

the whole Coulomb Green's function¹⁸ do not have any branch cuts if they are written in terms of ϵ . In this case, the region $|\epsilon| > 1$ corresponds to the second sheet. In the relativistic case there are, however, additional contributions which correspond to higher branch cuts

¹⁸ J. Schwinger, *J. Math. Phys.* **5**, 1606 (1964).

at $\eta^2 = 1$ and which are probably caused by the infinity of inelastic thresholds which accumulate at $\eta^2 = 1$.

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Remarks on the Dipole-Ghost Scattering States*

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By using Nagy's modified Lee model, it is explicitly shown that the dipole-ghost scattering states exist and can be expressed as superpositions of the eigenstates of the Hamiltonian. Thus the Ascoli-Minardi theorem, which guarantees the unitarity of the physical S matrix, ceases to be valid in dipole-ghost theories.

I. INTRODUCTION

IN order to construct convergent field theories, it is an attractive idea to extend the space spanned by state vectors to the so-called indefinite-metric Hilbert space. It is well known,¹ however, that the negative-norm states (ghosts) usually lead us to the difficulty of negative probability; that is, it is very hard to extract a unitary physical S matrix consistently.

An interesting possible way out of this difficulty was suggested by Heisenberg² in 1957. In order to justify the quantization proposed in his nonlinear spinor theory, he considered a special limiting case of the Lee model³ which contains two clothed V -particle states $|V_+\rangle$ (normal state) and $|V_-\rangle$ (ghost). One can adjust parameters in such a way that $|V_-\rangle$ tends to $|V_+\rangle$. In this limit one obtains only one eigenstate $|V_0\rangle$ of the Hamiltonian H :

$$(H - E)|V_0\rangle = 0, \quad \langle V_0|V_0\rangle = 0. \quad (1.1)$$

Another independent state $|D\rangle$ is no longer an eigenstate of H , but it satisfies an inhomogeneous equation

$$(H - E)|D\rangle = |V_0\rangle, \quad \langle V_0|D\rangle \neq 0. \quad (1.2)$$

The state $|D\rangle$ is usually called a dipole ghost. A model which contains dipole ghosts already in bare states was constructed by Froissart.⁴

The usefulness of dipole ghosts was clarified by Ascoli and Minardi⁵ and by Nagy.¹ Suppose that apart from dipole ghosts all linearly independent states have positive or zero norm. From the completeness assumption any state can be expressed as a superposition of the incoming physical states $|in, \beta\rangle$, the zero-norm states $|0(in, \gamma)\rangle$, and the corresponding dipole ghosts $|D(in, \gamma)\rangle$, where β and γ indicate sets of quantum numbers. For an arbitrary outgoing physical state $|out, \alpha\rangle$, therefore, we can write

$$|out, \alpha\rangle = \sum_{\beta} \varphi(\beta) |in, \beta\rangle + \sum_{\gamma} \chi(\gamma) |0(in, \gamma)\rangle + \sum_{\gamma} \psi(\gamma) |D(in, \gamma)\rangle. \quad (1.3)$$

Since $|out, \alpha\rangle$ is an eigenstate of the Hamiltonian while $|D(in, \gamma)\rangle$ is not and the states $|0(in, \gamma)\rangle$ are linearly independent, (1.3) will hold only when $\psi(\gamma) = 0$. Accordingly, the physical S matrix $\{\langle in, \beta | out, \alpha \rangle\}$ is unitary because the zero-norm states, which are orthogonal to all states other than dipole ghosts, give no contribution to the matrix elements. We call this result the Ascoli-Minardi theorem, though their theorem is actually more general (see Sec. III).

Some years ago, the present author^{6,7} proposed a manifestly covariant quantum electrodynamics in the general covariant gauge, which is a natural extension of the well-known Gupta-Bleuler theory.⁸ In this theory the longitudinal part and the scalar part of the electromagnetic field are quantized by means of dipole ghosts

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¹ K. Nagy, *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (P. Noordhoff, Groningen, Holland, 1966).

² W. Heisenberg, *Nucl. Phys.* **4**, 532 (1957).

³ T. D. Lee, *Phys. Rev.* **95**, 1329 (1954).

⁴ M. Froissart, *Nuovo Cimento Suppl.* **14**, 197 (1959).

⁵ R. Ascoli and E. Minardi, *Nuovo Cimento* **8**, 951 (1958); *Nucl. Phys.* **9**, 242 (1958).

⁶ N. Nakanishi, *Progr. Theoret. Phys. (Kyoto)* **35**, 1111 (1966).

⁷ N. Nakanishi, *Progr. Theoret. Phys. (Kyoto)* **38**, 881 (1967).

⁸ S. N. Gupta, *Proc. Phys. Soc. (London)* **A63**, 681 (1950); K. Bleuler, *Helv. Phys. Acta* **23**, 567 (1950).

except for the case of the Feynman gauge, namely, the Gupta-Bleuler case. Of course, the S matrix is gauge invariant and unitary if current is conserved. If current is *not* conserved, however, the S matrix is no longer gauge invariant, and it is *not unitary*. If we compare this fact with the Ascoli-Minardi theorem, the situation is quite paradoxical,⁷ though our quantization is made in the interaction picture but not in the Schrödinger picture.

According to Nagy,⁹ it seems that a similar question was raised in connection with the Lee model and the Froissart model by some people privately in 1968. Nagy,⁹ therefore, investigated what was wrong about the Ascoli-Minardi theorem by introducing a modified Lee model similar to the Froissart model. In his model, a dipole ghost exists in a one-particle sector, but the physical S matrix is *not* unitary in the $N\theta$ sector, in spite of the fact that the only possible source of negative probability is the above dipole-ghost particle. The reason for this is that the $N\theta$ sector is actually spanned by the eigenstates of H alone. Thus Nagy concluded that nothing was wrong with the Ascoli-Minardi theorem but the dipole-ghost situation was "unstable."

In the present paper we show that Nagy's conclusion is *not adequate* though his result is correct. In Nagy's modified Lee model, we explicitly demonstrate that *dipole-ghost scattering states do exist* in the $N\theta$ sector and that, contrary to common belief, *they can be expressed as superpositions of the eigenstates of the Hamiltonian*. Thus the Ascoli-Minardi theorem is wrong for the dipole-ghost case, because we cannot put $\psi(\gamma)=0$ in (1.3). In Sec. II we analyze Nagy's model, and in Sec. III we discuss the problem of dipole ghosts belonging to a continuous spectrum.

II. NAGY'S MODIFIED LEE MODEL

The Hamiltonian $H \equiv H_0 + H_1$ of Nagy's model is given by

$$H_0 = m_V V^* V + m_N N^* N + \int d\mathbf{k} \omega_{\mathbf{k}} \theta^*(\mathbf{k}) \theta(\mathbf{k}) + m(A^* B + B^* A) + \lambda A^* A, \quad (2.1)$$

$$H_1 = V^* \int d\mathbf{k} [g(\mathbf{k}) N + g_1(\mathbf{k}) A + g_2(\mathbf{k}) B] \theta(\mathbf{k}) + \text{H.c.}, \quad (2.2)$$

where $g(\mathbf{k})$, $g_1(\mathbf{k})$, and $g_2(\mathbf{k})$ are certain real functions of the momentum \mathbf{k} , $\omega_{\mathbf{k}}$ denotes the relativistic energy $(m_\theta^2 + \mathbf{k}^2)^{1/2}$ of the θ particle, and H.c. stands for the Hermitian conjugate. The commutation relations for field operators are as follows:

$$\{V, V^*\} = \{N, N^*\} = \{A, B^*\} = \{B, A^*\} = 1, \quad (2.3)$$

$$[\theta(\mathbf{k}), \theta^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'),$$

and all the other commutators or anticommutators vanish.

Let $|0\rangle$ be the bare vacuum state, that is,

$$V|0\rangle = N|0\rangle = A|0\rangle = B|0\rangle = \theta(\mathbf{k})|0\rangle = 0, \quad (2.4)$$

$$\langle 0|0\rangle = 1.$$

Let

$$|V\rangle \equiv V^*|0\rangle, \quad |N\rangle \equiv N^*|0\rangle, \quad (2.5)$$

$$|A\rangle \equiv A^*|0\rangle, \quad |B\rangle \equiv B^*|0\rangle.$$

Then it is straightforward to show that

$$H|0\rangle = (H - m_N)|N\rangle = (H - m)|A\rangle = 0, \quad (2.6)$$

$$(H - m)|B\rangle = \lambda|A\rangle,$$

together with

$$\langle A|A\rangle = \langle B|B\rangle = 0, \quad (2.7)$$

$$\langle A|B\rangle = \langle B|A\rangle = 1.$$

Thus for $\lambda \neq 0$, which we always assume, $|B\rangle$ is a dipole ghost.

Now we consider the $N\theta$ sector. An arbitrary state $|\Phi\rangle$ belonging to this sector can be expressed as

$$|\Phi\rangle = \alpha|V\rangle + \int d\mathbf{k} \varphi(\mathbf{k}) \theta^*(\mathbf{k}) |N\rangle + \int d\mathbf{k} \varphi_1(\mathbf{k}) \theta^*(\mathbf{k}) |A\rangle + \int d\mathbf{k} \varphi_2(\mathbf{k}) \theta^*(\mathbf{k}) |B\rangle. \quad (2.8)$$

Hence, from (2.1) and (2.2), we find

$$H|\Phi\rangle = \left\{ m_V \alpha + \int d\mathbf{k} [g(\mathbf{k}) \varphi(\mathbf{k}) + g_1(\mathbf{k}) \varphi_2(\mathbf{k}) + g_2(\mathbf{k}) \varphi_1(\mathbf{k})] \right\} |V\rangle + \int d\mathbf{k} [(m_N + \omega_{\mathbf{k}}) \varphi(\mathbf{k}) + \alpha g(\mathbf{k})] \theta^*(\mathbf{k}) |N\rangle + \int d\mathbf{k} [(m + \omega_{\mathbf{k}}) \varphi_1(\mathbf{k}) + \lambda \varphi_2(\mathbf{k}) + \alpha g_1(\mathbf{k})] \theta^*(\mathbf{k}) |A\rangle + \int d\mathbf{k} [(m + \omega_{\mathbf{k}}) \varphi_2(\mathbf{k}) + \alpha g_2(\mathbf{k})] \theta^*(\mathbf{k}) |B\rangle. \quad (2.9)$$

By means of (2.9), it is easy to construct the physical V state and the various scattering states.⁹

⁹ K. L. Nagy, Acta Phys. Acad. Sci. Hung. 28, 245 (1970).

The incoming or outgoing states for the $A\theta$ scattering are given by

$$\begin{aligned} |A\theta, \mathbf{p}\rangle \equiv & a_p |V\rangle + \int d\mathbf{k} \frac{-a_p g(\mathbf{k})}{m_N + \omega_k - m - \omega_p \pm i\epsilon} \theta^*(\mathbf{k}) |N\rangle \\ & + \int d\mathbf{k} \left[\delta(\mathbf{k} - \mathbf{p}) + \frac{-a_p g_1(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} \right. \\ & \left. + \frac{\lambda a_p g_2(\mathbf{k})}{(\omega_k - \omega_p \pm i\epsilon)^2} \right] \theta^*(\mathbf{k}) |A\rangle \\ & + \int d\mathbf{k} \frac{-a_p g_2(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} \theta^*(\mathbf{k}) |B\rangle, \quad (2.10) \end{aligned}$$

where

$$a_p \equiv -g_2(\mathbf{p})/h(m + \omega_p \mp i\epsilon), \quad (2.11)$$

with

$$\begin{aligned} h(z) \equiv & m_V - z - \int d\mathbf{k} \frac{g^2(\mathbf{k})}{m_N + \omega_k - z} \\ & - \int d\mathbf{k} \frac{2g_1(\mathbf{k})g_2(\mathbf{k})}{m + \omega_k - z} + \int d\mathbf{k} \frac{\lambda g_2^2(\mathbf{k})}{(m + \omega_k - z)^2}. \quad (2.12) \end{aligned}$$

It should be noted that $[h(E + i\epsilon)]^{-1}$ is nothing but the modified propagator of the V particle.

Likewise, the incoming or outgoing states for the $B\theta$ scattering are given by

$$\begin{aligned} |B\theta, \mathbf{p}\rangle \equiv & b_p |V\rangle + \int d\mathbf{k} \frac{-b_p g(\mathbf{k})}{m_N + \omega_k - m - \omega_p \pm i\epsilon} \theta^*(\mathbf{k}) |N\rangle \\ & + \int d\mathbf{k} \left[\lambda \frac{\omega_p}{|\mathbf{p}|} \delta'(\mathbf{k} - \mathbf{p}) + \frac{-b_p g_1(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} \right. \\ & \left. + \frac{\lambda b_p g_2(\mathbf{k})}{(\omega_k - \omega_p \pm i\epsilon)^2} \right] \theta^*(\mathbf{k}) |A\rangle \\ & + \int d\mathbf{k} \left[\delta(\mathbf{k} - \mathbf{p}) + \frac{-b_p g_2(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} \right] \theta^*(\mathbf{k}) |B\rangle, \quad (2.13) \end{aligned}$$

together with

$$b_p \equiv [-g_1(\mathbf{p}) + \lambda |\mathbf{p}|^{-1} \omega_p g_2'(\mathbf{p})]/h(m + \omega_p \mp i\epsilon), \quad (2.14)$$

where a prime stands for a differentiation with respect to the magnitude of the argument vector.

It is important to note that the incident wave of $|B\theta, \mathbf{p}\rangle$ contains $\delta'(\mathbf{k} - \mathbf{p})$, that is, it is not a pure plane wave. If we avoid using $\delta'(\mathbf{k} - \mathbf{p})$, which is not a measure (in the mathematical sense), we cannot find the eigenstates of H corresponding to the $B\theta$ scattering. Instead, we can construct the following states:

$$\begin{aligned} |D, \mathbf{p}\rangle \equiv & c_p |V\rangle + \int d\mathbf{k} \left[\frac{-c_p g(\mathbf{k})}{m_N + \omega_k - m - \omega_p \pm i\epsilon} + \frac{-a_p g(\mathbf{k})}{(m_N + \omega_k - m - \omega_p \pm i\epsilon)^2} \right] \theta^*(\mathbf{k}) |N\rangle \\ & + \int d\mathbf{k} \left[\frac{-c_p g_1(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} + \frac{\lambda c_p g_2(\mathbf{k}) - a_p g_1(\mathbf{k})}{(\omega_k - \omega_p \pm i\epsilon)^2} + \frac{2\lambda a_p g_2(\mathbf{k})}{(\omega_k - \omega_p \pm i\epsilon)^3} \right] \theta^*(\mathbf{k}) |A\rangle \\ & + \int d\mathbf{k} \left[\lambda^{-1} \delta(\mathbf{k} - \mathbf{p}) + \frac{-c_p g_2(\mathbf{k})}{\omega_k - \omega_p \pm i\epsilon} + \frac{-a_p g_2(\mathbf{k})}{(\omega_k - \omega_p \pm i\epsilon)^2} \right] \theta^*(\mathbf{k}) |B\rangle, \quad (2.15) \end{aligned}$$

where

$$c_p \equiv -\frac{\lambda^{-1} g_1(\mathbf{p}) + a_p h'(m + \omega_p \mp i\epsilon)}{h(m + \omega_p \mp i\epsilon)}, \quad (2.16)$$

and a_p is given by (2.11). It is straightforward to confirm that (2.15) satisfies the dipole-ghost equation

$$(H - m - \omega_p) |D, \mathbf{p}\rangle = |A\theta, \mathbf{p}\rangle. \quad (2.17)$$

Thus we have obtained two kinds of $B\theta$ scattering states. Therefore, they cannot be independent. Indeed, from (2.10), (2.13), and (2.15), we can prove that

$$|D, \mathbf{p}\rangle - \lambda^{-1} |B\theta, \mathbf{p}\rangle = |\mathbf{p}|^{-1} \omega_p \int d\mathbf{q} \delta'(\mathbf{p} - \mathbf{q}) |A\theta, \mathbf{q}\rangle. \quad (2.18)$$

In other words, a dipole-ghost scattering state $|D, \mathbf{p}\rangle$ can be expressed as a superposition of the eigenstates of H , provided that we permit to use δ' in the expansion

coefficients. (Since in the scattered wave we have to use distributions which are not measures, it is unreasonable to forbid one to use δ' .) Thus we may adopt either of $|B\theta, \mathbf{p}\rangle$ and $|D, \mathbf{p}\rangle$ to form a complete system of states. Since

$$\langle A\theta, \mathbf{q} | A\theta, \mathbf{p}\rangle = 0, \quad (2.19)$$

as is shown in the Appendix, it seems more natural to adopt $|D, \mathbf{p}\rangle$.

III. DISCUSSION

In Sec. II, we have shown that the dipole-ghost scattering states are not linearly independent of the eigenstates of H . This result is in contradiction with common

belief, but one should remember that the concept of dipole ghosts is usually introduced in a discrete spectrum. The concept of dipole ghosts in a continuous spectrum is much different from it. In a discrete spectrum, an eigenstate and a dipole ghost correspond to a simple pole and a double pole, respectively, of the Green function. In a continuous spectrum, however, both correspond to a cut, and there is no clear-cut distinction between them.

In order to understand the point more clearly, we consider a modification of Nagy's model. In (2.1), we replace A and B by $A(\mathbf{k})$ and $B(\mathbf{k})$, respectively, which satisfy

$$\begin{aligned} \{A(\mathbf{k}), B^*(\mathbf{k}')\} &= \{B(\mathbf{k}), A^*(\mathbf{k}')\} = \delta(\mathbf{k} - \mathbf{k}'), \\ \{A(\mathbf{k}), B(\mathbf{k}')\} &= \{A^*(\mathbf{k}), B^*(\mathbf{k}')\} = 0, \end{aligned} \quad (3.1)$$

and m by $E_p \equiv (m^2 + \mathbf{p}^2)^{1/2}$. Then we still have

$$\begin{aligned} (H - E_p)A^*(\mathbf{p})|0\rangle &= 0, \\ (H - E_p)B^*(\mathbf{p})|0\rangle &= \lambda A^*(\mathbf{p})|0\rangle. \end{aligned} \quad (3.2)$$

We can now, however, construct the following eigenstates of H corresponding to the B particle:

$$|B(\mathbf{p})\rangle \equiv \left[B^*(\mathbf{p}) - \lambda |\mathbf{p}|^{-1} E_p \int d\mathbf{q} \delta'(\mathbf{p} - \mathbf{q}) A^*(\mathbf{q}) \right] |0\rangle, \quad (3.3)$$

$$(H - E_p)|B(\mathbf{p})\rangle = 0. \quad (3.4)$$

Thus the same situation as in Sec. II exists also in a one-particle sector.

Of course, one can prohibit (3.3) by requiring the state to be also an eigenstate of the momentum operator. In one-particle states we can always assign a definite momentum, and therefore obtain a discrete spectrum.

The same also applies to any bare-particle state. This is exactly the situation encountered in the manifestly covariant quantum electrodynamics in the general covariant gauge.^{6,7} In the scattering states, however, the momentum of each particle is no longer a good quantum number. Therefore, in general, we conclude that it is a matter of taste which of the eigenstates and the dipole ghosts we adopt as the states belonging to a complete system in the scattering problem; that is, *the dipole-ghost scattering states are interchangeable with the scattering eigenstates*. Thus the Ascoli-Minardi theorem breaks down for the dipole-ghost situation.

The Ascoli-Minardi theorem is still valid for the complex-ghost situation; that is, if the last two terms of (1.3) are replaced by a superposition of complex ghosts (i.e., the eigenstates of H whose eigenvalues are not real), then the physical S matrix is unitary. As is well known,^{1,10} the trouble in the complex-ghost case comes out from the complex-ghost-pair states which have a real eigenvalue and negative norm. Recently, however, Lee¹¹ and Lee and Wick¹² have found that this difficulty can be overcome in relativistic field theories with real momenta. The reason for this is most clearly explained by the fact that the eigenvalue spectrum of the complex-ghost-pair states has a *two-dimensional* spread in the energy plane.¹³ Thus the complex-ghost-pair states having a real eigenvalue are of *measure zero*, and the Ascoli-Minardi theorem assures the unitarity of the physical S matrix. Though the above mechanism necessarily violates the Lorentz invariance of the theory,¹³ it is quite an interesting way of constructing convergent field theories. Since the analysis presented in this paper made it hopeless to construct a convergent, unitary field theory by using dipole ghosts, it is now very important to explore various possibilities of the Lee-Wick-type theories.

APPENDIX

We verify (2.19). From (2.10), we have

$$\begin{aligned} \langle A\theta, \mathbf{q} | A\theta, \mathbf{p} \rangle &= a_q^* a_p \left[\frac{-g_2(\mathbf{p})/a_p}{\omega_p - \omega_q \mp i\epsilon} + \frac{-g_2(\mathbf{q})/a_q^*}{\omega_q - \omega_p \pm i\epsilon} + 1 + \int d\mathbf{k} \frac{g^2(\mathbf{k})}{(m_N + \omega_k - m - \omega_q \mp i\epsilon)(m_N + \omega_k - m - \omega_p \pm i\epsilon)} \right. \\ &\quad \left. + \int d\mathbf{k} \frac{2g_1(\mathbf{k})g_2(\mathbf{k})}{(\omega_k - \omega_q \mp i\epsilon)(\omega_k - \omega_p \pm i\epsilon)} - \int d\mathbf{k} \frac{\lambda g_2^2(\mathbf{k})}{(\omega_k - \omega_q \mp i\epsilon)(\omega_k - \omega_p \pm i\epsilon)} \left(\frac{1}{\omega_k - \omega_p \pm i\epsilon} + \frac{1}{\omega_k - \omega_q \mp i\epsilon} \right) \right]. \end{aligned} \quad (A1)$$

We rewrite the integrands of the last three terms of (A1) by using an identity:

$$\frac{1}{(x \mp i\epsilon)(y \pm i\epsilon)} = \frac{1}{y - x \pm i\epsilon} \left(\frac{1}{x \mp i\epsilon} - \frac{1}{y \pm i\epsilon} \right). \quad (A2)$$

Then, on substituting (2.11) together with (2.12) in the first two terms of (A1), we immediately obtain (2.19).

¹⁰ R. Ascoli and E. Minardi, *Nuovo Cimento* **14**, 1254 (1959); S. Tanaka, *Progr. Theoret. Phys. (Kyoto)* **29**, 104 (1963).

¹¹ T. D. Lee, *Quanta: Essays in Theoretical Physics Dedicated to Gregor Wentzel* (University of Chicago Press, Chicago, 1970), p. 260.

¹² T. D. Lee and G. C. Wick, *Phys. Rev.* **D2**, 1033 (1970).

¹³ N. Nakanishi, *Phys. Rev. D* **3**, 811 (1971).