valid for all values of the argument[19]:

$$E_1(z)=-\gamma-\ln z+\int_0^1(1-e^{-zt})t^{-1}dt,$$

where γ is the Euler-Mascheroni constant. Now we consider the Nth partial sum of the infinite series in the Cheng representation:

$$\ln S_l^{(N)}=\sum_{n=0}^{N}\left[E_1((l-\alpha^{\rm I}+n\Delta^{\rm I})\xi)-E_1((l-\alpha^{\rm II}+n\Delta^{\rm II})\xi)\right]=-\ln\left(\frac{l-\alpha^{\rm I}}{l-\alpha^{\rm II}}\right)-\sum_{n=1}^{N}\ln\left(\frac{l-\alpha^{\rm I}+n\Delta^{\rm I}}{l-\alpha^{\rm II}+n\Delta^{\rm II}}\right)$$
$$+\int_0^1\left[e^{-(l-\alpha^{\rm II})\xi x}\left(\sum_{n=0}^{N}e^{-n\Delta^{\rm II}\xi x}\right)-e^{-(l-\alpha^{\rm I})\xi x}\left(\sum_{n=0}^{N}e^{-n\Delta^{\rm I}\xi x}\right)\right]\frac{dx}{x}.$$

[19] Reference 11, Eq. (5.1.11), p. 229.

This form is not convergent as we take $N\to\infty$. Rearranging the sum and the integral, we obtain

$$\ln S_l^{(N)} = -(N+1)\ln\left(\frac{\Delta^{\mathrm{I}}}{\Delta^{\mathrm{II}}}\right) - \ln\left[\frac{(l-\alpha^{\mathrm{I}})/\Delta^{\mathrm{I}}}{(l-\alpha^{\mathrm{II}})/\Delta^{\mathrm{II}}}\right] - \sum_{n=1}^{N}\ln\left[\frac{1+(l-\alpha^{\mathrm{I}})/n\Delta^{\mathrm{I}}}{1+(l-\alpha^{\mathrm{II}})/n\Delta^{\mathrm{II}}}\right]$$

$$+\int_0^1 \frac{1}{x}\left[e^{-(l-\alpha^{\mathrm{II}})\xi x}-1+(l-\alpha^{\mathrm{II}})\xi x\right]\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{II}}\xi x}\right)dx - \int_0^1 \frac{1}{x}\left[e^{-(l-\alpha^{\mathrm{I}})\xi x}-1+(l-\alpha^{\mathrm{I}})\xi x\right]\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{I}}\xi x}\right)dx$$

$$+\int_0^1 \frac{1}{x}\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{II}}\xi x} - \sum_{n=0}^{N} e^{-n\Delta^{\mathrm{I}}\xi x}\right)dx - (l-\alpha^{\mathrm{II}})\xi\int_0^1 \sum_{n=0}^{N} e^{-n\Delta^{\mathrm{II}}\xi x}dx + (l-\alpha^{\mathrm{I}})\xi\int_0^1 \sum_{n=0}^{N} e^{-n\Delta^{\mathrm{I}}\xi x}dx.$$

Substituting

$$\int_0^1 \sum_{n=0}^{N} e^{-n\Delta\xi x}dx = 1 - \frac{1}{\Delta\xi}\sum_{n=1}^{N}\frac{e^{-n\Delta\xi}-1}{n},$$

as well as adding and subtracting

$$\left(\frac{l-\alpha^{\mathrm{I}}}{\Delta^{\mathrm{I}}} - \frac{l-\alpha^{\mathrm{II}}}{\Delta^{\mathrm{II}}}\right)\left(\gamma - \sum_{n=1}^{N}\frac{1}{n}\right),$$

we obtain

$$\ln S_l^{(N)} = -(N+1)\ln\left(\frac{\Delta^{\mathrm{I}}}{\Delta^{\mathrm{II}}}\right) - \left\{\sum_{n=1}^{N}\left[\ln\left(\frac{1+(l-\alpha^{\mathrm{I}})/n\Delta^{\mathrm{I}}}{1+(l-\alpha^{\mathrm{II}})/n\Delta^{\mathrm{II}}}\right) - \frac{l-\alpha^{\mathrm{I}}}{n\Delta^{\mathrm{I}}} + \frac{l-\alpha^{\mathrm{II}}}{n\Delta^{\mathrm{II}}}\right]\right.$$

$$\left. + \ln\left[\frac{(l-\alpha^{\mathrm{I}})/\Delta^{\mathrm{I}}}{(l-\alpha^{\mathrm{II}})/\Delta^{\mathrm{II}}}\right] + \gamma\left(\frac{l-\alpha^{\mathrm{I}}}{\Delta^{\mathrm{I}}} - \frac{l-\alpha^{\mathrm{II}}}{\Delta^{\mathrm{II}}}\right)\right\} + \sum_{n=1}^{N}\left(\frac{l-\alpha^{\mathrm{II}}}{n\Delta^{\mathrm{II}}} - \frac{l-\alpha^{\mathrm{I}}}{n\Delta^{\mathrm{I}}}\right) + \gamma\left(\frac{l-\alpha^{\mathrm{I}}}{\Delta^{\mathrm{I}}} - \frac{l-\alpha^{\mathrm{II}}}{\Delta^{\mathrm{II}}}\right)$$

$$+\int_0^1 \frac{1}{x}\left[e^{-(l-\alpha^{\mathrm{II}})\xi x}-1+(l-\alpha^{\mathrm{II}})\xi x\right]\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{II}}\xi x}\right)dx - \int_0^1 \frac{1}{x}\left[e^{-(l-\alpha^{\mathrm{I}})\xi x}-1+(l-\alpha^{\mathrm{I}})\xi x\right]\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{I}}\xi x}\right)dx$$

$$-\frac{(l-\alpha^{\mathrm{I}})}{\Delta^{\mathrm{I}}}\sum_{n=1}^{N}\left(\frac{e^{-n\Delta^{\mathrm{I}}\xi}-1}{n}\right) + \frac{(l-\alpha^{\mathrm{II}})}{\Delta^{\mathrm{II}}}\sum_{n=1}^{N}\left(\frac{e^{-n\Delta^{\mathrm{II}}\xi}-1}{n}\right) + (\alpha^{\mathrm{II}}-\alpha^{\mathrm{I}})\xi - \int_0^1 \frac{dt}{t}\left(\sum_{n=0}^{N} e^{-n\Delta^{\mathrm{I}}\xi t} - \sum_{n=0}^{N} e^{-n\Delta^{\mathrm{II}}\xi t}\right).$$

Now notice that the expression in the curly brackets { } is the logarithm of the Nth partial product of Euler's infinite-product representation of the Γ function[20]:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right],$$

valid for $|z|<\infty$. Define

$$\Gamma_N(z) = \left\{ze^{\gamma z}\prod_{n=1}^{N}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right]\right\}^{-1}$$

so that

$$\lim_{N\to\infty}\Gamma_N(z) = \Gamma(z).$$

Also define

$$\ln_N(1-z) = -\sum_{n=1}^{N} z^n/n,$$

so that $\lim_{N\to\infty}\ln_N(1-z) = \ln(1-z)$ where the series converges.

Finally, rewriting the integral

$$\int_0^1 \frac{1}{x}\sum_{n=0}^{N}\left(e^{-n\Delta^{\mathrm{II}}\xi x} - e^{-n\Delta^{\mathrm{I}}\xi x}\right)dx = \int_{\Delta^{\mathrm{I}}}^{\Delta^{\mathrm{II}}} \frac{1}{x}\sum_{n=0}^{N} e^{-n\xi x}dx - (N+1)\ln\left(\frac{\Delta^{\mathrm{II}}}{\Delta^{\mathrm{I}}}\right),$$

[20] Reference 11, Eq. (6.1.3), p. 255.

we obtain for our expression

$$\ln S_l^{(N)} = \ln\left[\Gamma_N\left(\frac{l-\alpha^{\rm I}}{\Delta^{\rm I}}\right) \Big/ \Gamma_N\left(\frac{l-\alpha^{\rm II}}{\Delta^{\rm II}}\right)\right]$$

$$+\gamma\left(\frac{l-\alpha^{\rm I}}{\Delta^{\rm I}} - \frac{l-\alpha^{\rm II}}{\Delta^{\rm II}}\right)$$

$$+\int_0^1 \frac{1}{x}[e^{-(l-\alpha^{\rm II})\xi x} - 1 + (l-\alpha^{\rm II})\xi x](\sum_{n=0}^{N} e^{-n\Delta^{\rm II}\xi x})dx$$

$$-\int_0^1 \frac{1}{x}[e^{-(l-\alpha^{\rm I})\xi x} - 1 + (l-\alpha^{\rm I})\xi x](\sum_{n=0}^{N} e^{-n\Delta^{\rm I}\xi x})dx$$

$$+\int_{\Delta^{\rm I}}^{\Delta^{\rm II}} \frac{1}{x}\sum_{n=0}^{N} e^{-n\xi x}dx + \frac{l-\alpha^{\rm I}}{\Delta^{\rm I}}\ln_N(1-e^{-\Delta^{\rm I}\xi})$$

$$-\frac{l-\alpha^{\rm II}}{\Delta^{\rm II}}\ln_N(1-e^{-\Delta^{\rm II}\xi}) + (\alpha^{\rm II}-\alpha^{\rm I})\xi .$$

Now all the terms are convergent as $N\to\infty$. Using the expansion

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1},$$

we can write

$$\ln S_l = \lim_{N\to\infty} S_l^{(N)},$$

obtaining Eq. (48).

This expression can be continued to any point in the l plane except for the singularities of the Γ functions. We have arrived at an elastic-scattering "model" involving an infinite number of Regge trajectories.[21]

APPENDIX C

Using the Cheng representation, scattering cross sections are calculated for infinite numbers of evenly spaced Regge pole and zero trajectories. $\alpha^{\rm I}$ and $\alpha^{\rm II}$ refer to the leading pole and zero trajectories, respectively, while $\Delta^{\rm I}$ and $\Delta^{\rm II}$ refer to the spacings of their respective daughters. ξ is a kinematic variable indicating the position of the nearest crossed-channel cut (see text). Figures 2–4 show the variation in these cross sections as the leading zero trajectory crosses the real axis and

[21] These formulas simplify for real, integer spacing of daughter trajectories. In fact, one obtains a recursion relation relating the S matrix for a single trajectory S_l^1 to the S matrix for an infinite number of integer-spaced daughters S_l^∞ lying below a leading trajectory identical to the single trajectory of S_l^1: $S_{l+1}^\infty = S_l^\infty S_l^{1*}$. This relation is also valid for a slightly larger class of product representations. Transforming back to $\cos\theta$ space, one obtains for $A^\infty(\cos\theta)$ an integral equation involving a kernel given in terms of $A^1(\cos\theta)$

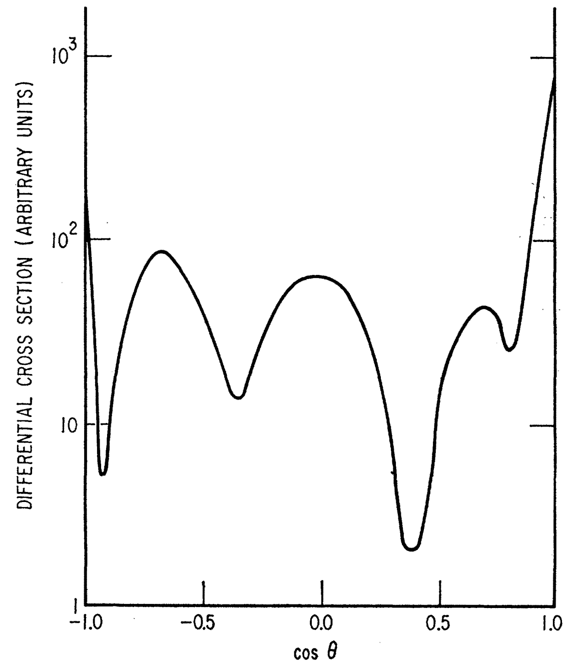

FIG. 2. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{\rm I}=4.1+0.4i$; $\alpha^{\rm II}=4.1-0.4i$; $\Delta^{\rm I}=\Delta^{\rm II}=1.2$; $\xi=0.35$. (See text for explanation of parameters.)

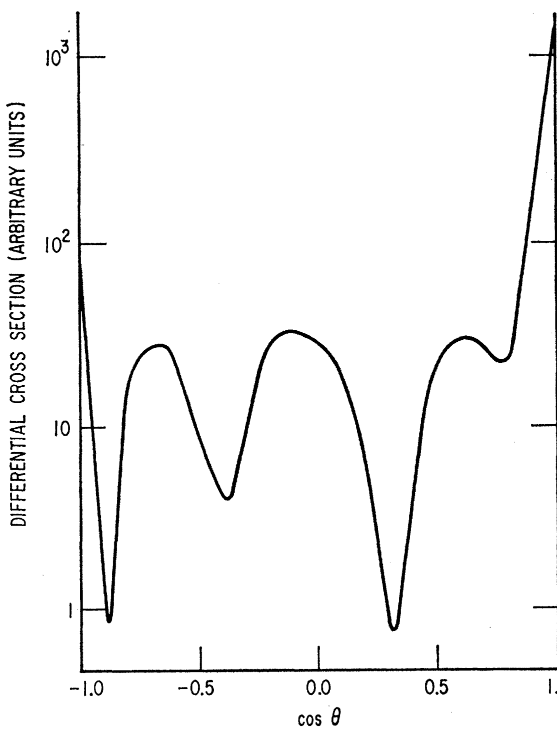

FIG. 3. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{\rm I}=4.1+0.4i$; $\alpha^{\rm II}=4.1-0.2i$; $\Delta^{\rm I}=\Delta^{\rm II}=1.2$; $\xi=0.35$.

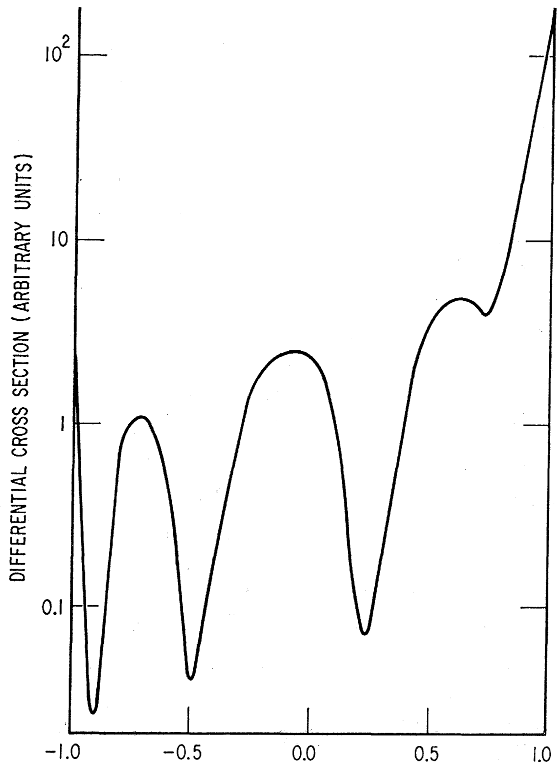

FIG. 4. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{I}=4.1+0.4i$; $\alpha^{II}=4.1+0.2i$; $\Delta^{I}=\Delta^{II}=1.2$; $\xi=0.35$.

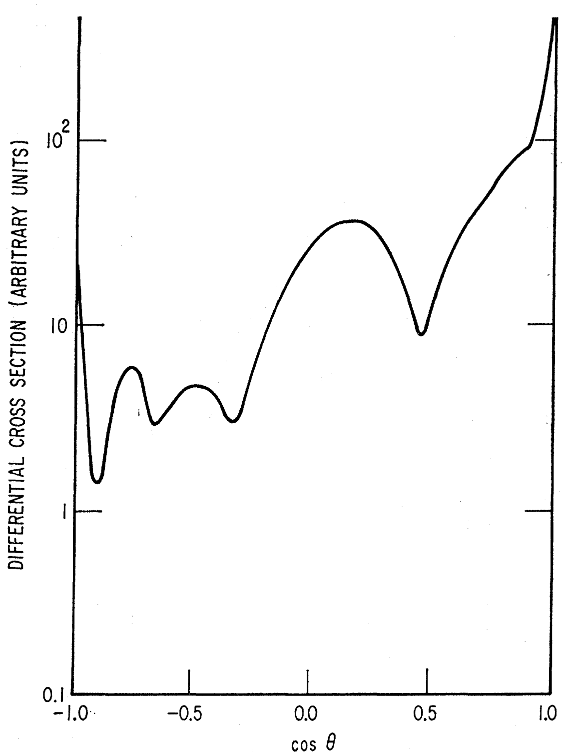

FIG. 5. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{I}=5.7+0.04i$; $\alpha^{II}=5.7-0.04i$; $\Delta^{I}=1.5+0.1i$; $\Delta^{II}=1.5-0.1i$; $\xi=0.29$.

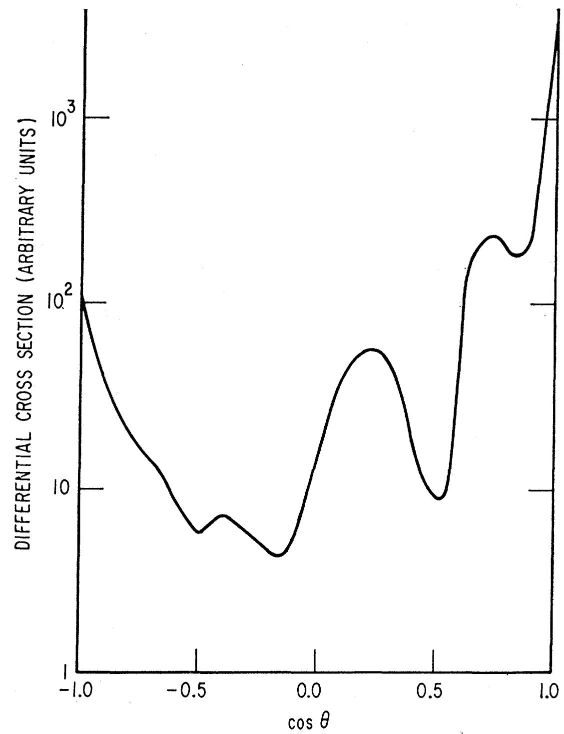

FIG. 6. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{I}=6.1+0.04i$; $\alpha^{II}=6.1-0.04i$; $\Delta^{I}=1.14+0.1i$; $\Delta^{II}=1.14-0.1i$; $\xi=0.28$.

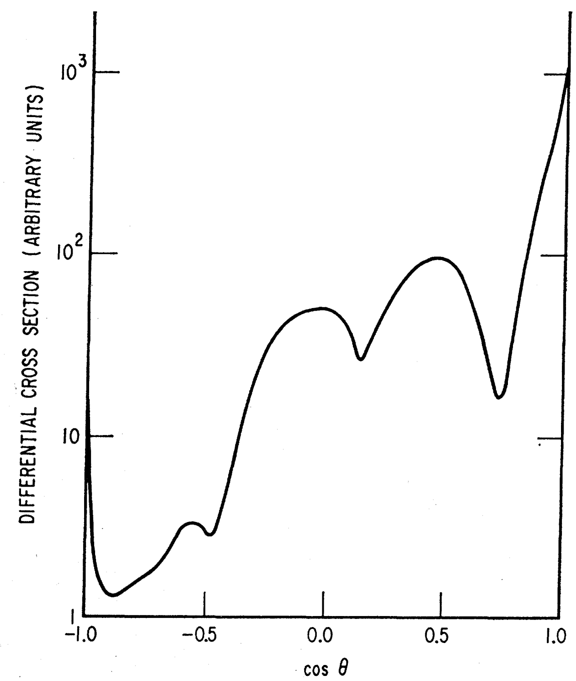

FIG. 7. Differential cross section for the Cheng representation with an infinite number of evenly spaced Regge poles. $\alpha^{I}=6.5+0.04i$; $\alpha^{II}=6.5-0.04i$; $\Delta^{I}=1.13+0.1i$; $\Delta^{II}=1.13-0.1i$; $\xi=0.27$.

approaches the leading pole trajectory. Figures 5–7 illustrate the cross-section behavior as Reα is increased and Δ is made complex.