# Relativistic Kepler Problem

CHRISTIAN FRONSDAL
*Department of Physics, University of California, Los Angeles, California 90024*
(Received 18 May 1970)

Relativistic quantum field theory is used as a starting point to construct a classical, completely relativistic theory of planetary orbits.

## INTRODUCTION

CONSIDER a quantum field theory of three scalar fields $\psi_1(x)$, $\psi_2(x)$, and $A(x)$, where $\psi_1$ and $\psi_2$ are associated with particles with masses $m_1$ and $m_2$, and $A(x)$ is massless. Let the interaction Lagrangian be $g\psi_1^*(x)\psi_1(x)A(x)+g\psi_2^*(x)\psi_2(x)A(x)$. In a very good approximation—at least if $g$ is small and in a region of energies near threshold—the two-particle sector of states having one massive particle of each kind, in either a bound or a scattering state, can be described by a relativistic Lippmann-Schwinger equation.[1,2] Let $T$ be the relativistic scattering amplitude for elastic scattering of the two massive particles from each other, and let $G$ be one of the two-particle free-field Green's functions. Then it is possible to construct a potential $V$ so that the following equation is satisfied to all orders of perturbation theory:

$$T=-V+VGT\,. \tag{1}$$

When $V$ is approximated by the first few terms of the perturbation expansion, then this equation can be solved for $T$ to give a nonperturbative approximation for the exact scattering amplitude.

For the two-particle Green's function $G$, we take[3]

$$G=-i\Delta_F(2)\hat{\Delta}(1)\,,\quad \hat{\Delta}=2\pi\epsilon(p_1)\delta(p_1^2-m_1^2)\,. \tag{2}$$

The $\delta$ function restricts the momentum of one of the particles to the mass shell, so that the scattering amplitude and the potential, as well as the two-particle wave function $\psi(p_1,p_2)$, need be defined on $p_1^2=m_1^2$ only. For $V$ we shall take the first term of the perturbation expansion:

$$V=g^2/(p_1-p_1')^2\,. \tag{3}$$

With this choice of $G$ and of $V$, one easily obtains the following equation for the wave function:

$$[(p-q)^2-m_2^2-\gamma/m_1r]\psi=0\,, \tag{4}$$

where $1/m_1r$ is the integral operator

$$\frac{1}{m_1r}\psi(p,q)=-\pi^{-2}\int\frac{(dq')}{(q-q')^2}\psi(p,q') \tag{5}$$

and $\gamma=-g^2/8\pi$. The volume element is $(dq)=d^4q\times\delta(q^2-m_1^2)\epsilon(q_0)$. The external and the relative coordinates and momenta have been defined as follows:

$$p=p_1+p_2\,,\quad q=p_1\,, \tag{6}$$

$$x=x_2\,,\qquad y=x_1-x_2\,. \tag{7}$$

The restriction $p_1^2=m_1^2$ involves the relative momentum only, so that this condition becomes a subsidiary condition:

$$(q^2-m_1^2)\psi=0\,. \tag{8}$$

The dynamics is then given by Eq. (4) alone; this equation can be derived from the Lagrangian

$$\int d^4p(dq)\psi^*L(p)\psi\,, \tag{9}$$

with

$$L(p)=(p-q)^2-m_2^2-\gamma/m_1r\,. \tag{10}$$

On this basis one can develop[2] the entire axiomatic structure of quantum mechanics, including the usual interpretation by means of the conservation laws. In particular, the normalization condition for bound states is, in the c.m. system,

$$\int(dq)\psi^*2(p_0-q_0)\psi\Big|_{\mathbf{p}=0}=1\,, \tag{11}$$

and the expectation value of an operator $A$ is defined by

$$\langle A\rangle=\int(dq)\psi^*2(p_0-q_0)A\psi\Big|_{\mathbf{p}=0}\,. \tag{12}$$

The wave equation (4) can be solved, and closed analytic expressions have been obtained for the scattering matrix.[2]

## EHRENFEST FORMULA

To every quantum-mechanical operator there corresponds a classical observable that will be denoted by the same symbol:

$$A=\lim_{h\to 0}\langle A\rangle\,. \tag{13}$$

The correspondence is a product-preserving linear map, since

$$\sum\psi\psi^*2(p_0-q_0)=1\,. \tag{14}$$

Poisson brackets between the classical observables

[1] C. Fronsdal and R. Huff, Phys. Rev. D (to be published).
[2] C. Fronsdal and L.-E. Lundberg, Phys. Rev. D 1, 3247 (1970).
[3] Our choice is different from that of Bethe and Salpeter, and our potential and off-shell $T$ matrix are also different.

are defined by

$$\{A,B\}=\lim_{\hbar\to 0}(i\hbar)^{-1}\langle[A,B]\rangle\,, \tag{15}$$

and this correspondence is a Lie algebra homomorphism.

Ehrenfest's formula for the time derivative of a classical observable is easily derived in the usual way, the Lagrange operator (10) taking the place of the Hamiltonian. Let $\psi(p)$ and $\psi(p')$ be two wave functions with slightly different four-momenta $p$ and $p'$. Then

$$0=\int(dq)\psi^*(p)L(p)A\psi(p')$$

$$=\int(dq)\psi^*(p)(\{L(p)-L(p')\}A$$

$$+[L(p'),A])\psi(p')\,. \tag{16}$$

To lowest order in $\hbar$ this is the equation of motion:

$$2(p_0-q_0)\dot{A}=\{L,A\}=\{p^2-2pq+\gamma/m_1r,\,A\}\,. \tag{17}$$

Note that the substitution $\mathbf{p}\to 0$ is to be carried out after evaluating the commutator in (15). The dot means, of course, the time derivative—the "time" being $t=t_2$, the variable conjugate to the total energy.

## KINEMATICS

The external variables of the system, which are constants of the motion in the absence of external forces, are the total linear momentum $p_\mu$ and the total angular momentum $L_{\mu\nu}$. We work in the c.m. frame, in which $\mathbf{p}=0$. Four-momenta $q_\mu$ and $p_\mu-q_\mu$ are ascribed to the individual particles, so that no portion of the linear momentum is "carried by the interaction." The total energy $p_0$ measures the binding energy, and $\mathbf{q}$ defines the state of motion of both particles. No additional energy variable is required—hence it is quite appropriate that $q_0$ is fixed in terms of $\mathbf{q}$. In the classical theory positive and negative energies are uncoupled and $q_0=+(\mathbf{q}^2+m_1^2)^{1/2}$.

The external position coordinate $\mathbf{x}$ was originally associated with particle 2, and the interpretation of $\mathbf{x}=\mathbf{x}_2$ as position coordinate for particle 2 is easily confirmed by the equations of motion:

$$\dot{\mathbf{x}}=-\mathbf{q}/(p_0-q_0)=\mathbf{p}_2/p_{20}\,, \tag{18}$$

which shows that the time derivative of $\mathbf{x}$ is the velocity of particle 2. The "time" is the 0 component of $x_\mu$; it is conjugate to the total energy.

The internal position coordinate $y_\mu$ is not unambiguously defined. By taking $x_{2\mu}$ as the external position coordinate, we have put particle 2 in a privileged position, and the presence of particle 1 is felt only through the interaction. The coordinate $y_\mu$ is represented in the quantum theory by the symbol $i\hbar\partial/\partial q^\mu$, but this is not an operator since the wave function is defined on the surface $q^2=m_1^2$ only. A partial definition of $y_\mu$ is provided by the total angular momentum tensor

$$L_{\mu\nu}=L_{\mu\nu}{}^x+s_{\mu\nu}\,. \tag{19}$$

Here $L_{\mu\nu}{}^x=-p_\mu x_\nu+p_\nu x_\mu$ is the external or "orbital" part of the angular momentum, and

$$s_{\mu\nu}=-q_\mu y_\nu+q_\nu y_\mu \tag{20}$$

is the internal or "spin" part. The operator $s_{\mu\nu}$ is well defined and corresponds to a well-defined classical observable, and (20) serves to define $y_\mu$. Note that if $y_\mu$ is defined by (20) and $x_{1\mu}=x_\mu+y_\mu$, then

$$L_{\mu\nu}=(-p_{1\mu}x_{1\nu}+p_{1\nu}x_{1\mu})+(-p_{2\mu}x_{2\nu}+p_{2\nu}x_{2\mu})\,, \tag{21}$$

so that the total angular momentum is ascribable to the motion of the two particles and the interaction does not carry any angular momentum either.

The solution of (20) is

$$y_\mu=m_1^{-2}q^\nu s_{\mu\nu}+c_\mu\,, \tag{22}$$

where $c_\mu$ is an arbitrary vector parallel to $q_\mu$. The motion of particle 1 is not subject to a complete and unambiguous determination. We have put ourselves in the role of an observer that travels with particle 2, and all measurements of the whereabouts of particle 1 must be done by means of signals that propagate with finite velocity. The world line can be determined, but neither the position $\mathbf{x}_1$ nor the "time" $x_{10}$ is unambiguously defined. This point is often overlooked and has led to much confusion. Recognizing that the time dependence of $x_{1\mu}$ is partly conventional, we are free to resolve the ambiguity to suit our convenience. The lack of complete predictability of $x_{1\mu}$ was built into the theory from the beginning, by restricting $p_{1\mu}$ to the mass shell. As was explained by Dirac,[4] the time dependence of the coordinates is determined only to the extent that the momenta are unrestrained.

In view of the indeterminacy of the relative coordinate, it is clear that the potential cannot be given as a function of $y_\mu$. In the quantum-mechanical formulation the potential is given as an operator $1/r$, and the classical potential is given in terms of the classical observable $r$. It is not necessary to have an intuitive definition of this quantity, but we do have to know all the Poisson brackets in which it is involved, and this requires that we know a number of commutators in the quantum theory. Note that $r$ is a Lorentz-invariant quantity; in the nonrelativistic limit it is just the spatial separation between the particles, and in the relativistic theory its properties are derived directly from the "one-graviton-exchange" potential of quantum field theory. The required commutation relations can be summarized as follows.[2] Define

$$s_{\mu 4}=-s_{4\mu}=m_1^{-1}(q^\nu s_{\mu\nu}+2i\hbar q_\mu)\,, \tag{23}$$

$$s_{\mu 5}=-s_{5\mu}=rq_\mu\,,\quad s_{45}=-s_{54}=m_1 r\,. \tag{24}$$

[4] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva U. P., New York, 1964).

Then

$$[s_{AB},s_{CD}]=-i\hbar(g_{AC}s_{BD}+g_{BD}s_{AC}-g_{AD}s_{BC}-g_{BC}s_{AD}), \quad (25)$$

where the indices take the values 0, 1, 2, 3, 4, and 5, and $g_{AB}$ is diagonal with signature $+-----+$. In the classical theory the second term in (23) is dropped, and (25) is turned into an expression for the Poisson bracket by striking the factor $i\hbar$.

From the definitions one can derive a number of kinematical identities, most compactly summarized as follows:

$$g^{AB}(s_{AC}s_{BD}+s_{BD}s_{AC})+2\hbar^2 g_{CD}=0, \quad (26)$$

$$\epsilon^{ABCDEF}s_{AB}s_{CD}=0. \quad (27)$$

It is convenient to re-express these identities in terms of the following quantities. First, let $m_\pm=m_1\pm m_2$, and $m_-^2<p^2<m_+^2$. This is the domain of stationary orbits; there is no difficulty in a parallel discussion of asymptotic orbits. Next, let

$$P=+[(m_+^2-p^2)(p^2-m_-^2)]^{1/2} \quad (28)$$

and

$$\lambda_4=(p^2+m_+m_-)/P, \quad (29)$$

$$\lambda_0=+(1+\lambda_4^2)^{1/2}=2m_1p_0/P. \quad (30)$$

Finally, define space vectors $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{B}$ by

$$L_i=\epsilon_{ijk}s_{jk}, \quad (31)$$

$$A_i=\lambda_0 s_{i4}-\lambda_4 s_{i0}, \quad (32)$$

$$B_i=\lambda_0 s_{i0}-\lambda_4 s_{i4}. \quad (33)$$

Then Eq. (27) reduces to

$$\mathbf{L}\cdot\mathbf{q}=\mathbf{L}\cdot\mathbf{A}=\mathbf{L}\cdot\mathbf{B}=0, \quad (34)$$

$$\mathbf{B}\times\mathbf{q}=(\lambda_0q_0-\lambda_4m_1)\mathbf{L}, \quad (35)$$

and the equations for the orbit and the hodograph:

$$\mathbf{A}\times\mathbf{B}=-s_{04}\mathbf{L}, \quad (36)$$

$$\mathbf{A}\times\mathbf{q}=(\lambda_0m_1-\lambda_4q_0)\mathbf{L}, \quad (37)$$

while Eq. (26) becomes

$$\mathbf{A}^2=s_{04}^2+r^2(\lambda_4q_0-\lambda_0m_1)^2, \quad (38)$$

$$\mathbf{B}^2=-s_{04}^2+r^2(\lambda_0q_0-\lambda_4m_1)^2, \quad (39)$$

$$\mathbf{L}^2=-s_{04}^2+r^2\mathbf{q}^2, \quad (40)$$

$$\mathbf{q}\cdot\mathbf{A}=(\lambda_0q_0-\lambda_4m_1)s_{04}, \quad (41)$$

$$\mathbf{q}\cdot\mathbf{B}=(\lambda_0m_1-\lambda_4q_0)s_{04}, \quad (42)$$

$$\mathbf{A}\cdot\mathbf{B}=r^2(\lambda_0q_0-\lambda_4m_1)(\lambda_0m_1-\lambda_4q_0). \quad (43)$$

The content of this section is completely independent of the potential and could be taken to describe the kinematics of two particles in general. The complicated set of Poisson brackets and kinematical identities is not *ad hoc*; it generates itself from the variables $r$ and $q_\mu$. The interpretation of the auxiliary variables $\mathbf{A}$ and $\mathbf{B}$ will result from the equations of motion.

## DYNAMICS

Taking a number of different choices of $A$ in Eq. (17) and using (25), one easily obtains

$$(p_0-q_0)\dot{\mathbf{x}}_2=-\mathbf{q}, \quad m_1r(p_0-q_0)\dot{r}=p_0s_{04}, \quad (44)$$

$$2m_1^2r^3(p_0-q_0)\dot{q}_\mu=-\gamma s_{\mu4}, \quad (45)$$

$$2m_1(p_0-q_0)\dot{s}_{\mu4}=(p^2+m_+m_-)q_\mu-2m_1^2p_\mu, \quad (46)$$

and of course $\dot{p}_\mu=\dot{L}_{\mu\nu}=0$. The complete dynamics includes in addition the equation

$$L(p)=p_2^2-m_2^2-\gamma/m_1r=0, \quad (47)$$

which is the analog of the nonrelativistic $E-H=0$. Equation (34) shows that all the events take place in a plane that is normal to the constant vector $\mathbf{L}$. From (44)–(46) it easily follows that $\mathbf{A}$ is a constant; this is the relativistic analog of the Runge-Lenz vector.[5] From the kinematical identities one now sees that the vector $\mathbf{B}$ moves on an ellipse with major axis parallel to $\mathbf{A}$; if

$$B_1=\mathbf{B}\cdot\mathbf{A}/|\mathbf{A}|, \quad B_2=\pm|\mathbf{B}\times\mathbf{A}|/|\mathbf{A}| \quad (48)$$

then

$$(B_1/n)^2+(B_2/L)^2=1, \quad (49)$$

where $L$ is the magnitude of $\mathbf{L}$ and $n$ is the "principal quantum number":

$$n=+(\mathbf{A}^2+\mathbf{L}^2)^{1/2}. \quad (50)$$

Equation (47) gives the numerical value of $n$:

$$n=-\gamma/P. \quad (51)$$

We now determine the motion of particle 2.

From Eqs. (44)–(46) one easily finds that $\mathbf{x}_2+(2m_1/P)\mathbf{B}$ is a constant. Hence $\mathbf{x}_2$ moves on an ellipse. Taking the center of the ellipse as the origin of the coordinate system we get

$$\mathbf{x}_2=-(2m_1/P)\mathbf{B}. \quad (52)$$

The foci of the orbit are at the points $\mathbf{x}_2=\pm(2m_1/P)\mathbf{A}$. Next we may notice that

$$\tfrac{1}{2}(\mathbf{x}_2-\alpha\mathbf{A})\times\dot{\mathbf{x}}_2=m_1^2\mathbf{L}/p_0(p^2-m_1^2-m_2^2)=\text{const}, \quad (53)$$

provided the constant $\alpha$ has the value

$$\alpha=\frac{2m_1^2}{p_0P}\frac{p^2-m_+m_-}{p^2-m_1^2-m_2^2}. \quad (54)$$

Hence the areal velocity is constant if seen from the point $\mathbf{x}_2=\alpha\mathbf{A}$. This completes the determination of the motion of particle 2.

[5] The following manipulations were greatly facilitated by a comparison with the work of G. Györgyi, Nuovo Cimenti **53A**, 717 (1968).

We have stressed the ambiguity that exists in defining the space coordinates of particle 1, represented by the term $c_\mu$ in (22) about which we know only that it is parallel to $q_\mu$. We now choose $c_\mu=0$ because it leads to simple formulas, but no intuitive reasoning can be applied to the coordinate $x_{1\mu}$ that is obtained in this way. Thus

$$y_\mu=(x_1-x_2)_\mu=m_1^{-2}q^\nu s_{\mu\nu}=m_1^{-1}s_{\mu 4} \tag{55}$$

or

$$\mathbf{x}_1=\frac{p^2-m_1^2-m_2^2}{m_1P}\mathbf{B}+\left(\frac{2p_0}{P}\right)\mathbf{A}. \tag{56}$$

This shows that particle 1 also moves on an ellipse, with foci at $\mathbf{x}_1=(p^2-m_1^2-m_2^2\pm 2m_1p_0)\mathbf{A}/m_1P$. The areal velocity is constant if seen from the point $\mathbf{x}_1=\beta\mathbf{A}$, where

$$\beta=(p^2+m_+m_-)/p_0P\,. \tag{57}$$

There is one linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ that is a constant of the motion, namely,

$$\frac{(p^2-m_1^2-m_2^2)\mathbf{x}_2+2m_1^2\mathbf{x}_1}{p^2+m_+m_-}=\frac{4p_0m_1^2}{P(p^2+m_+m_-)}\mathbf{A}, \tag{58}$$

and this may be called the center of mass. Notice that five points that coincide in the nonrelativistic theory are slightly separated; they are a focus of each ellipse, the two points from which the areal velocities are seen as constants, and the "center of mass." Note that these results depend on the choice $c_\mu=0$ in Eq. (22). One benefit of this choice is a very nice formula for the potential, namely, $r^2=-y_\mu^2$ or

$$\begin{aligned}1/r&=[-(x_1-x_2)_\mu^2]^{-1/2}\\&=1/[(\mathbf{x}_1-\mathbf{x}_2)^2-(t_1-t_2)^2]^{1/2}\,.\end{aligned} \tag{59}$$

Note that $t=t_2$ and that $t_1$ is a known function of $t$, given by

$$t_1-t=m_1^{-2}q^\nu s_{0\nu}=m_1^{-1}s_{04} \tag{60}$$

and the solution of Eqs. (44)–(46). The reality of $r$ is of course guaranteed by the equations of motion, but the result that the relative position coordinate is spacelike ($y_\mu^2=-r^2<0$) carries no absolute significance since it depends on choosing the "gauge" in which $c_\mu=0$. It is our opinion that the ambiguity in the definition of the relative coordinate is intrinsic to the problem and that some works have been hampered by imprecise formulations following from overlooking this important point.

## CONCLUSIONS

Motivated by entirely different considerations, we have stumbled on a theory of classical relativistic mechanics of two interacting particles. Unlike previous works,[6–8] that were formulated entirely within the classical framework and started with intuitive arguments concerning the nature and the properties of the coordinates, the theory presented here was derived from quantum field theory. This allows a determination of the potential to any desired accuracy and guarantees the relevance of the theory for actual physical phenomena, but it does not necessarily conform to any *a priori* ideas that one might have about classical relativistic mechanics. In a future report[9] we intend to build up the same theory by intuitive classical arguments—this will allow a detailed comparison with other approaches,[6–8] although an appeal to quantum field theory will always have to be made to determine the potential.

So far, the following features have emerged. There exists a special potential for which the orbits can be determined analytically. This potential is a natural extension of the Newtonian $1/r$ potential and is directly related to the one-graviton-exchange potential of quantum field theory, and the corresponding orbits are ellipses. Corrections to this special potential come from the tensor nature of the gravitational field as well as field-theoretical effects that are analogous to the fine structure of hydrogen. The field-theoretical corrections include nonlinear effects such as must be included if Einstein's theory of gravitation is treated as a spin-2 field theory in flat space. In a future work we plan to evaluate the advance of the perihelion of Mercury and draw the parallel between this and the fine structure of hydrogen.

---

[6] E. P. Wigner, in *Proceedings of the First Coral Gables Conference on Fundamental Interactions at High Energy* (Freeman, San Francisco, 1969).

[7] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).

[8] A. Schild, Phys. Rev. **131**, 2762 (1963).

[9] C. Fronsdal, UCLA Report, 1971 (unpublished).