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Nonpolynomial Lagrangians with Derivative Interactions

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The techniques for computing S-matrix elements for nonpolynomial scalar-field Lagrangians with derivative interactions are presented. To second order in the interaction Lagrangian, it is shown that all the dependence arising from the derivative part is completely separated out as operators acting on integrals identical to those obtained in a nonderivative theory. The Fourier transforms of self-energy graphs for a class of nonlocal interaction Lagrangians are taken in the massless case. The on-mass-shell contributions are determined by the analytic continuation of the coefficients appearing in the series expansion of the Lagrangian. As special examples, two Lagrangians which are isoscalar analogs of chiral $SU(2)\times SU(2)$ Lagrangians are treated. The possible equivalence of on-mass-shell matrix elements for Lagrangians related by nonlinear field transformations is discussed.

I. INTRODUCTION

THE partial-summation method of Efimov[1] and Fradkin[2] for treating nonpolynomial Lagrangians has been extended by Delbourgo, Salam, and Strathdee[3] to Lagrangians with derivative interactions. In this paper we develop the techniques required for calculating S-matrix elements for these Lagrangians.

We consider a one-component scalar-field Lagrangian given by

$$L(\phi,\partial_\mu\phi)=\tfrac{1}{2}:(\partial_\mu\phi)(\partial_\mu\phi):-\tfrac{1}{2}m^2:\phi^2:+L_{\rm int}(\phi,\partial_\mu\phi)\,, \quad (1.1)$$

where the interaction Lagrangian is of the form

$$L_{\rm int}(\phi,\partial_\mu\phi)=h:u(\phi):+g:(\partial_\mu\phi)(\partial_\mu\phi)v(\phi):. \quad (1.2)$$

Here $u(\phi)$ and $v(\phi)$ are taken to be arbitrary functions of ϕ which have a Taylor-series expansion about $\phi=0$; g and h are coupling constants. The normal ordering in (1.2) is defined by expanding $u(\phi)$ and $v(\phi)$ and then normally ordering each term.

General expressions for the S-matrix elements to second order in $L_{\rm int}$ are derived in Sec. II, and in Sec. III we take the Fourier transforms of the self-energy graphs assuming zero-mass fields. All the usual difficulties[1–5] of constructing field theories with nonpolynomial Lagrangians without derivative interactions are also present here. In particular, the ambiguity arising from the summation techniques cannot be completely eliminated by unitarity and analyticity arguments. This is discussed in detail for the self-energy graphs.

Our future interest lies in comparing the predictions of chiral Lagrangians in the so-called "tree approximation" to a field-theoretic approach where one computes lower-order perturbation graphs (loops). The above Lagrangian contains as a special case, when $h=0$, the isoscalar analogs of chiral $SU(2)\times SU(2)$ meson Lagrangians. In Sec. IV the self-energy diagrams for these Lagrangians are treated in two coordinate systems. In both examples there arise additional ultraviolet infinities. These infinities prevent a direct on-mass-shell comparison of the S-matrix elements in the different coordinate systems. It is possible to obtain the same on-mass-shell matrix elements only with a special choice of the renormalization constants.

II. S-MATRIX ELEMENTS: SECOND-ORDER PERTURBATION THEORY

Following the discussion of Delbourgo, Salam, and Strathdee,[3] we derive in this section general S-matrix elements in second-order perturbation theory for the interaction Lagrangian (1.2). The matrix elements for the nonderivative part of (1.2),

$$hL_i(\phi)=h:u(\phi):\,, \quad (2.1)$$

have been derived by several authors[1–5] and in order to extend their results to the interaction Lagrangian (1.2) we first derive the S-matrix contributions for the derivative interaction Lagrangian,

$$gL_I(\phi,\partial_\mu\phi)=g:\partial_\mu\phi\partial_\mu\phi v(\phi):. \quad (2.2)$$

The contribution from the product of $L_i(\phi)$ and $L_I(\phi,\partial_\mu\phi)$ can then be simply deduced by the same methods and is given towards the end of this section.

For notational convenience we introduce the concept of a "five-vector" defined by

$$\phi_N(z)\equiv\left(\phi(z),\frac{\partial}{\partial z_\mu}\phi(z)\right)\equiv(\phi(z),\phi_\mu(z))\,. \quad (2.3)$$

The second-order term in the S-matrix expansion

$$S=\sum_n\frac{i^n}{n!}S^{(n)} \quad (2.4)$$

is given by

$$S^{(2)}=g^2\int d^4z_1d^4z_2T^*\{L_I(\phi_N(z_1))L_I(\phi_N(z_2))\}\,, \quad (2.5)$$

where the modified time-ordering operator T^* is defined

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[1] G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **44**, 2107 (1963) [Soviet Phys. JETP **17**, 1417 (1963)]; Nuovo Cimento **32**, 1046 (1964); Nucl. Phys. **74**, 657 (1965); Trieste Report Nos. I, II, and III, 1969 (unpublished).

[2] E. S. Fradkin, Nucl. Phys. **49**, 624 (1963).

[3] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969).

[4] M. K. Volkov, Ann. Phys. (N. Y.) **49**, 202 (1968).

[5] B. W. Lee and B. Zumino, Nucl. Phys. **B13**, 671 (1969).

such that the order of time ordering and differentiation is inverted in taking vacuum expectation values of the following kind[6]:

$$\langle T^*\{\phi_\mu(x_1)\phi(x_2)\}\rangle=\Delta_{\mu 1}(x_1-x_2)\equiv\frac{\partial}{\partial x_{1\mu}}\Delta(x_1-x_2)\,,$$

$$\langle T^*\{\phi_\mu(x_1)\phi_\nu(x_2)\}\rangle=\Delta_{\mu 1,\nu 2}(x_1-x_2)\equiv\frac{\partial}{\partial x_{1\mu}}\frac{\partial}{\partial x_{2\nu}}\Delta(x_1-x_2)\,,\quad(2.6)$$

where

$$\langle T\{\phi(x_1)\phi(x_2)\}\rangle=\Delta(x_1-x_2)\,.\quad(2.7)$$

As usual we take the propagators to be regularized by the Pauli-Villars method or by an equivalent method, and we let the regulator mass $M\to\infty$ at the end. We do not denote the regularized propagator by a different symbol.

Expanding $S^{(2)}$ into normal-ordered products, one obtains

$$S^{(2)}=g^2\int d^4z_1d^4z_2\sum_{m,n=0}^{\infty}S_{m;n}(\Delta(z_1-z_2)):\frac{\phi^m(z_1)}{m!}\frac{\phi^n(z_2)}{n!}:+2S_{m+1,\mu;n}(\Delta(z_1-z_2)):\phi_\mu(z_1)\frac{\phi^m(z_1)}{m!}\frac{\phi^n(z_2)}{n!}:$$
$$+S_{m+1,\mu;n+1,\nu}(\Delta(z_1-z_2)):\phi_\mu(z_1)\frac{\phi^m(z_1)}{m!}\phi_\nu(z_2)\frac{\phi^n(z_2)}{n!}:+2S_{m+2,\mu\rho;n}(\Delta(z_1-z_2)):\phi_\mu(z_1)\phi_\rho(z_1)\frac{\phi^m(z_1)}{m!}\frac{\phi^n(z_2)}{n!}:$$
$$+2S_{m+2,\mu\rho;n+1,\nu}(\Delta(z_1-z_2)):\phi_\mu(z_1)\phi_\rho(z_1)\frac{\phi^m(z_1)}{m!}\phi_\nu(z_2)\frac{\phi^n(z_2)}{n!}:$$
$$+S_{m+2,\mu\rho;n+2,\nu\sigma}(\Delta(z_1-z_2)):\phi_\mu(z_1)\phi_\rho(z_1)\frac{\phi^m(z_1)}{m!}\phi_\nu(z_2)\phi_\sigma(z_2)\frac{\phi^n(z_2)}{n!}:.\quad(2.8)$$

The coefficient functions in the normal-product expansion (2.8) can be written in the form[3]

$$S_{mK;nL}(\Delta)=\exp\left(\frac{\partial}{\partial\phi_{1,M}}\Delta_{MN}\frac{\partial}{\partial\phi_{2,N}}\right)\left(\frac{\partial}{\partial\phi_{1,K}}\right)^m\times\left(\frac{\partial}{\partial\phi_{2,L}}\right)^n L_I(\phi_{1,M})L_I(\phi_{2,N})\Bigg|_{\phi_{1,M}=0;\,\phi_{2,N}=0}\,,\quad(2.9)$$

where

$$\Delta_{MN}=\begin{pmatrix}\Delta & \Delta_{\nu 2}\\ \Delta_{\mu 1} & \Delta_{\mu 1,\nu 2}\end{pmatrix}.\quad(2.10)$$

The indices m and n refer to the number of external lines (including derived scalar lines) at vertices 1 and 2 and K, L are the five-vector labels.

Substituting the five-dimensional Laplace transform $\tilde{L}_I(\zeta_M)$ of $L_I(\phi_M)$, where

$$L_I(\phi_M)=\int_0^\infty d^5\zeta_{(M)}e^{-\phi_M\zeta_N}\tilde{L}_I(\zeta_M)\,,\quad(2.11)$$

into Eq. (2.9) and noting that

$$\frac{\partial}{\partial\phi_M}L_I(\phi_M)\Bigg|_{\phi\to 0}=\int_0^\infty d^5\zeta_{(M)}(-\zeta_M)\tilde{L}_I(\zeta_M)e^{-\phi_M\zeta_M}\,,\quad(2.12)$$

we obtain

$$S_{mK;nL}(\Delta)=\int_0^\infty\int_0^\infty d^5\zeta_{1,M}d^5\zeta_{2,N}(-\zeta_{1,K})^m(-\zeta_{2,L})^n\times\tilde{L}_I(\zeta_{1,M})\tilde{L}_I(\zeta_{2,N})\exp(\zeta_{1,M}\Delta_{MN}\zeta_{2,N})\,.\quad(2.13)$$

[6] P. T. Matthews, Phys. Rev. **75**, 1270 (1949); F. J. Dyson, *ibid.* **83**, 608 (1951).

Taking L_I to be given by Eq. (2.2) gives

$$\tilde{L}_I(\zeta_M)=\tilde{v}(\zeta)\frac{\partial^2}{\partial\zeta_\mu\partial\zeta_\mu}\delta^4(\zeta_\mu)\,,\quad(2.14)$$

where $\tilde{v}(\zeta)$ is the Laplace transform of $v(\phi)$. Substituting into Eq. (2.13), we obtain for the case of no external derived scalar lines

$$S_{m;n}(\Delta)=\int_0^\infty\int_0^\infty d\zeta_1d\zeta_2d^4\zeta_{1(\mu)}d^4\zeta_{2(\nu)}(-\zeta_1)^m(-\zeta_2)^n\times\tilde{v}(\zeta_1)\tilde{v}(\zeta_2)\left[\frac{\partial^2}{\partial\zeta_{1,\mu}\partial\zeta_{1,\mu}}\delta^4(\zeta_{1\mu})\right]\left[\frac{\partial^2}{\partial\zeta_{2,\nu}\partial\zeta_{2,\nu}}\delta^4(\zeta_{2,\nu})\right]\times\exp(\zeta_1\Delta\zeta_2+\zeta_{1\mu}\Delta_{\mu 1}\zeta_2+\zeta_1\Delta_{\nu 2}\zeta_{2,\nu}+\zeta_{1\mu}\Delta_{\mu 1,\nu 2}\zeta_{\nu 2})\,.\quad(2.15)$$

Higher-order S-matrix elements can be obtained by a generalization of this equation. The vector integrations are performed by partial integration to give

$$S_{m;n}(\Delta)=\int_0^\infty\int_0^\infty d\zeta_1d\zeta_2(-\zeta_1)^m(-\zeta_2)^n\tilde{v}(\zeta_1)\tilde{v}(\zeta_2)\times[2\Delta_{\mu 1,\nu 2}\Delta_{\mu 1,\nu 2}+4\Delta_{\mu 1}\Delta_{\mu 1,\nu 2}\Delta_{\nu 2}\zeta_1\zeta_2+\Delta_{\mu 1}\Delta_{\mu 1}\Delta_{\nu 2}\Delta_{\nu 2}(\zeta_1\zeta_2)^2]e^{\zeta_1\Delta\zeta_2}\quad(2.16)$$
$$=\Theta\int_0^\infty\int_0^\infty d\zeta_1d\zeta_2(-\zeta_1)^m(-\zeta_2)^n\times\tilde{v}(\zeta_1)\tilde{v}(\zeta_2)e^{\zeta_1\Delta\zeta_2}\equiv I_{m;n}(\Delta)\,,$$

where the operator Θ is defined by

$$\Theta=2\Delta_{\mu 1,\nu 2}\Delta_{\mu 1,\nu 2}+4\Delta_{\mu 1}\Delta_{\mu 1,\nu 2}\Delta_{\nu 2}(\partial/\partial\Delta)+\Delta_{\mu 1}\Delta_{\mu 1}\Delta_{\nu 2}\Delta_{\nu 2}(\partial/\partial\Delta)^2.\quad(2.17)$$

All other graphs can be derived in a similar manner but they are more immediately obtained by partial differentiation of $S_{m;n}(\Delta)$ with respect to the $\Delta_{\mu 1}$, $\Delta_{\nu 2}$, and $\Delta_{\mu 1,\nu 2}$ propagators. From the general formula (2.9) it follows that

$$S_{m+1,\mu;n+1}(\Delta)=\frac{\partial}{\partial\Delta_{\mu 1}}S_{m;n}(\Delta),$$

$$S_{m+1,\mu;n+1,\nu}(\Delta)=\frac{\partial}{\partial\Delta_{\mu 1,\nu 2}}S_{m;n}(\Delta),$$

$$S_{m+2,\mu\rho;n+2}(\Delta)=\frac{\partial}{\partial\Delta_{\mu 1}}\frac{\partial}{\partial\Delta_{\rho 1}}S_{m;n}(\Delta), \qquad (2.18)$$

$$S_{m+2,\mu\rho;n+2,\nu}(\Delta)=\frac{\partial}{\partial\Delta_{\mu 1,\nu 2}}\frac{\partial}{\partial\Delta_{\rho 1}}S_{m;n}(\Delta),$$

$$S_{m+2,\mu\rho;n+2,\nu\sigma}(\Delta)=\frac{\partial}{\partial\Delta_{\mu 1,\nu 2}}\frac{\partial}{\partial\Delta_{\rho 1,\sigma 2}}S_{m;n}(\Delta).$$

In addition, there is the symmetry relation

$$S_{mK;nL}(\Delta)=S_{nL:mK}(\Delta). \qquad (2.19)$$

Performing the propagator differentiations which act only on the Θ operator, we obtain the following formulas for all second-order contributions:

$$\begin{aligned}S_{m,\mu;n}(\Delta)&=[4\Delta_{\mu 1,\nu 2}\Delta_{\nu 2}\\&\quad+2\Delta_{\mu 1}\Delta_{\nu 2}\Delta_{\nu 2}(\partial/\partial\Delta)]I_{m;n}(\Delta),\\S_{m,\mu\rho;n}(\Delta)&=2g_{\mu\rho}\Delta_{\nu 2}\Delta_{\nu 2}I_{m;n}(\Delta),\\S_{m+1,\mu;n+1,\nu}(\Delta)&=4[\Delta_{\mu 1,\nu 2}\\&\quad+\Delta_{\mu 1}\Delta_{\nu 2}(\partial/\partial\Delta)]I_{m;n}(\Delta),\\S_{m+1,\mu\rho;n+1,\nu}(\Delta)&=4g_{\mu\rho}\Delta_{\nu 2}I_{m;n}(\Delta),\\S_{m+2,\mu\rho;n+2,\nu\sigma}(\Delta)&=4g_{\mu\rho}g_{\nu\sigma}I_{m;n}(\Delta).\end{aligned} \qquad (2.20)$$

In practice it is also very useful to note the identity

$$S_{m+l;l}\equiv(\partial/\partial\Delta)^{l}S_{m;0}. \qquad (2.21)$$

We make the observation that all graphs are written in the form of an operator acting on $I_{m;n}(\Delta)$ *integrals. These integrals are identical to those which one obtains for second-order diagrams with m external lines at one vertex and n at the other, using a nonderivative Lagrangian* $v(\phi)$.[1–3] *Thus all the dependence coming from the derivative part of the Lagrangian has been completely separated out.*

To take into account the contributions from the product of the two Lagrangians $L_i(\phi_1)$ and $L_I(\phi_2,\phi_{2,\mu})$ defined in Eqs. (2.1) and (2.2), respectively, we again expand the modified time-ordering operator into a series of normal-ordered products. The corresponding coefficient functions in the expansion are $\hat{S}_{m;n}(\Delta)$, $2\hat{S}_{m+1;n+1,\nu}(\Delta)$, and $\hat{S}_{m+2;n+2,\nu\rho}(\Delta)$, where

$$\begin{aligned}\hat{S}_{m;n}(\Delta)&=\exp\left[\frac{\partial}{\partial\phi_1}\left(\Delta\frac{\partial}{\partial\phi_2}+\Delta_{\nu 2}\frac{\partial}{\partial\phi_{2,\nu}}\right)\right]\left(\frac{\partial}{\partial\phi_1}\right)^m\\&\quad\times\left(\frac{\partial}{\partial\phi_2}\right)^n L_i(\phi_1)L_I(\phi_{2,N})\bigg|_{\phi_1=0;\,\phi_{2,N}=0}\\&=\Delta_{\nu 2}\Delta_{\nu 2}\int_0^\infty\int_0^\infty d\zeta_1 d\zeta_2(-\zeta_1)^{m+2}(-\zeta_2)^n\\&\quad\times\tilde{u}(\zeta_1)\tilde{v}(\zeta_2)\exp(\zeta_1\Delta\zeta_2)\\&\equiv\Delta_{\nu 2}\Delta_{\nu 2}\hat{I}_{m;n}(\Delta),\end{aligned} \qquad (2.22)$$

$\tilde{u}(\zeta_1)$ being the Laplace transform of $u(\phi_1)$,

$$\begin{aligned}S_{m+1;n+1,\nu}(\Delta)&=\frac{\partial}{\partial\Delta_{\nu 2}}\hat{S}_{m,n}(\Delta)\\&=2\Delta_{\nu 2}\hat{I}_{m;n}(\Delta)\end{aligned} \qquad (2.23)$$

and

$$\hat{S}_{m+2;n+2,\nu\rho}(\Delta)=2g_{\nu\rho}\hat{I}_{m;n}(\Delta). \qquad (2.24)$$

There are also similar contributions with $L_i(\phi)$ and $L_I(\phi_2,\phi_{2\mu})$ interchanged. The $\hat{I}_{m;n}$ integrals are again identical to those which occur, in second order, for a nonderivative Lagrangian corresponding to a Lagrangian $u(\phi)$ at one vertex and $v(\phi)$ at the other. Both the $I_{m;n}$ and $\hat{I}_{m;n}$ integrals can easily be evaluated.[1–3] For the remainder of this paper we only consider the contributions from the derivative-interaction Lagrangian.

We shall consider the case when $v(\phi)$ is a linear combination of expressions of the form

$$w(\phi)=(f^2\phi^2)^\alpha/(1-f^2\phi^2)^\beta, \qquad (2.25)$$

where α and β are integers. The restriction $\beta>\alpha\geqslant 0$ is also made since with this condition we shall see that we meet no difficulties with over-all ultraviolet divergences. We note that any expression of the form (2.25) can be written as a sum of similar terms satisfying the condition $\beta>\alpha\geqslant 0$ together with a polynomial in ϕ^2. The polynomial in ϕ^2 can then be treated separately.

Expanding $w(\phi^2)$ binomially, we have

$$w(\phi)=\sum_{r=}^{\infty}\frac{(\beta-\alpha+r-1)\cdots(r-\alpha+1)}{(\beta-1)!}(f^2\phi^2)^r. \qquad (2.26)$$

Since $v(\phi)$ is some linear combination of the $w(\phi^2)$'s, we deduce that

$$v(\phi)=\sum_{r=0}^{\infty}c(r)(f^2\phi^2)^r, \qquad (2.27)$$

where $c(r)$ is a polynomial in r. It is now easily shown[1–3] that for $v(\phi)$ given by (2.27),

$$I_{2n;0}(\Delta)=\sum_{r=0}^{\infty} c_1(r+n)c_2(r)(2r+2n)!\times(f_1{}^2)^{r+n}(f_2{}^2)^r\Delta^{2r}, \quad (2.28)$$

where for later convenience we have distinguished between the two vertices by subindices 1 and 2. $I_{2n+1;0}(\Delta)$ is zero, and the general integral $I_{2n+l;l}$ can be obtained by l differentiations with respect to Δ [Eq. (2.21)]. Thus all second-order graphs may be derived from Eq. (2.28) with use of the appropriate operators defined in (2.20).

III. FOURIER TRANSFORMS

In this section we derive the Fourier transforms of the self-energy graphs in the zero-mass case. As has been discussed in great detail by Efimov[1] and by Salam and Strathdee,[7] the Fourier transform is first taken in the Symanzik region in p space ($p^2<0$) and the results obtained are then analytically continued to timelike values of p^2. For $p^2<0$ one continues the propagators $\Delta(x)$ into Euclidean x space. Hence, in the zero-mass case, one obtains

$$\Delta(x)=-1/4\pi^2x^2, \quad (3.1)$$

where $x^2=-x_0{}^2-\mathbf{x}^2$.

We consider the interaction Lagrangian

$$gL_I(\phi,\partial_\mu\phi)=g\!:\partial_\mu\phi\partial_\mu\phi v(\phi)\!:, \quad (3.2)$$

where $v(\phi)$ is of the form defined in Eq. (2.27). The second-order self-energy contributions $S_{1;1}$, $S_{1,\mu;1}$, $S_{1,\mu;1,\nu}$, $S_{2;0}$, $S_{2,\mu;0}$, and $S_{2,\mu\nu;0}$ are given by (2.16) and (2.20). Explicitly we have

$$\begin{aligned}
S_{1;1}&=\Theta\frac{\partial}{\partial\Delta}I_{0;0}(\Delta),\\
S_{1,\mu;1}&=\left(4\Delta_{\mu1,\nu2}\Delta_{\nu2}+2\Delta_{\mu1}\Delta_{\nu2}\Delta_{\nu2}\frac{\partial}{\partial\Delta}\right)\frac{\partial}{\partial\Delta}I_{0;0}(\Delta),\\
S_{1,\mu;1,\nu}&=4\left(\Delta_{\mu1,\nu2}+\Delta_{\mu1}\Delta_{\nu2}\frac{\partial}{\partial\Delta}\right)I_{0;0}(\Delta),\\
S_{2;0}&=\Theta I_{2;0}(\Delta),\\
S_{2,\mu;0}&=\left(4\Delta_{\mu1,\nu2}\Delta_{\nu2}+2\Delta_{\mu1}\Delta_{\nu2}\Delta_{\nu2}\frac{\partial}{\partial\Delta}\right)I_{2;0}(\Delta),\\
S_{2,\mu\nu;0}&=2g_{\mu\nu}\Delta_{\rho2}\Delta_{\rho2}I_{2;0}(\Delta)
\end{aligned} \quad (3.3)$$

where, from Eq. (2.28),

$$\begin{aligned}
I_{0;0}(\Delta)&=\sum_{r=0}^{\infty} c(r)c(r)(2r)!f^{4r}\Delta^{2r},\\
I_{2;0}(\Delta)&=f_1{}^2\sum_{r=0}^{\infty} c(r+1)c(r)(2r+2)!f^{4r}\Delta^{2r},
\end{aligned} \quad (3.4)$$

where $f^4=f_1{}^2f_2{}^2$.

[7] A. Salam and J. Strathdee, Phys. Rev. D 1, 3296 (1970).

From Eq. (3.1) it follows that

$$\begin{aligned}
\Delta_{\mu1}(x)&=\tfrac{1}{2}(4\pi)^2\Delta^2x_\mu,\\
\Delta_{\mu1\nu2}(x)&=-\tfrac{1}{2}(4\pi)^2\Delta^2(g_{\mu\nu}-4x_\mu x_\nu/x^2),
\end{aligned} \quad (3.5)$$

where $x=x_1-x_2$. Hence the Θ operator defined in Eq. (2.17) takes on the following form:

$$\Theta=(4\pi)^4\Delta^4\left[6+6\Delta\frac{\partial}{\partial\Delta}+\Delta^2\left(\frac{\partial}{\partial\Delta}\right)\right]^2. \quad (3.6)$$

Thus

$$\begin{aligned}
S_{1;1}(\Delta)&\equiv S_{1;1}(\Delta,f^4)\\
&=\Theta\sum_{r=0}^{\infty} c(r)^2 2r(2r)!f^{4r}\Delta^{2r-1}\\
&=(4\pi)^4\sum_{r=0}^{\infty} c(r)^2 2r(2r+2)!f^{4r}\Delta^{2r+3}\\
&=\frac{(4\pi)^4}{2i}\int_C\frac{dz}{\sin\pi z}c(z)^2 2z\\
&\qquad\times\Gamma(2z+3)(e^{\pm i\pi}f^4)^z\Delta^{2z+3},
\end{aligned} \quad (3.7)$$

where we have written the sum as a Sommerfeld-Watson integral with the contour C taken counterclockwise around the poles on the positive z axis including the point $z=0$. Details of such a procedure may be found in Refs. 4 and 7. Volkov[4] has discussed the restrictions imposed on the coefficients $c(z)^2$.

The invalidity of Carlson's theorem for the formal power series in (3.7) implies that the analytic continuation of the coefficients $c(r)^2$ from the positive integers to complex values z is not, in general, unique. For example, additional terms of the form

$$d(r)\sin\pi r\,\Delta^r$$

with undetermined coefficients $d(r)$ may be added. Following Volkov,[4] it is, however, possible in certain cases to obtain a unique analytic continuation of $c(r)^2$. For instance, the requirement

$$\lim_{r\to\infty}(2r)^{-a}|(2r)!c(r)^2|^{1/2r}=A, \quad (3.8)$$

with $0\leqslant a<2$ and $A>0$, determines $c(z)^2$ uniquely and sets $d(z)\equiv0$. Condition (3.8) determines a class of nonlocal interactions[4] and, with our restriction that $c(r)$ is a polynomial in r, the Lagrangians (and in particular the chiral Lagrangians) which we are considering fall into this class. In the case of more general coefficients, the presence of the terms $d(r)\sin\pi r\,\Delta^r$ would lead, after taking the Fourier transforms, to an undetermined entire function in the energy for the self-energy graphs. These coefficients $d(r)$ clearly play the role of an infinite set of renormalization constants. It is of prime importance that these are identically zero for nonlocal interactions (3.8), in particular for chiral Lagrangians which are not written in exponential coordinates.

We also point out at this stage that with our restrictions on $c(r)$ we can show, by treating the divergent series in (3.7) as an asymptotic series, that the ultraviolet behavior is given by the pole at $r=-1$,

$$S_{1;1}(\Delta)\sim -(4\pi)^4 2c(-1)^2\,\Delta/f^4 + O(\Delta^{-1}) \quad \text{as} \quad \Delta\to\infty\,. \tag{3.9}$$

Hence it is lower than Δ^2 and therefore the S matrix has no over-all ultraviolet divergence.

The integral (3.7) has a cut in f^4 from 0 to $+\infty$. The Fourier transform will first be taken for negative values of f^4 and the result then analytically continued to positive physical values of f^4 with an averaging procedure determined by unitarity. For the massless case the Fourier transform of $\Delta^z(x)$ is given by[7]

$$D(p^2,z)=i\int d^4x\; e^{ipx}\Delta^z(x) = \frac{-(16\pi)^{-1}}{\sin\pi z\;\Gamma(z)\Gamma(z-1)}\left(\frac{-p^2}{16\pi^2}\right)^{z-2}, \tag{3.10}$$

valid initially in the strip $0<\mathrm{Re}z<2$ and outside it by analytic continuation. In order to take the Fourier transform of $S_{1;1}(\Delta,-f^4)$, the Sommerfeld-Watson contour is deformed to lie in the strip $1<\mathrm{Re}(2z+3)<2$ along the imaginary axis. This can be done without picking up additional pole contributions since no over-all ultraviolet divergences are present. One obtains for $s=p^2<0$ and in the $\mathrm{Re}f^4>0$ half of the $(-f^4)$ plane

$$F(s,-f^4)=i\int d^4x\; e^{ipx}S_{1;1}(\Delta(x),-f^4) = -8i\pi^3\int_{\alpha+i\infty}^{\alpha-i\infty}\frac{dz}{\sin\pi z}\,\frac{c(z)^2 2z}{\sin2\pi z\;\Gamma(2z+2)} \times f^{4z}\left(\frac{-s}{16\pi^2}\right)^{2z+1}, \tag{3.11}$$

where $-1<\mathrm{Re}\alpha<-\frac{1}{2}$. We next collapse the contour back around the positive real axis and pick up the residues of the first- and second-order poles to obtain

$$F(s,-f^4)=-c(0)^2 s +8\pi^3\sum_{r=0}(-)^r\frac{(2r-1)c(r-\frac{1}{2})^2 f^{4r-2}}{(2r)!}\left(\frac{s}{16\pi^2}\right)^{2r} -s\sum_{r=1}^{\infty}\frac{(-)^r(2r)c(r)^2}{(2r+1)!}\left(\frac{f^2s}{16\pi^2}\right)^{2r}\left[\tfrac{1}{2}\ln f^4+\ln\left(\frac{-s}{16\pi^2}\right) + \frac{1}{2}\frac{d}{dz}\ln c(z)^2\Big|_{z=r} -\psi(2r+2)+\frac{1}{2r}\right]. \tag{3.12}$$

Here we point out that the original x-space sum (3.7) contains only odd powers of Δ while the evaluation of the Sommerfeld-Watson contour integral, after taking the Fourier transform, also yields terms arising from initially nonexistent even powers of Δ. The mathematical reason for this is that the Fourier transform (3.10) of $\Delta^z(x)$ has itself simple poles at the integers $z=2,3,4,\ldots$, and therefore changes—as a renormalization—the original simple poles under the Sommerfeld-Watson integral (3.7) into a series of double poles while introducing simple poles for the even powers of Δ.

Analytic continuation of $F(s,-f^4)$ to positive values of the coupling constant f^4 from below and above the cut in the f^4 plane determines the physical amplitude to be

$$F(s,f^4,b)=\alpha F(s,-f^4e^{i\pi})+\beta F(s,-f^4e^{-i\pi}), \tag{3.13}$$

where

$$\alpha+\beta=1\,,\quad \mathrm{Re}(\alpha-\beta)=0\,. \tag{3.14}$$

The second equation in (3.14) follows from unitarity. Thus

$$\alpha=\tfrac{1}{2}(1-ib)\,,\quad \beta=\tfrac{1}{2}(1+ib)\,, \tag{3.15}$$

where b is an arbitrary real constant. Therefore the Fourier transform of the self-energy diagram $S_{1;1}(\Delta)$ is given by

$$F(s,f^4,b)=-c(0)^2 s +b\frac{8\pi^3}{f^2}c(-\tfrac{1}{2})^2-b8\pi^3\sum_{r=0}^{\infty}\frac{(r+1)c(\frac{1}{2}r+\frac{1}{2})^2}{(r+2)!}\left(\frac{f^2s}{16\pi^2}\right)^{r+2} -s\sum_{r=1}^{\infty}\frac{(2r)c(r)^2}{(2r+1)!}\left(\frac{f^2s}{16\pi^2}\right)^{2r}\left[\ln\left(\frac{-sf^2}{16\pi^2}\right) +\frac{1}{2}\frac{d}{dz}\ln c(z)^2\Big|_{z=r}-\psi(2r+2)+\frac{1}{2r}\right], \tag{3.16}$$

where $f^2=+(f_1^2f_2^2)^{1/2}$.

The amplitude $F(s,f^4,b)$ may be written in the form

$$F(s,f^4,b)=F_1(s,f^4)+bF_2(s,f^4)\,, \tag{3.17}$$

where $F_2(s,f^4)$ is an entire function of s. In the limit $s\to 0$, i.e., on the mass shell, one obtains

$$bF_2(s,f^4)=b\frac{8\pi^3}{f^2}c(-\tfrac{1}{2})^2 \quad \text{at} \quad s=0\,, \tag{3.18}$$

while

$$F_1(s,f^4)=0 \quad \text{at} \quad s=0\,. \tag{3.19}$$

Similarly the p-space contributions from $2S_{1,\mu;1}(\Delta)$ and $S_{1,\mu;1,\nu}(\Delta)$ can be evaluated and are found to be $-4F(s,f^4,b)$ and $+4F(s,f^4,b)$, respectively, where $F(s,f^4,b)$ is as given in Eq. (3.16). Thus the contributions from these two graphs cancel. We stress that without our restrictions on $c(r)$ this relation is not necessarily valid.

The self-energy contribution $S_{2;0}(\Delta)$ is a p^2-independent constant. From Eqs. (3.3), (3.4), and (3.6),

$$S_{2;0}(\Delta)=f_1^2\Theta\sum_{r=0}^{\infty}c(r+1)c(r)(2r+2)!f^{4r}\Delta^{2r}$$
$$=f_1^2(4\pi)^4\sum_{r=0}^{\infty}c(r+1)c(r)(2r+2)(2r+3)!\times f^{4r}\Delta^{2r+4}. \quad (3.20)$$

The asymptotic behavior for large Δ is given by the coefficient at $r=-1$ and normally gives a behavior $\sim\Delta^2$ which would lead to a logarithmic divergence. However, because of the factor $2r+2$ which has arisen by application of the Θ operator, the asymptotic behavior is lower than Δ^2. Performing as before the Fourier transform by Sommerfeld-Watson technique, only the energy-independent term has to be taken into account. We immediately obtain the expression

$$G(f^4)=(16\pi^2/f_2^2)\,c(0)c(-1) \quad (3.21)$$

for the Fourier transform of $S_{2;0}(\Delta)$. We note that here no ambiguity parameter arises.

The contribution from $S_{2,\mu;0}(\Delta)$ always vanishes trivially in x space since it is of the form

$$\delta^4(x)\int S_{2,\mu;0}(\Delta(z))d^4z, \quad (3.22)$$

where

$$S_{2,\mu;0}(\Delta(z))=z_\mu f(z^2) \quad (3.23)$$

and the integral (3.22) then vanishes by symmetry.

Finally we have

$$S_{2,\mu\nu;0}(\Delta)=2g_{\mu\nu}16\pi^2f_1^2\sum_{r=0}^{\infty}c(r+1)c(r)(2r+2)!\times f^{4r}\Delta^{2r+3}, \quad (3.24)$$

which again exhibits no over-all ultraviolet divergences. Taking the Fourier transform, we obtain the expression

$$H(s,f^4)=s2b\pi c(\tfrac{1}{2})c(-\tfrac{1}{2})f_1^2/f^2, \quad (3.25)$$

which clearly vanishes on the mass shell. All final results are to be evaluated with $f_1^2=f_2^2$. Then with the definition $f^2=+(f_1^2f_2^2)^{1/2}$, it follows that

$$f_i^2/f^2=\pm1 \quad \text{for} \quad f_i^2\gtrless0, \quad i=1,2. \quad (3.26)$$

Thus the self-energies from all second-order diagrams for Lagrangians falling into the class that we have considered may be simply determined by substituting into Eqs. (3.16), (3.21), and (3.25) the coefficients $c(r)$ appearing in the expansion of $v(\phi)$, the scalar part of the Lagrangian. Let us remark that the use of the Sommerfeld-Watson method in taking Fourier transforms of divergent series, especially with reference to Eqs. (3.21) and (3.25), has a formal character. However, the results given here can also be obtained using other methods.[5]

The Lagrangian can be shown to be equivalent to a free-field Lagrangian. If, therefore, we wish to require that the sum of all second-order self-energy graphs vanishes on the mass shell for zero-mass particles, then this implies

$$bF_2(f^4)+G(f^4)=0 \quad \text{at} \quad s=0 \quad (3.27)$$

and allows the ambiguity parameter b to be uniquely determined as

$$b=-\frac{2}{\pi}\frac{c(0)c(-1)}{c(-\frac{1}{2})^2}\frac{f^2}{f_2^2}. \quad (3.28)$$

It is seen that zero self-energy (to second order) on the mass shell implies, in general, $b\neq0$. An extremely interesting point to note is that the coefficient $c(-1)$ appearing in the numerator of the expression for b also appears in Eq. (3.9) for the leading term in the ultraviolet behavior of $S_{1;1}(\Delta)$. For the scalar part of the Lagrangian given by Eq. (2.25), we see that this coefficient $c(-1)$ always vanishes if $\beta-2\geqslant\alpha\geqslant0$, i.e., for a theory where the scalar part of the Lagrangian $\sim M^{-4}$ or better applying the usual power-counting method of Dyson.

When the self-energy graphs contain additional ultraviolet divergences, then it is not possible to deduce from Eq. (3.27) a unique value for b. This will be the case for the chiral Lagrangians.

IV. CHIRAL LAGRANGIANS AND ULTRAVIOLET INFINITIES

We now apply our results to chiral meson Lagrangians without isospin which are of the form

$$L(\phi)=\tfrac{1}{2}:g(\phi)\partial_\mu\phi\partial_\mu\phi:, \quad (4.1)$$

where $g(\phi)$ is a metric on the circle S^1. We consider two different coordinate systems on S^1. Coordinate system I is obtained by restricting the coordinates of the plane R^2 to a circle of radius $1/\lambda$, giving

$$g^{\mathrm{I}}(\phi)=1/(1-\lambda^2\phi^2). \quad (4.2)$$

Coordinate system II yields

$$g^{\mathrm{II}}(\phi)=1/[(1+\tfrac{1}{4}\lambda^2\phi^2)^2], \quad (4.3)$$

and is the stereographic coordinate system on S^1. In the case of chiral $SU(2)\times SU(2)$, these two coordinate systems are generalized to coordinates on the three-dimensional sphere S^3 embedded in the Euclidean space R^4.

It is known that in the massless case, scalar Lagrangians of the type given in (4.1) can be reduced to the usual free massless scalar-field Lagrangian

$$L(\psi)=\tfrac{1}{2}:\partial_\mu\psi\partial_\mu\psi: \quad (4.4)$$

by the transformation

$$\psi=G(\phi), \quad (4.5)$$

where

$$G(\phi)=\int g(\phi)^{1/2}d\phi. \tag{4.6}$$

The Lagrangians L^{I} and L^{II} may be generated from the free Lagrangian $L(\psi)$ of Eq. (4.4) by the respective transformations

$$\psi=-(1/\lambda)\sin^{-1}(\lambda\phi) \tag{4.7}$$

and

$$\psi=(2/\lambda)\tan^{-1}(\tfrac{1}{2}\lambda\phi). \tag{4.8}$$

These transformations from a free-field theory are of course not possible for the chiral $SU(2)\times SU(2)$ theory. The two Lagrangians L^{I} and L^{II} are also related by a coordinate transformation of the field since Lagrangian $L^{\text{II}}(\phi)$ can be obtained from the Lagrangian $L^{\text{I}}(\phi)$ by the transformation

$$\phi\to\phi/(1+\tfrac{1}{4}\lambda^2\phi^2). \tag{4.9}$$

Subtracting from the total Lagrangian the free part, one obtains the following two interaction Lagrangians:

$$L_{\text{int}}{}^{\text{I}}(\lambda^2,\phi)=\tfrac{1}{2}:(\partial_\mu\phi)(\partial_\mu\phi)\left(\frac{1}{1-\lambda^2\phi^2}-1\right): \tag{4.10}$$

and

$$L_{\text{int}}{}^{\text{II}}(\kappa^2,\phi)=\tfrac{1}{2}:(\partial_\mu\phi)(\partial_\mu\phi)\left(\frac{1}{(1+\kappa^2\phi^2)^2}-1\right):, \tag{4.11}$$

where

$$\kappa^2=\tfrac{1}{4}\lambda^2. \tag{4.12}$$

Note that the two interaction Lagrangians are also related by differentiation with respect to the coupling constant. We have

$$L_{\text{int}}{}^{\text{II}}(\kappa^2)=\frac{\partial}{\partial\kappa^2}\kappa^2L_{\text{int}}{}^{\text{I}}(-\kappa^2). \tag{4.13}$$

This relation would allow one to deduce, in perturbation theory, all the Green's functions of $L_{\text{int}}{}^{\text{II}}$ to any order from the corresponding Green's functions of $L_{\text{int}}{}^{\text{I}}$ and is one of our reasons for having distinguished between the couplings arising at each vertex.

We see that in both coordinate systems the total Lagrangians can be dealt with as special cases of Sec. III. Simply by noting that $g^{\text{I}}(\phi)\sim M^{-2}$ and $g^{\text{II}}(\phi)\sim M^{-4}$, we can immediately tell that the b-independent contribution to the on-mass-shell self-energy, i.e., that of $S_{2;0}$, will be zero for coordinate system II but nonzero for coordinate system I. Explicitly, the respective coefficients and couplings are

$$\begin{aligned}c^{\text{I}}(r)&=\tfrac{1}{2}, & f_1{}^2&=f_2{}^2=\lambda^2,\\ c^{\text{II}}(r)&=\tfrac{1}{2}(r+1), & f_1{}^2&=f_2{}^2=-\kappa^2,\end{aligned} \tag{4.14}$$

and from Eqs. (3.18) and (3.21) the second-order contributions for the total Lagrangians are

$$\begin{aligned}\tilde{S}^{\text{I}}(s=0,\lambda^2)&=2\pi^3b/\lambda^2+4\pi^2/\lambda^2,\\ \tilde{S}^{\text{II}}(s=0,\kappa^2)&=\pi^3b/2\kappa^2\ (=2\pi^3b/\lambda^2).\end{aligned} \tag{4.15}$$

Note that, with $\lambda^2=4\kappa^2$, the $S_{1;1}$ on-mass-shell contributions are equal in both coordinate systems.

To consider the interaction Lagrangians (4.10) and (4.11), we need only evaluate the additional contributions resulting from the subtraction of the free part from the total Lagrangian. Explicitly, the scalar integrals for the interaction Lagrangian reduce to

$$\begin{aligned}I_{0;0}(\Delta)&=\sum_{r=0}^{\infty}c(r)^2(2r)!f^{4r}\Delta^{2r}-1,\\ I_{2;0}(\Delta)&=f_1{}^2\sum_{r=0}^{\infty}c(r+1)c(r)(2r+2)!f^{4r}\Delta^{2r}-f_1{}^2c(1),\end{aligned} \tag{4.16}$$

where the additional contributions from the free part are just the single terms subtracted off from the infinite sums. These sums simply yield the results already given in Eq. (4.15). Applying the appropriate operators as defined in Eqs. (2.16) and (2.20) to obtain expressions for the x-space second-order contributions, we see that the additional terms only contribute to $S_{1,\mu,1,\nu}(\Delta)$, $S_{2;0}(\Delta)$, and $S_{2,\mu\nu;0}(\Delta)$. The respective expressions for these additional terms are

$$S'_{1,\mu;1,\nu}(\Delta)=-\Delta_{\mu1,\nu2}, \tag{4.17}$$

$$\begin{aligned}S'_{2;0}(\Delta)&=-2f_1{}^2c(1)\Delta_{\mu1,\nu2}\Delta_{\mu1,\nu2}\\ &=-6f_1{}^2(4\pi)^4c(1)\Delta^4,\end{aligned} \tag{4.18}$$

$$\begin{aligned}S'_{2,\mu\nu;0}(\Delta)&=-2g_{\mu\nu}f_1{}^2c(1)\Delta_{\rho2}\Delta_{\rho2}\\ &=2g_{\mu\nu}f_1{}^2(4\pi)^2c(1)\Delta^3.\end{aligned} \tag{4.19}$$

Here we notice the appearance of ultraviolet divergences. In Eq. (4.18) there is a quartic divergence and in Eq. (4.19) a quadratic divergence. In removing these divergences finite parts will remain, which we denote by c_4 and c_2, respectively. Thus the Fourier transforms of (4.17)–(4.19) together yield the following additional contribution to the self-energy:

$$\tilde{S}'(s)=s+c_4+c_2s. \tag{4.20}$$

Thus, on the mass shell we are left with the undetermined constant c_4 arising from the removal of the quartic divergence. Clearly this constant is not necessarily the same in both coordinate systems. Hence Eqs. (4.15) must be amended to

$$\begin{aligned}\tilde{S}^{\text{I}}(s=0,\lambda^2)&=2\pi^3b/\lambda^2+4\pi^2/\lambda^2+c_4,\\ \tilde{S}^{\text{II}}(s=0,\lambda^2)&=2\pi^3b/\lambda^2+c_4{}',\end{aligned} \tag{4.21}$$

which are the final on-mass-shell second-order self-energies.

V. CONCLUSIONS

We have explicitly given the techniques required to calculate second-order diagrams for nonlinear scalar Lagrangians with derivative couplings. The on-mass-shell contributions to the self-energy for the massless case are determined by the analytically continued expansion coefficients $c(z)$ at the critical points $z=-1$, $-\frac{1}{2}$, 0 and yield for a theory with no over-all ultraviolet divergences the general self-mass contribution

$$\delta\mu^2=-g^2\left[b\frac{8\pi^3}{(f_1{}^2f_2{}^2)^{1/2}}c(-\tfrac{1}{2})^2+\frac{16\pi^2}{f_2{}^2}c(0)c(-1)\right],$$

where b is a real parameter. Restricting ourselves to the class of nonlocal interactions considered in this paper, it is clear that the second-order self-mass $\delta\mu^2$ will be an invariant for those field transformations

$$\phi\rightarrow\phi'=G(\phi)$$

which leave unchanged

$$\frac{c(-\frac{1}{2})^2}{(f_1{}^2f_2{}^2)^{1/2}}\quad\text{and}\quad\frac{c(0)c(-1)}{f_2{}^2}.$$

In general, however, there are additional ultraviolet divergences which introduce extra renormalization parameters and then this statement becomes more complicated.

In the case of chiral Lagrangians, the finite self-energy graph $S_{1;1}(\Delta)$ gives the same result on the mass shell for both coordinate systems; the invariance of $c(-\frac{1}{2})^2/(f_1{}^2f_2{}^2)^{1/2}$ is quite remarkable. However, the $S_{2;}0(\Delta)$ graphs are infinite. Hence in this model, using the Efimov-Fradkin method of partial summation of perturbation theories, the theorem of Coleman, Wess, and Zumino,[8] that coordinate transformations leave invariant the on-mass-shell results of S-matrix elements with a fixed number of loops, cannot be checked directly because the S-matrix elements are infinite to each order in $L_{\text{int}}(\phi)$. This theorem must be implemented by the requirement of a coordinate-independent choice of the parameter b and the renormalization parameters. Coordinate independence to second order can be guaranteed by a suitable choice of the renormalization parameters c_4 and c_4'.

The techniques developed here can be extended to the physically important self-energy diagrams of chiral $SU(2)\times SU(2)$ Lagrangians. This will be dealt with in a forthcoming paper.

Note added in proof. We wish to point out an unresolved problem, concerned with the possible connection of the normal ordering prescription that we have employed to the recent work of Charap. He deals with chiral Lagrangians which are not normally ordered and shows the necessity of modifying the T^* product to obtain the correct Feynman rules. This problem is being examined.

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