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Direct-Channel Reggeization of Strong-Interaction Scattering Amplitudes. I*

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The construction of Reggeized strong-interaction amplitudes is discussed in a series of three papers. In this paper (Paper I), the amplitude is expanded in a partial-wave series with the individual partial-wave amplitudes represented as products over direct-channel Regge-pole contributions. Unitarity bounds automatically hold for direct-channel processes. Such representations are shown to account for the essential features of both elastic and inelastic scattering, in terms of the Regge trajectories on several Riemann sheets. It is shown that difficulties usually associated with rising trajectories are naturally avoided in this formulation.

I. INTRODUCTION

A REASONABLE model of strong-interaction scattering must have the following properties:

(1) Unitarity in all channels.

(2) Correct positioning of direct- and crossed-channel singularities.

(3) Correct energy and angular behavior in the neighborhood of singularities. Phase shifts should have the proper threshold behavior, and resonances should have the proper spin.

(4) Regge asymptotic behavior in any channel for large values of the channel energy.

(5) A simple and accurate treatment of inelasticity.

(6) An explicit and workable mathematical procedure for relating the parameters describing one channel to those of another channel.

Decomposing the amplitude into a set of partial-wave amplitudes expressed in terms of phase shifts, it is trivial to meet the requirement of direct-channel unitarity, at least below the first inelastic threshold. Expressing these phase shifts in terms of direct-channel Regge trajectories results in an amplitude which has Regge behavior in the crossed-channel energy variables. Assembling a large enough ensemble of trajectories which rise with energy, one can simulate Regge behavior in the direct-channel energy variable. This article is the first of a series of three papers which investigate the problems associated with practical utilization of a model defined in terms of direct-channel Reggeized phase shifts. Of special concern are the difficulties associated with making such a model consistent with the above six properties.

In this paper, we discuss some properties of such Reggeized representations. We show how direct-channel inelasticity can be related to the behavior of the trajectories on their various Riemann sheets. We also investigate the consequences of large numbers of infinitely rising trajectories, in terms of the forward scattering peak as well as nonforward scattering.

Some of the results discussed in Paper I have been pointed out by other authors in isolated contexts. They are presented again in order to demonstrate how they fit naturally into the mathematical framework presented in these papers, and to provide the background necessary for the exposition in Paper II which culminates in an extremely general Reggeized representation derived in Paper III.

In Paper II, we exhibit mathematical techniques which can be used to continue analytically our partial-wave series, determining the discontinuity across the crossed-channel cuts and the residues of the crossed-channel poles. We illustrate these methods by discussing applications to some unitary Regge representations previously proposed by other authors.

Using these techniques, one could perform bootstrap calculations, imposing approximate unitarity in the crossed channels and obtaining the proper spin and mass of both direct- and crossed-channel resonances, all by manipulating the parameters describing the direct-channel phase shifts.

In Paper II, we also investigate some of the properties of a specific model with an infinite number of evenly spaced daughter trajectories, using one of the older representations.

With some care in the choice of Regge representations, the direct-channel and crossed-channel singularities can be properly positioned, and correct direct-channel phase-shift threshold behavior obtained, *provided* the Regge trajectory functions are constrained to have correct threshold behavior. This fact has been demonstrated previously by several other authors.[1] In Paper III, we exhibit a representation with the additional feature that one obtains correct threshold behavior of the direct-channel trajectories, by the imposition of reasonable behavior of the whole amplitude in the neighborhood of the nearest crossed-channel singularity. Applying some of the mathematical

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[1] See, e.g., Hung Cheng, Phys. Rev. **144**, 1237 (1966); W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, *ibid.* **140**, B1595 (1965).

techniques of Paper II, we perform a simple bootstrap calculation to illustrate our new representation; in so doing, we obtain a relationship between the derivative of the Regge trajectory, the self-consistent bound-state mass, and the external mass.

Earlier workers derived the asymptotic behavior of the partial-wave projection of the S matrix for large values of $|l|$. In the case where $\mathrm{Re}\,l>\frac{1}{2}$,[2]

$$\ln S_l \sim -i\rho(s)K(s)\pi^{1/2}e^{-(l+\frac{1}{2})\xi(s)}q_s^{-2}[2l\sinh\xi(s)]^{-1/2}, \quad (1)$$

where $\cosh\xi(s)=1+M_x^2/(2q_s^2)$. M_x is the mass and $\rho(s)$ is the strength of the nearest crossed-channel singularity. $K(s)$ is a kinematic factor. In the case where $\mathrm{Re}\,l<-\frac{1}{2}$, for a wide class of potential scattering problems, it has been demonstrated[3] that the behavior is

$$S_l \sim e^{2i\pi(l+\frac{1}{2})}. \quad (2)$$

This asymptotic behavior precludes the possibility of pushing the Regge background integral to infinity in the left half-plane and replacing it by a sum over Regge poles lying to the left of $\mathrm{Re}\,l=-\frac{1}{2}$.[4] Such a process would not be convergent, which implies that the background integral is not simply related to the Regge poles that lie to the left of the background integral contour.

Consider an amplitude Reggeized in the s channel. If this function is represented, according to the Sommerfeld-Watson transformation, by a sum of Regge poles and a background integral, then satisfactory behavior can be obtained for high-energy t-channel scattering in the forward direction. Difficulties arise in using such approximations to represent the scattering amplitude for low-energy t-channel processes, and for the s channel in general. In particular, s-channel unitarity cannot be rigorously imposed on such sums in any simple way, and the position and discontinuity of the t-channel cut is incorrect. Making adjustments between the Regge-pole terms and the background integral, one can correct the position of the t-channel cut,[5] but the elastic unitarity condition

$$\mathrm{Im}\,a_l(s)=K(s)|a_l(s)|^2, \quad l=0,1,2,\ldots \qquad s\geq 4M^2 \quad (3)$$

remains a hopeless tangle of Regge-pole and background terms unless the amplitude is approximated by a small number of terms in the Sommerfeld-Watson expansion. The background integral is non-negligible in these cases. Therefore, the Sommerfeld-Watson method is not a particularly well-suited description for these regions.

The most natural way around these difficulties is to use a product form for S_l:

$$S_l=e^{G(l,s)}\prod_n \{[l-\alpha_n^*(s)]/[l-\alpha_n(s)]\}e^{\phi_{l,n}(s)}, \quad (4)$$

where the product is taken over the direct-channel Regge poles of the amplitude. $G(l,s)$ is an entire function (for the case of potential scattering) and the $\{\phi_{l,n}\}$ are convergence factors.[6] The usefulness of such representations depends on a judicious choice of $G(l,s)$ and the $\{\phi_{l,n}(s)\}$, as well as on the availability of efficient computational techniques for summing the partial-wave series described in these terms. This requirement implies that alternatives to the Sommerfeld-Watson transformation are necessary. Paper II is devoted to such methods.

Paper III represents the convergence factors $\{\phi_{l,n}(s)\}$ in an extremely general fashion, consistent with analyticity and proper asymptotic behavior in the right half l plane. It is then shown that for large values of the direct-channel energy variable, families of crossed-channel Regge poles appear in a natural way, out of the representation which began in terms of Regge poles explicit in the direct channel alone.

An important consequence of the mathematics of Paper II and the representation of Paper III is that one obtains a set of integral equations, for which the solutions are automatically unitary and Reggeized in the direct channel, allowing in a simple way for Reggeization in the crossed channel and for imposition of crossing symmetry. All remaining partial-wave series are exactly summable in closed form, so that the resulting model equations are to be solved directly in terms of the Regge trajectories and the continuous functions (developed in Paper III) which specify the particular representations.

II. PHASE-SHIFT REGGEIZATION

It is convenient to define a new function:

$$\psi_{l,n}=\ln\{[l-\alpha_n^*(s)]/[l-\alpha_n(s)]\}+\phi_{l,n}(s), \quad (5)$$

which may be regarded as being the contribution of the nth s-channel Regge pole to the lth phase shift, with the reservation that such an identification is dependent on the decomposition which produces the convergence factors $\{\phi_{l,n}\}$, which are not unique. The only finite

[2] This relation is obtained from the well-known asymptotic behavior of the partial-wave amplitude: $a_l=(S_l-1)/K(s)$.

[3] Hung Cheng and Tai Tsun Wu, Phys. Rev. 144, 1232 (1966).

[4] See Ref. 3 and Hung Cheng (Ref. 1); also N. N. Khuri, Phys. Rev. 130, 429 (1963); S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1962).

[5] N. N. Khuri (Ref. 4).

[6] Earlier investigations of product representations were made by B. R. Desai and R. G. Newton, Phys. Rev. 129, 1445 (1963), by Hung Cheng (Ref. 1), and by Abbe *et al.* (Ref. 1). For a discussion of relativistic bootstrap calculations using one particular (modified) form of the Cheng representation, see W. J. Abbe *et al.*, Phys. Rev. 141, 1513 (1966); W. J. Abbe and G. A. Gary, *ibid.* 160, 1510 (1967). Some properties of the modification of Abbe *et al.* are discussed in Papers II and III of this series, to illustrate mathematical techniques developed by this author for the study of infinite-product representations, in the general case. Desai and Newton discuss the justification of an infinite-product representation of the S matrix in the case of potential theory. Cheng and Abbe *et al.* analyze approximations derived from certain particular choices of the functions $G(l,s)$ and $\{\phi_{l,n}\}$. Paper III discusses a very general representation of these functions.

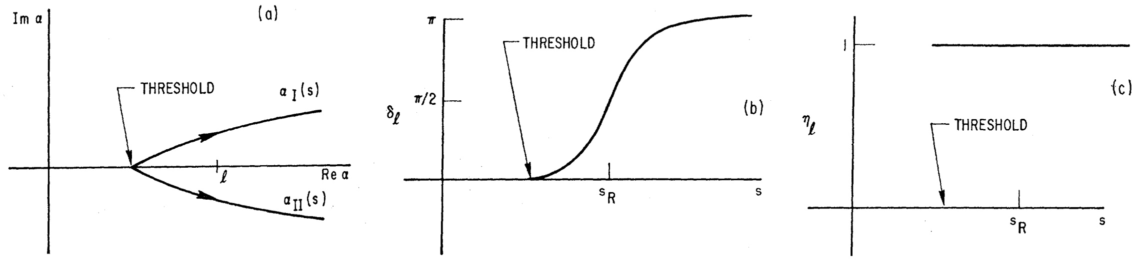

FIG. 1. (a) Pole and zero trajectories in the complex plane for the case of a completely elastic resonance. (b) Phase shift as a function of energy for the pole and zero trajectories in (a). (c) Elasticity as a function of energy for the pole and zero trajectories in (a).

singularities allowed for $\psi_{l,n}$ are logarithmic singularities at the Regge pole and Regge zero. The requirement that the Regge singularity in the amplitude be a simple pole fixes the normalization of $\psi_{l,n}$. This fact has a bearing on the size of $\rho(s)$, the t-channel residue of the whole amplitude, as will be made clear in Paper II.

$\psi_{l,n}(s)$ is not defined as an analytic function of s unless we replace $\alpha_n^*(s)$ by the analytic function agreeing with $\alpha_n^*(s)$ for real s above threshold. This function we denote by $\alpha_n^{II}(s)$; it is the analytic continuation of $\alpha_n(s)$ defined by taking a counterclockwise circuit around the threshold branch point onto the second physical sheet. To avoid any confusion, we denote the function $\alpha_n(s)$ on the first physical sheet by $\alpha_n^{I}(s)$. We can then express $\psi_{l,n}(s)$ in terms of the boundary values of the analytic functions:

$$\psi_{l,n}(s)=\lim_{\epsilon\to 0}\ln\{[l-\alpha_n^{II}(\sigma)]/[l-\alpha_n^{I}(\sigma)]\}+\phi_{l,n}(s),\quad \sigma=s+i\epsilon,\quad s\geq 4M^2. \tag{6}$$

III. TREATMENT OF INELASTIC THRESHOLDS

In generalizing to relativistic scattering problems, we propose to retain the product form explicitly exhibiting the pole and zero behavior of S_l in the l plane. Below the elastic threshold, $\mathrm{Im}\alpha_n^{I}=0$; at the elastic threshold, $\mathrm{Im}\alpha_n^{I}=\mathrm{Im}\alpha_n^{II}=0$ and $\mathrm{Re}\alpha_n^{I}=\mathrm{Re}\alpha_n^{II}$; for real s above the elastic threshold and below the first inelastic threshold, $\alpha_n^{I*}=\alpha_n^{II}$; above the first inelastic threshold, $\alpha_n^{I*}\neq\alpha_n^{II}$. Evidently, we need to know the Regge trajectory function defined on two sheets in order to determine the behavior of the scattering amplitude for a process with an open inelastic channel, or for a description of the amplitude below the elastic threshold.

In principle, we could determine the Regge trajectory on the second sheet by analytic continuation of the trajectory on the first sheet. In practice, however, knowledge of the analytic structure of the $\{\alpha_n(s)\}$ is as remote as the complete solution of the scattering problem with many-body production processes. As a consequence, a practical Regge parametrization of two-body elastic scattering will involve specification of a new set of trajectory functions for every new threshold; to the extent of our ability to determine them, these new functions will be arbitrary, within certain limits.

As an illustration of the relation between the multi-sheeted Regge trajectory and the behavior of the partial-wave amplitudes, consider a partial-wave amplitude in an energy region for which only one resonance dominates the amplitude:

$$S_l=\eta_l e^{2i\delta_l}\approx Z_l[l-\alpha^{II}(s)]/[l-\alpha^{I}(s)]. \tag{7}$$

Evidently,

$$\eta_l=|[l-\alpha^{II}(s)]/[l-\alpha^{I}(s)]|\,|Z_l| \tag{8}$$

and

$$\delta_l=\tfrac{1}{2}\arg(Z_l)+\tfrac{1}{2}\arg[l-\alpha^{II}(s)]-\tfrac{1}{2}\arg[l-\alpha^{I}(s)]. \tag{9}$$

If this energy is below the first inelastic threshold, then $\alpha^{I*}=\alpha^{II}$ and $|Z_l|=1$ identically. Figure 1 shows the phase shift and trajectories representative of this case. As the energy rises through resonance, the two trajectory functions pass on either side of $\mathrm{Re}\alpha=l$, with $\mathrm{Im}\alpha^{I}(s)=-\mathrm{Im}\alpha^{II}(s)$.

Consequently, the elasticity factor remains constant: $\eta_l=1$, and the phase shift follows the pattern characteristic of an elastic resonance, rising from $\delta_l=\frac{1}{2}\arg Z_l$

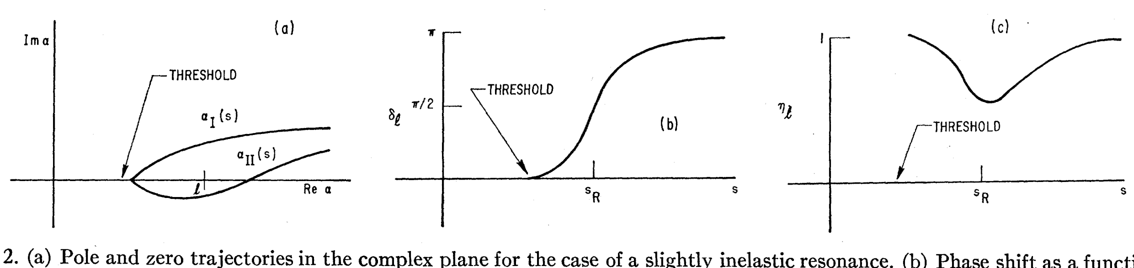

FIG. 2. (a) Pole and zero trajectories in the complex plane for the case of a slightly inelastic resonance. (b) Phase shift as a function of energy for the pole and zero trajectories in (a). (c) Elasticity as a function of energy for the pole and zero trajectories in (a).

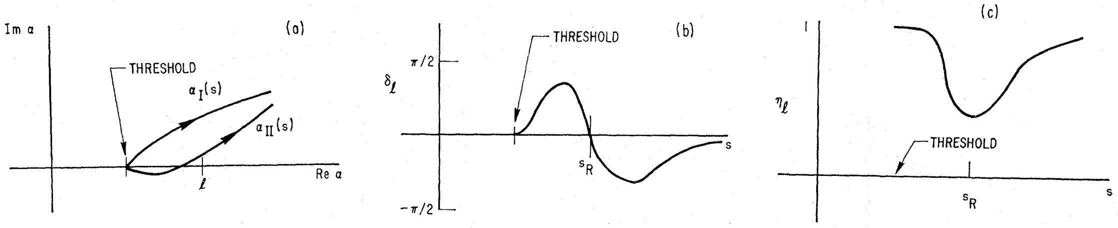

FIG. 3. (a) Pole and zero trajectories as a function of energy for the case of a very inelastic resonance. (b) Phase shift as a function of energy for the pole and zero trajectories in (a). (c) Elasticity as a function of energy for the pole and zero trajectories in (a).

through $\delta_l=\frac{1}{2}\arg Z_l+\frac{1}{2}\pi$ at resonance, and reaching $\delta_l=\frac{1}{2}\arg Z_l+\pi$ at energies well past the resonance.

Now consider the case where the resonance energy is above one or more inelastic thresholds. There are two cases which appear quite different in terms of the behavior of the phase shift, but which appear very similar when interpreted in terms of the behavior of the multisheeted Regge trajectory.

First, there is the case where the two trajectory functions pass on opposite sides of $\mathrm{Re}\alpha=l$. Figure 2 illustrates this case. For simplicity, let us assume that $\mathrm{Re}\alpha^{\mathrm{I}}=\mathrm{Re}\alpha^{\mathrm{II}}$. The condition $\eta_l<1$ then implies $|\mathrm{Im}\alpha^{\mathrm{I}}|>|\mathrm{Im}\alpha^{\mathrm{II}}|$. In order to have a rising phase shift as the energy goes through resonance, we must have $\mathrm{Im}\alpha^{\mathrm{I}}>0$. Evidently, the elasticity will have a dip at resonance. The phase shift will rise through $\delta_l=\frac{1}{2}\arg Z_l+\frac{1}{2}\pi$ to $\delta_l=\frac{1}{2}\arg Z_l+\pi$ at energies past resonance, just as in the completely elastic case.

The second possibility is for the two trajectories to pass on the same side of $\mathrm{Re}\alpha=l$. In this case, Fig. 3 depicts the expected behavior. Again, the elasticity will dip at resonance. Slightly before resonance, the phase shift will rise to a value smaller than $\delta_l=\frac{1}{2}\arg Z_l+\frac{1}{2}\pi$; exactly at resonance it will pass downward through $\delta_l=\frac{1}{2}\arg Z_l$, going lower at energies slightly past resonances; finally, the phase shift will approach the value it had for energies well before resonance: $\delta_l=\frac{1}{2}\arg Z_l$.

Evidently, the difference between a "slightly inelastic" and a "very inelastic" resonance is simply that the trajectory on the second sheet passes $\mathrm{Re}\alpha=l$ on different sides of the real axis. Roughly speaking, the first elastic threshold causes equal and opposite increments in the imaginary parts of α^{I} and α^{II}, whereas any inelastic threshold causes increases in $\mathrm{Im}\alpha^{\mathrm{I}}$ greater than the increase in $-\mathrm{Im}\alpha^{\mathrm{II}}$. As more inelastic channels open, $\mathrm{Im}\alpha^{\mathrm{I}}$ and $\mathrm{Im}\alpha^{\mathrm{II}}$ come to have the same sign.

Examples of the three different kinds of resonances can be taken from elastic $N\pi$ scattering at low energy. Up to about 2000-MeV kinetic energy, this scattering has been analyzed in terms of phase shifts and elasticity factors.[7] First, consider scattering in the $I=\frac{3}{2}$ channel. In the P_{33} amplitude, there appear to be two resonances; one, the well-known 1238 MeV at about 200 MeV above threshold, is completely elastic; the other, at about 900 MeV above threshold, is very inelastic, and shows up as a dip in the elasticity factor. In the F_{37} amplitude, there is a very inelastic resonance at about 1400 MeV above threshold, which produces a dip in the elasticity factor and an oscillation of the phase shift about zero (this resonance is presumably the Regge recurrence of the 1238 resonance). An example of a slightly inelastic resonance can be seen in the $I=\frac{1}{2}$ channel; in the D_{13} amplitude at about 600 MeV, the phase shift rises through 90° and there is a large dip in the elasticity factor. In a schematic fashion, Fig. 4 shows Regge trajectories which will account for this behavior.

Unitarity constrains the behavior of the trajectory on the two sheets. For example, if we assume $\mathrm{Re}\alpha^{\mathrm{I}}=\mathrm{Re}\alpha^{\mathrm{II}}$ and that a one-trajectory approximation is valid, then the unitarity bound on this approximate amplitude implies $|\mathrm{Im}\alpha^{\mathrm{II}}|\leq|\mathrm{Im}\alpha^{\mathrm{I}}|$.[8] The precise effect of unitarity is modified by the presence of other trajectories. In any approximation, the consequences of unitarity will appear to depend on the form of the functions $G(l,s)$ and $\{\phi_{l,n}\}$ in our amplitude.

IV. INFINITELY RISING TRAJECTORIES

Some authors have speculated on possible difficulties with amplitudes described in terms of infinitely rising trajectories.[9] Such problems are easily avoided in the inelastic generalizations of the product representations. As more inelastic channels open, the $\{\alpha_n^{\mathrm{II}}\}$ cross the real axis and the resonances cease causing the phase shifts to jump by π. Furthermore, since the effect of a Regge pole is negated by a Regge zero, any of the other difficulties appearing to arise from infinitely rising Regge trajectories could be eliminated by the corresponding poles and zeros approaching coincidence. Since $\alpha^{\mathrm{II}}(s)\to\alpha^{\mathrm{I}}(s)$ does not imply $\mathrm{Im}\alpha(s)\to 0$, there is considerable freedom left in the allowed representations for the Regge trajectory functions.

Consider the partial-wave amplitudes for α fixed and $l\to\infty$, $\mathrm{Re}l>-\frac{1}{2}$. We must impose $\ln S_l\sim O(e^{-l\xi}/\sqrt{l})$.

[7] C. Lovelace, CERN Report No. TH. 837, 1967 (unpublished).

[8] It has been pointed out to this author that the relationship between the behavior of the Regge-trajectory functions on the various sheets and the phase shift near resonance was discussed by Loyal Durand, III, Phys. Rev. **166**, 1680 (1968). Dr. Peter Kaus has informed the author that he also has noticed this interpretation of phase-shift behavior in terms of Regge-pole and zero trajectories.

[9] S. Mandelstam, Phys. Rev. **166**, 1539 (1968); N. N. Khuri, Phys. Rev. Letters **18**, 1094 (1967).

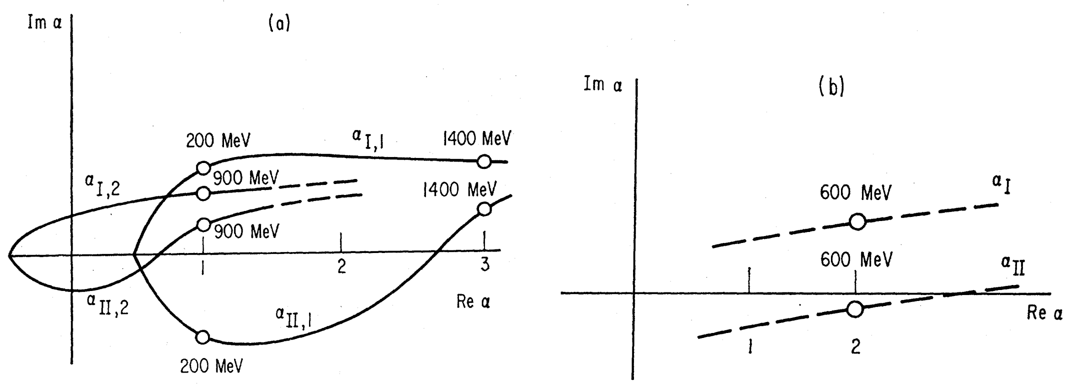

FIG. 4. (a) Schematic Regge pole and zero trajectories which account for some of the low-energy $N\pi$ phase-shift results in the P_{33} and F_{37} amplitudes. $\alpha=J-\frac{1}{2}$. (b) Schematic Regge pole and zero trajectories which account for some of the low-energy $N\pi$ phase-shift results in the D_{13} amplitude. $\alpha=J+\frac{1}{2}$.

A simple way to ensure this behavior would be to choose the $\{\phi_{l,n}\}$ so that

$$\phi_{l,n}(s)=\phi_n{}^{\mathrm{I}}(l,s)-\phi_n{}^{\mathrm{II}}(l,s)\,, \tag{10a}$$

$$G(l,s)=0\,, \tag{10b}$$

and

$$\phi_n{}^i(l,s)-\ln[l-\alpha_n{}^i(s)]\sim O(e^{-l\xi}/\sqrt{l})\,,\quad i=\mathrm{I},\,\mathrm{II} \quad \mathrm{Re}l>-\tfrac{1}{2}\,,\quad l\to\infty\,. \tag{10c}$$

We can specialize to a convenient class of representations without losing too much generality as follows:

$$\phi_n{}^i(l,s)=g(l-\alpha_n{}^i(s))\,,\quad i=\mathrm{I},\mathrm{II}\,. \tag{11}$$

Taking $\alpha_n{}^i(s)$ as the origin for the l plane, we can write the asymptotic condition as follows:

$$g(z)-\ln z \underset{z\to+\infty}{\sim} O(e^{-z\xi}/\sqrt{z})\,, \tag{12a}$$

$$z=l-\alpha\,. \tag{12b}$$

If we assume that $g(z)$ is an entire function, we have constructed a representation which has only Regge singularities and automatically has the correct asymptotic behavior in the right half l plane, regardless of the number of terms kept in the product representation.

In the case of realistic particle scattering, it is often supposed that the real part of a Regge trajectory rises indefinitely with rising energy. To study the effect of this behavior on our representation, we fix l and allow α to rise, causing the variable $z=l-\alpha$ to become large and negative. If $g(z)$ is an entire function, then exponential falloff for large positive z would imply exponential growth for large negative z. At first sight, this class of representations would therefore appear to have the unfortunate feature that partial-wave amplitudes are influenced more by remote Regge trajectories than by nearby trajectories. We expect the opposite effect to be the actual case: Since the leading Regge trajectories correspond to intermediate states with high spins, these trajectories should become decoupled from the amplitude.

In our approximate model, the full contribution of a single Regge pole would be

$$[(l-\alpha^{\mathrm{II}})/(l-\alpha^{\mathrm{I}})]\exp[g(l-\alpha^{\mathrm{I}})-g(l-\alpha^{\mathrm{II}})]\,. \tag{13}$$

This formula indicates that the high-energy difficulties of our model would be resolved if any of the following three conditions were satisfied:

First, $\alpha(s)$ might rise but still grow more slowly than $\sqrt{s}$. Since $\xi\sim s^{-1/2}$, the product $\xi\alpha$ would remain bounded.

Second, $\alpha^{\mathrm{II}}(s)$ might approach $\alpha^{\mathrm{I}}(s)$ rapidly enough for $|\alpha^{\mathrm{I}}(s)-\alpha^{\mathrm{II}}(s)|\sim O(|e^{-\alpha\xi}|)$.[8] The remote trajectories would fade away as the pole trajectory approaches coincidence with the zero trajectory. This situation would contrast with potential theory, where the trajectories turn around rather than disappear.

Third, the relativistic problem allows Regge cuts as well as poles. Cuts in $g(z)$ might shield lower partial waves from distant trajectories. For example,[10]

$$h(z)=\int_0^1\frac{dt}{t}\{1-\exp[-(\tfrac{1}{2}z\xi t+\tfrac{1}{2}(z^2\xi^2+iA)^{1/2}t)]\}-\gamma-\ln z \tag{14a}$$

$$=g(z)-\ln z \tag{14b}$$

is a function falling off exponentially for $\mathrm{Re}z>0$ and not blowing up exponentially for $\mathrm{Re}z<0$. In this particular case, the contribution from the distant Regge pole would reduce to $(l-\alpha^{\mathrm{II}})/(l-\alpha^{\mathrm{I}})$.

A priori, we have no idea how to approximate $\phi_{l,n}(s)$ when l is in the midst of a cluster of Regge poles and cuts. Presumably, the $\{\phi_{l,n}(s)\}$ would be developed in a full dynamical calculation as discussed in Paper III; considerations such as the foregoing would be helpful, however, in arriving at a reasonable first approximation.

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[10] γ is the Euler-Mascheroni constant and A is a small positive number.

V. DISTRIBUTION OF IMPORTANT TRAJECTORIES AT LARGE ENERGY

In order to get a rough picture of the distribution of s-channel Regge poles, we consider the partial-wave amplitudes of $s^{\alpha(t)}$ for large s. Assuming a linear approximation for $\alpha(t)$, and using a well-known expansion of $e^{a\cos\theta}$ in a series of Legendre polynomials, one obtains approximate partial-wave amplitudes a_l which are proportional to Bessel functions of half-integer order. For small l we can use an expansion for Bessel functions of large argument[11]:

$$\ln(a_l/a_0)=\ln(I_{l+1/2}(a)/I_{1/2}(a)) \quad (15a)$$

$$=\ln\left(1-\frac{\mu-1}{8a}+\frac{(\mu-1)(\mu-9)}{2!(8a)^2}-\cdots\right) \quad (15b)$$

$$\cong -l(l+1)/2a, \quad (15c)$$

where $\mu=4(l+\frac{1}{2})^2$ and $a=2q_s^2\alpha'(0)\ln s$ and the approximation is valid for $l(l+1)\ll 2a$.

Examining the behavior of the amplitudes around $l^2\cong 2a$, we consider the function $\ln(I_{(2a)^{1/2}+\delta}(a)/I_{1/2}(a))$.

For large a we can replace the numerator by an asymptotic expansion of Bessel functions of large order[12]:

$$I_\nu(\nu z)\sim e^{\nu\eta}(2\pi\nu)^{-1/2}(1+z^2)^{-1/4}\left(1+\sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right), \quad (16a)$$

$$\eta=(1+z^2)^{1/2}+\ln\{z/[1+(1+z^2)^{1/2}]\}, \quad (16b)$$

$$t=(1+z^2)^{-1/2}, \quad (16c)$$

where $u_k(t)$ is a kth-order polynomial in t.

The expansion holds uniformly with respect to z in the sector $|\arg z|\leq\frac{1}{2}\pi-\epsilon$, $\epsilon>0$. Making the replacements $\nu z=a$, $\nu=(2a)^{1/2}+\delta$ and $z=(a/2)^{1/2}-\frac{1}{2}\delta$, we obtain

$$\ln[I_{(2a)^{1/2}+\delta}(a)/I_{1/2}(a)]\cong -1-\delta(2/a)^{1/2}, \quad (17)$$

which is the result we would have obtained from a naive extrapolation of our formula for small l. For $l\cong(2a)^{1/2}$, we have demonstrated that

$$\ln[a_l(s)/a_0(s)]\cong -1-2[l-(2a)^{1/2}]/(2a)^{1/2}. \quad (18)$$

For $[l-(2a)^{1/2}]/(2a)^{1/2}\gg 1$ it is easy to show[13] that the partial-wave amplitudes decrease at an even faster rate. Figure 5 shows a plot of $\ln[a_l(s)/a_0(s)]$ vs $l/(2a)^{1/2}$, illustrating our conclusion that the significant partial waves are concentrated in the region $l\lesssim(2a)^{1/2}=L(s)$.

[11] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), Eq. (9.7.1), p. 377.

[12] Reference 11, Eq. (9.7.7), p. 378.

[13] For example, by considering the uniform asymptotic expansion for large orders, making the replacement $a=\nu z$, holding a fixed and allowing ν to increase.

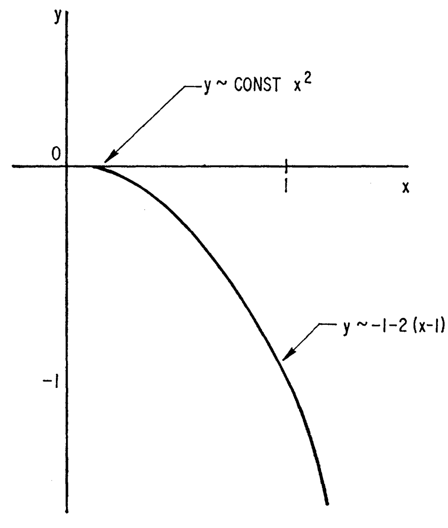

FIG. 5. $y=\ln[a_l(s)/a_0(s)]$ vs $x=l/(2a)^{1/2}$, where a_l is the lth partial-wave amplitude for $s^{\alpha(t)}$ and $a=2q_s^2\alpha'(0)\ln s$.

A more intuitive way to arrive at these results derives from a study of the Legendre polynomials near the forward direction. We can get a rough estimate of the maximum order required if we determine the order of a Legendre polynomial whose largest zero coincides with the angle in which the exponential $s^{\alpha(t)}$ decreases by one e-fold[14]:

$$L(s)\cong 1.2[\alpha'(0)s\ln s]^{1/2}, \quad (19)$$

which is consistent with our previous estimate.[15] This conclusion is the same as commonly given in discussions of impact-parameter representations of the scattering amplitude.

We now can abstract several results germane to the developments described later in Papers II and III. First, we point out a relationship between the derivative of the t-channel trajectory and the value of the lowest exchange mass. For $l\cong L(s)$, the partial waves are decreasing with an exponential factor

$$\exp\{-(l-L)/[4q_s^2\alpha'(0)\ln s]^{1/2}\} \quad (20)$$

for large s. It is appropriate to compare this behavior with the exponential factor obtained in our earlier discussion of large-l behavior:

$$\exp[-(l-L)\xi(s)]\cong\exp[-(l-L)(M_x^2/q_s^2)^{1/2}], \quad (21)$$

where M_x is the mass of the lowest-mass exchanged

[14] E. W. Hobson, *Spherical and Ellipsoidal Harmonics* (Chelsea, New York, 1965), p. 407.

[15] If one attributes the forward peak in t to exponential behavior in the Reggeized couplings (as is sometimes believed to be the case in exchanges involving the Pomeranchon, which some people believe to have a flat trajectory), then our argument is modified in a nonessential way: $\alpha'(0)s\ln s$ is replaced by the coefficient of t in the exponential.

system. Equality is evidently impossible:

$$M_x{}^2 \neq [4\alpha'(0)\ \ln s]^{-1}. \tag{22}$$

This failure is not particularly significant because the important properties of the partial-wave expansion of $s^{\alpha(t)}$ are related to the lower partial waves. $s^{\alpha(t)}$ has no t-channel cut for linear $\alpha(t)$ and therefore its Legendre coefficients necessarily have incorrect behavior as $l \to \infty$. $\ln s$ is not a rapidly varying function; if we were to speculate on a correct form of the above equation we might assume

$$M_x{}^2 \cong \text{const}/\alpha'(0) \tag{23}$$

in order to join the intermediate-l behavior to the required asymptotic behavior in l. In Paper II we will see that this equation arises in a more reasonable calculation.

Now if we assume that the scattering is determined by a collection of singularities in the l plane and that these singularities are confined to a region left of an effective "leading trajectory" at $l = L(s)$, we can make the identification $L(s) \sim \text{const}\, s^{1/2}$ to the nearest power in s.

The growth rate for $L(s)$ is consistent with the exponential behavior in t which is dictated by the crossed-channel Regge formalism, again, to the nearest power in s. This growth rate is a constraint on the behavior of the $\{\alpha_n(s)\}$ *only in an average sense.* As we pointed out earlier, it does not necessarily follow that *all* trajectories are strictly constrained by this asymptotic behavior; the influence of the higher trajectories, rising, for example, linearly with s, could be reduced by the associated Regge poles and Regge zeros approaching coincidence. $L(s)$ would then be the position of the highest important trajectory, roughly speaking.

We have given general arguments that an amplitude can be parametrized by direct-channel Regge poles and cuts and still have behavior appropriate to crossed-channel Reggeization. It should be clear, however, that such amplitudes necessitate a large collection of direct-channel Regge singularities; this observation means that a *simple* bootstrap could work only at values of the Mandelstam variables for which the amplitude is dominated by a single trajectory in all channels. In the following papers, we require that a direct-channel Reggeized amplitude reduce, for low energy, to a conventional fixed-spin amplitude, which manifests appropriate behavior near poles and threshold branch points. In this sense, the calculation would resemble old-fashioned bootstraps involving fixed-spin particles, although there may be significant differences in the treatment of distant singularities.

VI. HIGH-ENERGY BEHAVIOR AT NONFORWARD ANGLES

It is instructive to consider another consequence of direct-channel Reggeization: the nature of the deviations from idealized crossed-channel Regge behavior. In the forward direction, all Legendre polynomials have the same sign, while away from the forward direction, the Legendre polynomials of different order oscillate with different frequencies and will tend to cancel. In the previous discussion, we constructed a forward peak by imposing constraints on an otherwise unspecified set of partial-wave amplitudes. As we consider increasing angles, conformation to exponential behavior implies constraints on higher moments of our partial-wave amplitudes (or, equivalently, on higher derivatives of the amplitude in the forward direction). Recall that we have assumed that there is a fundamental change in the nature of the partial-wave amplitudes as l increases past $L(s)$, the "leading edge" of the region of important resonances. It would be possible to arrange the set of Regge trajectories and cuts to produce some desired set of amplitudes a_l for $l \lesssim L(s)$ but there would be little possibility of flexibility in a_l for $l > L(s)$. As a consequence, for large-angle scattering, there would be deviation from the behavior expected from a simple crossed-channel Reggeized model. This deviation would be an oscillating term similar to a Legendre polynomial of order roughly equal to L.

The magnitude of these deviations is related to the discontinuity in the behavior of the partial-wave amplitudes between the resonance and nonresonance regions. At low energies, a Regge pole close to the real axis can produce discernible resonances, assuming that the effect of the Regge pole is not diminished by the presence of a nearby Regge zero on the same side of the real axis. If the Regge singularities were not of sufficient strength to produce resonant effects in low-energy scattering amplitudes, however, it is plausible that on the real axis there would be a fairly smooth transition across $l = L$, in which case there would be small deviation from angular behavior smoothly extrapolated from the crossed-channel Regge-pole approximation. Experimentally, this appears to be the case. For example, high-energy $p\bar{p}$ scattering has dips and peaks at nonforward angles, whereas pp scattering does not[16]; our qualitative picture would associate these results with the presence of resonances in $p\bar{p}$ scattering and the absence of resonances in pp scattering, whereas the conventional crossed-channel Regge picture would associate these dips with interference between terms from several crossed-channel trajectories. The reader should verify for himself that the stated growth rate

[16] See, e.g., V. Barger, review paper, in *Proceedings of the CERN Topical Conference on High-Energy Collisions of Hadrons, 1968* (CERN, Geneva, 1968), Vol. I, Fig. 17. Our discussion would seem to indicate that pp scattering has trajectories not associated with easily identified resonances. In our original product representation, we allowed explicit poles in l associated with multiplicative factors $\exp[\phi_{l,n}(s)]$ as well as an over-all multiplicative factor $\exp[G(l,s)]$. The latter factor could be used to account for some of the behavior of the amplitude above inelastic threshold, and would contain terms due to Regge cuts, Regge poles not explicitly included, and background effects not properly taken care of by the factors $\phi_{l,n}(s)$. In a practical calculation one might use as a first approximation a few leading Regge trajectories whose behavior could be determined roughly from experiment, absorbing the rest of the amplitude into the factor $\exp[G(l,s)]$ with some reasonable parametrization.

for $L(s)$ ensures that dips and peaks in the amplitude occur at points of constant t for rising s, at least to the nearest power of s. There is evidently a certain similarity to the duality approach in this discussion, although there is a distinct difference in the mathematical details, as will be clear in the subsequent papers. Paper II describes calculations made with a product representation using a large number of trajectories, in which the aforementioned properties for nonforward angles are exhibited explicitly.

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Direct-Channel Reggeization of Strong-Interaction Scattering Amplitudes. II

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Direct-channel Reggeized strong-interaction scattering amplitudes are defined in terms of a set of direct-channel phase shifts and elasticity factors. These quantities are functions of the positions of direct-channel Regge poles and certain convergence factors, which are related to crossed-channel behavior. Two mathematical techniques are discussed which can be used to continue these partial-wave expansions outside their ordinary region of convergence. Unphysical values of angular momentum are not used in the continuation methods. The numerical method requires the partial-wave amplitudes specified at positive-integer values of l only. The analytic method can be used to calculate precisely the discontinuity across the crossed-channel cut, and thus would be valuable in further theoretical work using these representations. Applications of these methods are discussed, with reference to two earlier phase-shift Reggeization schemes.

I. INTRODUCTION

In principle, all that is necessary for a complete specification of the total scattering amplitude is knowledge of the set of s-channel partial-wave amplitudes $\{a_l(s)\}$ for non-negative integer l along with a viable method of analytic continuation of the partial-wave series in $\cos\theta_s$. We seek an improvement of the Sommerfeld-Watson transformation as a practical continuation technique for this purpose: It is clumsy in dealing with direct-channel unitarity and in determining the behavior of the amplitude around nearby crossed-channel singularities, and there is no simple way to approximate the background integral.

We shall, however, utilize certain information abstracted from the Sommerfeld-Watson method. For example, we restrict our attention to a class of representations for the scattering amplitude for which some direct-channel Regge poles are manifest, thereby treating whole sequences of observed resonances in a unified way. We also incorporate non-Regge information, insisting on certain asymptotic behavior of our amplitudes, independent of the level of approximation.

In order to satisfy unitarity we have introduced a "Regge zero" as well as a Regge pole, and thereby have lost the possibility of independently specifying a Regge residue.[1] Near a Regge pole, in fact,

$$S_l(s)\cong\exp[\phi_n(\alpha_n{}^{\mathrm{I}},s)+G(\alpha_n{}^{\mathrm{I}},s)]\left(\frac{\alpha_n{}^{\mathrm{I}}-\alpha_n{}^{\mathrm{II}}}{l-\alpha_n{}^{\mathrm{I}}}\right)\times\prod_{m\neq n}\left[\left(\frac{\alpha_n{}^{\mathrm{I}}-\alpha_m{}^{\mathrm{II}}}{\alpha_n{}^{\mathrm{I}}-\alpha_m{}^{\mathrm{I}}}\right)\exp[\phi_m(\alpha_n{}^{\mathrm{I}},s)]\right]\quad\text{for}\quad l\cong\alpha_n{}^{\mathrm{I}}.\quad(1)$$

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This paper is the second in a series of three papers dealing with direct-channel Reggeization of partial-wave amplitudes. It is concerned with mathematical techniques useful for calculations with such amplitudes. An introductory discussion and additional references are contained in Paper I.[1]

II. ANALYTIC METHOD FOR CONTINUATION OF PARTIAL-WAVE SERIES

We shall use product representations, in which $\ln S_l$ is specified in terms of physically relevant quantities. In this case, $\ln S_l$ is a simpler function than S_l, from the point of view of practical analytic methods. Nevertheless, we must continue the sum:

$$A(s,\cos\theta_s)=[2iK(s)]^{-1}\sum_{l=0}^{\infty}(2l+1)\times[\exp(\ln S_l(s))-1]P_l(\cos\theta_s).\quad(2)$$

Let us therefore consider a class of methods for the continuation of functions defined in terms of Legendre series. It will be obvious that these methods, with appropriate modifications, can be used for expansions in terms of any set of orthogonal functions.

A discussion of the significance of the continuation methods is given in Paper III, where they are used to develop a set of equations to impose crossing and Reggeized behavior.

Consider the analytic continuation of

$$f(z)=\sum_{l=0}^{\infty}(2l+1)[e^{bl}-1]P_l(z)\quad(3)$$

[1] S. P. Creekmore, preceding paper, Phys. Rev. D **3**, 1400 (1971).