Feynman Functional Integrals for Systems of Indistinguishable Particles

MICHAEL G. G. LAIDLAW* AND CÉCILE MORETTE DEWITT
Department of Physics, University of North Carolina, Chapel Hill, North Carolina 27514
(Received 1 April 1970; revised manuscript received 16 November 1970)

The theory of path integration is extended to include systems whose configuration space is multiply connected, and it is seen that there are as many distinct propagators as there are scalar representations of the associated fundamental group. It is shown that the configuration space for a system of indistinguishable particles is multiply connected. There are only two propagators for this system, giving bosons and fermions, and showing that the Feynman formalism excludes parastatistics.

INTRODUCTION

IT has been pointed out by Schulman[1] that the path integral in a multiply connected space[2] has a novel feature arising from the fact that paths in different homotopy classes cannot be continuously deformed one into the other. For, while the partial probability amplitude K^α for a transition via paths in a particular homotopy class may be obtained in principle by performing the path integration over all paths belonging to this class, the total probability amplitude is ambiguous to the extent that these partial amplitudes may appear in the final sum with unknown weight factors:

$$K=\sum_\alpha \chi(\alpha)K^\alpha.$$

The allowed values for these weight factors have been found by Schulman for the multiply connected spaces $SO(2)$ and $SO(3)$. We shall prove here that in general they must form a scalar unitary representation of the fundamental group of the space.[3] We shall apply this result to a system of indistinguishable particles.

DETERMINATION OF WEIGHT FACTORS

Assume that the configuration space X for a physical system is a multiply connected, arcwise connected, locally arcwise connected, and locally simply connected topological space. Let (a,b) be any two points in X, and denote by $\Omega(X,a,b)$ the set of all paths $q(a,b)$ in X from a to b. Homotopy between paths in $\Omega(X,a,b)$ is an equivalence relation. Let $[q(a,b)]$ be the set of all paths in $\Omega(X,a,b)$ which are homotopic to $q(a,b)$. These equivalence classes are called homotopy classes, and the set of all such homotopy classes will be denoted by $\Pi(X,a,b)$. Composition of paths gives rise to a "multiplication" of elements, since it can be shown that if $[q_1]\in\Pi(X,a,b)$ and $[q_2]\in\Pi(X,b,c)$, then $[q_1][q_2]=[q_1q_2]\in\Pi(X,a,c)$. If $a=b$ (i.e., the paths begin and end at the same point), then the set $\Pi(X,a,a)\equiv\Pi(X,a)$, together with this multiplication, forms a group, known as the fundamental group of X based at a.

Let x be some fixed point in X, and denote by $e, \alpha, \beta, \ldots, \gamma$ the elements of $\Pi(X,x)$. Let $C(a)$ be an arbitrarily chosen path[4] from x to a for every $a\in X$; then this construction induces a mapping f_{ab} from $\Pi(X,x)$ to $\Pi(X,a,b)$ for every (a,b) in X.

$$f_{ab}:\ \Pi(X,x)\to\Pi(X,a,b),$$

by

$$f_{ab}(\alpha)=[C^{-1}(a)]\alpha[C(b)].$$

It is readily shown that f_{ab} is a 1-1 mapping and that it is compatible with "multiplication" of homotopy classes, i.e.,

$$f_{ab}(\alpha)f_{bc}(\beta)=f_{ac}(\alpha\beta).$$

Roughly speaking, the mapping f_{ab} labels each homotopy class of paths from a to b with an element of the fundamental group coherently for every a, b in the space X. While this labeling is not arbitrary, neither is it unique since it depends upon the choice of the homotopy mesh of paths C.

The dynamics of the system will be given by some Lagrangian in X. The partial probability amplitude $K^\alpha(b,t_b;a,t_a)$ will be obtained by performing the path integration over all paths in the homotopy class $f_{ab}(\alpha)$ which start at time t_a and end at time t_b. The following four properties are satisfied by these partial amplitudes:

I. $K^\gamma(c,t_c;a,t_a)=\sum_{\alpha\beta=\gamma}\int K^\beta(c,t_c;b,t_b)K^\alpha(b,t_b;a,t_a)db.$

II. For every point $a\in X$ there exists an open set $U\subset X$ containing a such that if $a'\in U$, then

$$K^\alpha(a',t';a,t)\to 0 \quad\text{as}\quad t'\to t,\quad t'\neq t$$

for all but one and only one homotopy class.

III. Under a change in the homotopy mesh $C\to\bar{C}$, $K^\alpha\to\bar{K}^\alpha$, where $\bar{K}^\alpha=K^{\lambda\alpha\mu}$ for some λ, $\mu\in\Pi(X,x)$.

IV. Linear independence.

Proof:

I. Every path $q\in f_{ac}(\gamma)$ may be split into the pair (q_1,q_2), where $q_1\in f_{ab}(\alpha)$ and $q_2\in f_{bc}(\beta)$ for some point b and some α, β such that $\alpha\beta=\gamma$. Hence the need to sum

* Work supported in part by the National Science Foundation and the National Aeronautics and Space Administration.
[1] L. Schulman, Phys. Rev. **176**, 1558 (1968).
[2] I. M. Singer and John A. Thorpe, *Lecture Notes on Elementary Topology and Geometry* (Scott, Foresman, New York, 1967), Chap. 3.
[3] This result is conjectured in a recent report by L. S. Schulman, received after this work was written.
[4] This can always be done because X is arcwise connected.

over all α, β such that $\alpha\beta=\gamma$ as well as to integrate over b.

II. Let a be any point in X; then, since X is locally simply connected, there exists an open set $V\subset X$ containing a such that if $b\in V$ then all paths from a to b in V are in the same homotopy class. Let $U\subset X$ be an open set containing a such that $S_c(a,a')/\hbar<\epsilon$ if and only if $a'\in U$, where S_c is the least minimum action (for each homotopy class there may be a minimum action) between (a,t) and (a',t') and ϵ is positive nonzero number. Suppose $q(a,a')$ is a path from (a,t) to (a',t'), part of which lies outside U. Then we can write $q(a,a')=q(a,b)q(b,a')$, where b is not in U and $t<t_b<t'$. Hence, $S[q(a,a')]>S[q(a,b)]$. But $S[q(a,b)]>\hbar\epsilon$, both because b is not in U and because the time interval (t_b-t) is smaller than $(t'-t)$. Thus $S[q(a,a')]>\hbar\epsilon$. Therefore, a path $q_c(a,a')$ for which the action takes on its least minimum value[5] lies inside U. Let us choose ϵ sufficiently small such that $U\subset V$; then the path $q_c(a,a')$ is unique. It is well known[6] that when the time interval is small, the only paths which contribute significantly to the path integral are those which lie close[7] to the classical path q_c. But those paths which lie close to q_c are in V and hence are all in the same homotopy class. As the time interval approaches zero, all other contributions to the path integral become vanishingly small and we have the desired result.

III. Consider a change in the mesh $C\to\bar{C}$:

$$\begin{aligned}\bar{f}_{ab}(\alpha)&=[\bar{C}^{-1}(a)]\alpha[\bar{C}(b)]\\&=[C^{-1}(a)][C(a)][\bar{C}^{-1}(a)]\alpha[\bar{C}(b)][C^{-1}(b)][C(b)]\\&=[C^{-1}(a)]\lambda\alpha\mu[C(b)]\\&=f_{ab}(\lambda\alpha\mu),\end{aligned}$$

where

$$\lambda=[C(a)\bar{C}^{-1}(a)]\in\Pi(X,x),$$
$$\mu=[\bar{C}(b)C^{-1}(b)]\in\Pi(X,x);$$
$$\bar{K}^{\alpha}(b,t_b;a,t_a)=K^{\lambda\alpha\mu}(b,t_b;a,t_a).$$

IV. Suppose that the partial amplitudes are not linearly independent functions. Then we can find a set of complex numbers $\{A_\alpha\}$, not all of which are zero, such that

$$\sum_\alpha A_\alpha K^\alpha(b,t_b;a,t_a)=0\quad\text{for all } a,b\in X.$$

However, this condition is not sufficient for linear dependence because, with a change in mesh, the coefficients would be attached to the partial amplitudes in a different order. The sufficient condition is

$$\sum_\alpha A_\alpha K^{\lambda\alpha\mu}(b,t_b;a,t_a)=0\quad\text{for all } a,b\in X$$
$$\text{and for all } \lambda,\mu\in\Pi(X,x).$$

Let $t_b\to t_a$; then property II implies that $A_\alpha=0$ for all α. This contradicts the initial supposition, and so the partial amplitudes are linearly independent.

The total probability amplitude is the weighted sum of the partial amplitudes

$$K(b,t_b;a,t_a)=\sum_\alpha \chi(\alpha)K^\alpha(b,t_b;a,t_a).$$

We are assuming for the present that a set of weight factors can be found such that any change in the mesh will not alter the absolute value of K. We may then write

$$|K(b,t_b;a,t_a)|=|\sum_\alpha \chi(\alpha)K^{\lambda\alpha\mu}(b,t_b;a,t_a)|$$
$$\text{for all } \lambda,\mu\in\Pi(X,x).$$

Property II then implies that $|\chi(\alpha)|=1$ for all α.

In order that the total probability amplitude propagate the states of the system in a well-defined fashion, it is necessary that

$$|K(c,t_c;a,t_a)|=\left|\int K(c,t_c;b,t_b)K(b,t_b;a,t_a)db\right|.$$

Therefore,

$$e^{i\theta}K(c,t_c;a,t_a)$$
$$=\sum_{\alpha,\beta}\chi(\alpha)\chi(\beta)\int K^\beta(c,t_c;b,t_b)K^\alpha(b,t_b;a,t_a)db$$

for some phase factor $e^{i\theta}$. Change the mesh to all points $b\in X$ except for $b=a$ and $b=c$ such that $\bar{f}_{ab}(\alpha)=f_{ab}(\alpha\gamma)$ and $\bar{f}_{bc}(\beta)=f_{bc}(\gamma^{-1}\beta)$. Sum both sides over γ and use property I.

$$Ne^{i\theta}\sum_{\alpha\beta=\delta}\chi(\delta)K^{\alpha\beta}(c,t_c;a,t_a)=\sum_{\alpha,\beta}\chi(\alpha)\chi(\beta)K^{\alpha\beta}(c,t_c;a,t_a),$$

where N is the number of elements in the fundamental group. Using the linear independence of the partial amplitudes, we obtain

$$e^{i\theta}N\chi(\delta)=\sum_{\alpha\beta=\delta}\chi(\alpha)\chi(\beta).$$

Therefore,

$$N=|\sum_{\alpha\beta=\delta}\chi(\alpha)\chi(\beta)|\leq\sum_{\alpha\beta=\delta}|\chi(\alpha)\chi(\beta)|=N$$

$$\Rightarrow\quad e^{i\theta}\chi(\alpha\beta)=\chi(\alpha)\chi(\beta).$$

The over-all phase of the total amplitude is unimportant, and so without loss in generality we may set $\theta=0$.

$$\chi(\alpha)\chi(\beta)=\chi(\alpha\beta),\quad\text{with}\quad|\chi(\alpha)|=1.$$

[5] Such a path exists because X is locally arcwise connected.

[6] Cécile Morette DeWitt, Ann. Institut Henri Poincaré (A) **XI**, 153 (1969).

[7] A path $q(a,a')$ will be close to the classical path if $S[q]-S(q_c)<\hbar\delta$, where δ is some small nonzero positive number. By choosing δ sufficiently small, one can always arrange to have paths which lie close to the classical path inside V.

But this says that the weight factors form a scalar unitary representation of the fundamental group,

$$\chi(\alpha)=D(\alpha).$$

Conversely, given a one-dimensional unitary representation $\{D(\alpha)\}$ of the fundamental group at a base point, how does one assign the elements of this set to each partial amplitude? Attaching, say, $D(\alpha)$ to K^α, $D(\beta)$ to K^β, etc., implies that one knows how to label the partial amplitudes by the elements of the fundamental group. The labeling of the homotopy classes, hence the labeling of the partial amplitudes by the elements of the fundamental group, depends on the choice of mesh and therefore is not unique. This ambiguity cannot be removed since it reflects the fact that there is no canonical isomorphism between the fundamental groups at two different base points. We shall prove that the arbitrariness affects only the over-all phase factor of the total amplitude. This proof will justify the assumption made to determine the weight factors.

Proof:

Suppose that a given mesh C has labeled a given partial amplitude K^α. We can change the mesh so that the new mesh $\bar{C}$ gives to this amplitude any label we choose, say, β. Let us choose $\bar{C}$ such that

$$[C(a)\bar{C}^{-1}(a)]=\lambda\,,\quad [\bar{C}(b)C^{-1}(b)]=\mu\,,$$

where

$$\beta=\lambda\alpha\mu\,.$$

This choice of λ and μ then completely determines the labeling of the other partial amplitudes.

Thus if a mesh has given for K the expression

$$K=\sum_\alpha D(\alpha)K^\alpha\,,$$

another mesh will give

$$\bar{K}=\sum_\alpha D(\alpha)K^{\lambda\alpha\mu}$$
$$=D(\lambda^{-1}\mu^{-1})K\,,$$
$$|\bar{K}|=|K|\,,$$

and the assignment of elements of $\{D(\alpha)\}$ to the partial amplitudes by any mesh gives the same physical results. Let us call "equivalent" those labelings which give the same physical result; equivalent labelings differ from one another by pre- and postmultiplication by fixed elements.

It is interesting to note that the partial amplitudes have no physical meaning; only the total amplitude has physical meaning, a proposition which led Feynman to say, "sum over all paths."

Finally, we shall prove that $|K|$ as a function of the endpoints a and b is continuous, provided, of course, there is no physical discontinuity at a or b, although the partial amplitudes may not be continuous. Consider a mesh C and two points b and b' such that

$$K(b,a)\underset{b\to b'}{\not\to}K(b',a)\,.$$

For every α,

$$K^\alpha(b,a)\underset{b\to b'}{\longrightarrow}K^{\alpha\beta}(b',a)\,,$$

where

$$\beta=[\bar{C}(b')C^{-1}(b')]\,,$$

$\bar{C}(b')$ being a change in the mesh such that

$$\bar{K}^\alpha(b,a)\underset{b\to b'}{\longrightarrow}\bar{K}^\alpha(b',a)\,.$$

Consequently, the lack of continuity of the partial amplitudes at b' affects the total amplitude exactly like a change of mesh at b', namely, multiplies it by an over-all phase factor. In particular, let us move a and b to x; the paths from a to b become closed paths at x, and the assignment of weight factors to the partial amplitudes is then unique. Thus any mesh construction leads to equivalent total amplitudes which are related by continuity to a uniquely defined amplitude.

In conclusion, there are as many distinct propagators as there are unitary scalar representations. Furthermore, if over some interval of time the evolution of the system is given by a propagator associated with a certain representation, then the system will continue to evolve with the same propagator indefinitely (or until some radical change in the system alters the topology of its configuration space). The proof that transitions between the different means of propagation do not occur rests on the orthogonality theorem in the theory of group representations. This implies that any property of the system which depends only upon the means of propagation will be conserved.

SYSTEMS OF INDISTINGUISHABLE PARTICLES

We shall require that the configuration space X for a system of particles, whether distinguishable or indistinguishable, be in a 1-1 correspondence with the states of the system, and that points which represent the coincidence of two or more particles be excluded. Whether or not two point particles can simultaneously occupy the same point in space is not a question that we wish to settle here. We are only saying that by excluding points of coincidence from the configuration space, the resulting topology leads to meaningful physical results without any further assumptions.

Let $Y(n,m)$ be the set of all n-tuples $(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ of m-vectors, no two of which coincide.

$$Y(n,m)=\{y=(\mathbf{x}_1,\ldots,\mathbf{x}_n)\,;\,\mathbf{x}_i=(x_i^1,\ldots,x^m)\text{ and }\mathbf{x}_i\neq\mathbf{x}_j\}\,.$$

We observe that for $n\geq 2$,

$Y(n,1)$ is not connected,
$Y(n,2)$ is multiply connected,
$Y(n,m)$ is simply connected; $m\geq 3$.

For a physical system of n *distinguishable* particles,

$$X=Y(n,3)\,.$$

For a physical system of n *indistinguishable* particles,

$$X=Y(n,3)/S_n\,.$$

The quotient space X is the set of equivalence classes of points in $Y(n,3)$ under permutations belonging to the symmetric group S_n, and is given the identification topology under the natural projection p: $Y(n,3)\to X$. We observe that S_n acts effectively on $Y(n,3)$; that is to say, given any point $y\in Y$ and any element $\alpha\in S_n$ except the identity, then $\alpha(y)\neq y$ (this is true because we have excluded points of coincidence). Because $Y(n,3)$ is simply connected and S_n act effectively, $(Y(n,3),p)$ is a universal covering space for X and the fundamental group of X is isomorphic to S_n.[8,9] There are only two scalar unitary representations of the symmetric group.

$D^1(\alpha)=+1$ for all α,

$D^2(\alpha)=\pm1$ according as α is an even or odd permutation,

$$K^{\text{Bose}}=\sum_\alpha D^1(\alpha)K^\alpha\,,$$

$$K^{\text{Fermi}}=\sum_\alpha D^2(\alpha)K^\alpha\,.$$

[8] Peter Hilton, *Algebraic Topology—An Introductory Course* (Courant Institute of Mathematical Sciences, New York University, New York, 1969), p. 67.

[9] Edwin H. Spanier, *Algebraic Topology* (McGraw-Hill, New York, 1966), pp. 87–89.

ACKNOWLEDGMENTS

We are much indebted to Raoul Bott, who patiently helped us make sense out of intuitively formulated and incomplete ideas, and to Yvonne Choquet-Bruhat, who did not give up the physical meaning when the mathematics could not provide any guidance. We also wish to thank R. P. Walker for helpful conversations on homotopy theory. Credit is due to the Battelle Rencontres for the interactions from which this work has evolved.

PHYSICAL REVIEW D VOLUME 3, NUMBER 6 15 MARCH 1971

The Photon as a Composite State of a Neutrino-Antineutrino Pair

PRATUL BANDYOPADHYAY AND PROBHAS RAY CHAUDHURI
Indian Statistical Institute, Calcutta, India

(Received 22 September 1970)

The possibility that the photon may be a composite state of a neutrino-antineutrino pair has been examined from field-theoretic considerations on the basis of the photon-neutrino weak-coupling theory. It is shown that the difficulties which generally crop up in quantum electrodynamics to describe a photon as a composite state of an electron-positron pair using the $Z_3=0$ condition are avoided in neutrino dynamics (photon-neutrino weak interaction). In view of this, we conclude that the photon may be taken as a composite state of a neutrino-antineutrino pair when the composite character is described by the vanishing of the wave-function renormalization constant and the nonvanishing of a certain composite coupling constant.

I. INTRODUCTION

THE possibility that the photon may be a composite state of a neutrino-antineutrino pair was first suggested by de Broglie.[1] However, this simple picture of the photon was found not to obey Bose statistics because of the underlying Fermi statistics of its components. In fact, if two such photons were in the state with momentum **p**, two-component neutrinos (and antineutrinos) of these photons would be in the state with the same momentum **k**. In view of this, Jordan[2] suggested a model of the photon, composed of two neutrinos each being a superposition of states with different momenta. This assumption, with the proper choice of the superposition coefficients, provided the correct statistics for the theory. After this, Kronig[3] succeeded in constructing the photon field out of that of neutrinos. However, Pryce[4] has shown that the Kronig theory is not invariant under the group of spatial rotations about the direction of the photon momentum as an axis.

In recent times, several authors revived the discussions on the neutrino theory of photons. Barbour, Bietty, and Touschek[5] argued that a photon in neutrino theory is always longitudinally polarized. Ferretti[6] suggested that the photon be considered as a limiting state of a bound system of two nonzero mass particles with a given angular momentum when the binding energy as well as the mass tends to zero. Perkins[7] considered the usual four-component solutions with

[1] L. de Broglie, Compt. Rend. **195**, 862 (1932); **199**, 813 (1934).

[2] P. Jordan, Z. Physik **93**, 464 (1935).

[3] R. Kronig, Physica **3**, 1120 (1936).

[4] M. H. L. Pryce, Proc. Roy. Soc. (London) **165**, 247 (1938).

[5] I. M. Barbour, A. Bietti, and B. F. Touschek, Nuovo Cimento **28**, 453 (1963).

[6] B. Ferretti, Nuovo Cimento **33**, 265 (1964).

[7] W. A. Perkins, Phys. Rev. **137**, B1291 (1965).