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Stationary States of a Spin-1 Particle in a Constant Magnetic Field*

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Shay and Good's wave equation is solved for a spin-1 particle with arbitrary magnetic dipole moment. Simultaneous eigenfunctions of the following three operators are used: p_z, the component of $-i\nabla$ in the direction of the field; J_z, the component of $\mathbf{x}\times\mathbf{p}+\mathbf{s}$; and R_0^2, the operator for the square of the distance to the center of the orbit in the projection of the motion perpendicular to the field. Explicit formulas for the allowed energies in terms of the quantum numbers are found. Determination of the wave functions is reduced to a set of linear algebraic equations.

I. INTRODUCTION

THE problem of finding the states of a particle moving in a constant magnetic field has been solved for many different situations. The non-quantum-mechanical and the nonrelativistic quantum-mechanical solutions have long been known. The relativistic quantum-mechanical solutions for spin 0 and for spin $\frac{1}{2}$ with normal magnetic moment have been completely worked out. Johnson and Lippmann, in particular, have studied the spin-$\frac{1}{2}$ problem and reviewed the earlier contributions.[1] Recently the problem for spin $\frac{1}{2}$ with anomalous magnetic moment has also been solved.[2] In the present work the states of motion of a relativistic spin-1 particle with anomalous magnetic moment are found and discussed.

The spin-1 particle is described by the wave equation given recently by Shay and Good.[3] It is reasonable to assume that this equation applies to a spin-1 particle because it has so many of the correct properties. In particular, it leads to an invariant way to take matrix elements, and it allows for arbitrary magnetic dipole moment and electric quadrupole moment of the particle. However, although the deuteron in an external electromagnetic field is a realizable experimental situation, there are no available experimental tests of what wave equation applies relativistically. Another possibility, for example, is the Corben-Schwinger equation[4]; it differs from the Shay-Good equation in terms that depend quadratically on the external fields.

The method used here is to expand the six-component wave function in terms of a complete set of functions that are simultaneously eigenfunctions of the following three operators: p_z, the component of $-i\nabla$ in the direction of the field; J_z, the component of $\mathbf{x}\times\mathbf{p}+\mathbf{s}$ in the direction of the field; and R_0^2, the operator for the square of the distance to the center of the orbit in the projection of the motion perpendicular to the field. The dependence of these functions on the cylindrical coordinates r, φ, and z is completely determined, and

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[1] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828 (1949); **77**, 702 (1950).
[2] I. M. Ternov, V. G. Bagrov, and V. Ch. Zhukovskii, Moscow University Phys. Bull. **21**, 21 (1966).
[3] D. Shay and R. H. Good, Jr., Phys. Rev. **179**, 1410 (1969).
[4] H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1940).

when they are used the problem of finding the stationary-state solutions of the wave equation reduces to that of solving a six-by-six set of linear algebraic equations. It is not a matrix eigenvalue problem, but nevertheless there are six normalizable solutions. Explicit formulas for the energy values in terms of the quantum numbers of the three operators are derived. The technique is general and can be applied in the spin-0 and spin-$\frac{1}{2}$ problems also, as reported in Appendices B and C.

The calculation is done for an arbitrary g factor of the particle and it is interesting to ask if the results in any way prefer a special value of g. In the nonrelativistic problem for any spin, there is additional degeneracy when $g=2$. In the relativistic problem for spin $\frac{1}{2}$, the two spin orientations are degenerate when $g=2$ and the helicity operator $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$ resolves the degeneracy. In the present spin-1 problem, $g=2$ is again the special value, but it only implies a partial spin-orientation degeneracy and an operator that resolves the degeneracy was not found.

II. NOTATION

The wave equation is

$$W\psi=0\,, \qquad (1)$$

where

$$W=\pi_\alpha\pi_\beta\gamma_{\alpha\beta}+\pi_\alpha\pi_\alpha+2m^2+\tfrac{1}{12}e\lambda\gamma_{5,\alpha\beta}F_{\alpha\beta}+\left(\frac{eq}{6m^2}\right)\gamma_{6,\alpha\beta,\mu\nu}\left(\frac{\partial F_{\alpha\beta}}{\partial x_\mu}\right)\pi_\nu \qquad (2)$$

is the wave equation operator. Here π_α is $-i\partial/\partial x_\alpha-eA_\alpha$ and $F_{\alpha\beta}$ is the field tensor

$$F_{\alpha\beta}=\frac{\partial A_\beta}{\partial x_\alpha}-\frac{\partial A_\alpha}{\partial x_\beta}\,,$$

$$F_{ij}=\epsilon_{ijk}B_k\,,\quad F_{i4}=-F_{4i}=-iE_i\,.$$

The Latin indices run from 1 to 3 and the Greek indices from 1 to 4, with $x_4=it$. Factors of c and $\hbar$ are omitted and e is considered to be a positive number. The constants λ and q are real and adjust the sizes of the intrinsic moments. From former work[3] it is known that the particle's g factor is $\frac{1}{2}(1+\lambda)$. The q term does not contribute in the present problem when the field is constant. The wave function has six components, acted on by the 6×6 γ matrices.

Properties of the matrices are summarized in Ref. 3. It is convenient to work in terms of

$$\boldsymbol{\alpha}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix},\quad \mathbf{s}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & \mathbf{s}\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad (3)$$

where 0 and 1 on the right-hand side are the zero and unit 3×3 matrices, and the components of $\mathbf{s}$ on the right are the usual spin-1 matrices

$$s_1=\tfrac{1}{2}\sqrt{2}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix},\quad s_2=\tfrac{1}{2}\sqrt{2}i\begin{bmatrix}0&-1&0\\1&0&-1\\0&1&0\end{bmatrix},$$

$$s_3=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

In these terms the γ matrices needed are

$$\gamma_{44}=\beta\,,$$

$$\gamma_{i4}=\gamma_{4i}=-i\beta\alpha_i\,,$$

$$\gamma_{ij}=\beta(\delta_{ij}-s_is_j-s_js_i)\,,$$

$$\gamma_{5,ij}=-6\epsilon_{ijk}s_k\,.$$

The problem is to find the stationary-state solutions of the wave equation, meaning those with time dependence e^{-iEt}. With only the static magnetic field considered, π_4 is $-\partial/\partial t$ and may be replaced by iE. The equation for the stationary states is then

$$[(1+\beta)(\boldsymbol{\pi}\cdot\boldsymbol{\pi}-E^2)-2\beta(\mathbf{s}\cdot\boldsymbol{\pi})^2+2E\beta\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+2m^2-(\beta+\lambda)e\mathbf{s}\cdot\mathbf{B}]\psi=0\,. \qquad (4)$$

The vector potential $\mathbf{A}$ which is used is $(-\frac{1}{2}By,\ \frac{1}{2}Bx,\ 0)$. This gives the field $\mathbf{B}=(0,0,B)$ in the 3-direction.

III. SIMULTANEOUS EIGENFUNCTIONS OF p_z, J_z, AND R_0^2

The operators are p_z,

$$J_z=xp_y-yp_x+s_z \qquad (5)$$

and

$$R_0^2=x_0^2+y_0^2\,, \qquad (6)$$

where $\mathbf{p}$ is $-i\boldsymbol{\nabla}$, and where

$$x_0=\tfrac{1}{2}x+p_y/eB\,, \qquad (7a)$$

$$y_0=\tfrac{1}{2}y-p_x/eB\,. \qquad (7b)$$

Usefulness of the operators x_0 and y_0 in this type of problem is well known. Classically, x_0 and y_0 are the coordinates of the center of the circle that the orbit makes when projected on to the xy plane. In cylindrical coordinates r, φ, and z, the operators have the forms

$$p_z=-i\frac{\partial}{\partial z}\,,\qquad J_z=-i\frac{\partial}{\partial\varphi}+s_z\,, \qquad (8)$$

$$R_0^2=-\frac{1}{e^2B^2}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)-\frac{i}{eB}\frac{\partial}{\partial\varphi}+\frac{r^2}{4}\,, \qquad (9)$$

and it is evident here that they commute.

As is easily verified, simultaneous eigenfunctions of the three operators are

$$\psi_{p_z\mu n}=\begin{bmatrix} c_1\rho^{-(\mu/2)+(1/2)}L_n^{-\mu+1}(\rho)e^{i(\mu-i)\varphi} \\ c_2\rho^{-\mu/2}L_n^{-\mu}(\rho)e^{i\mu\varphi} \\ c_3\rho^{-(\mu/2)-(1/2)}L_n^{-\mu-1}(\rho)e^{i(\mu+1)\varphi} \\ c_4\rho^{-(\mu/2)+(1/2)}L_n^{-\mu+1}(\rho)e^{i(\mu-1)\varphi} \\ c_5\rho^{-\mu/2}L_n^{-\mu}(\rho)e^{i\mu\varphi} \\ c_6\rho^{-(\mu/2)-(1/2)}L_n^{-\mu-1}(\rho)e^{i(\mu+1)\varphi} \end{bmatrix} e^{-(\rho/2)+ip_z z}, \quad (10)$$

where the c's are constants, ρ is $\frac{1}{2}eBr^2$, and $L_n^{-\mu}(\rho)$ is the Laguerre polynomial in the notation of Magnus, Oberhettinger, and Soni.[5] The quantum numbers label the eigenvalues in the following way:

$$-i\frac{\partial}{\partial z}\psi_{p_z\mu n}=p_z\psi_{p_z\mu n}, \quad (11a)$$

$$J_z\psi_{p_z\mu n}=\mu\psi_{p_z\mu n}, \quad (11b)$$

$$eBR_0^2\psi_{p_z\mu n}=(2n+1)\psi_{p_z\mu n}. \quad (11c)$$

Here p_z may be any real number, positive or negative; μ may be any integer, positive or negative; $n=0, 1, 2, \ldots$, if $\mu<0$, and $n=\mu+1, \mu+2, \mu+3, \ldots$, if $\mu\geq 0$.

IV. ALLOWED VALUES OF ENERGY

The solutions of Eq. (1) for constant magnetic field are postulated to be of the form

$$\psi=\psi_{p_z\mu n}e^{-iEt},$$

where the c's are to be determined. It is necessary that these functions satisfy Eq. (4).

To simplify one writes $\mathbf{s}\cdot\boldsymbol{\pi}$ as

$$\mathbf{s}\cdot\boldsymbol{\pi}=\begin{bmatrix} p_z & \pi_-/\sqrt{2} & 0 \\ \pi_+/\sqrt{2} & 0 & \pi_-/\sqrt{2} \\ 0 & \pi_+/\sqrt{2} & -p_z \end{bmatrix},$$

where

$$\pi_\pm=\pi_x\pm i\pi_y$$
$$=e^{\pm i\varphi}\left(\frac{1}{i}\frac{\partial}{\partial r}\pm\frac{1}{r}\frac{\partial}{\partial\varphi}\mp\tfrac{1}{2}ieBr\right)$$
$$=e^{\pm i\varphi}(\tfrac{1}{2}eB\rho)^{1/2}\left(\frac{2}{i}\frac{\partial}{\partial\rho}\pm\frac{1}{\rho}\frac{\partial}{\partial\varphi}\mp i\right).$$

The properties of the Laguerre polynomials lead to the result

$$\mathbf{s}\cdot\boldsymbol{\pi}\begin{bmatrix} a\rho^{-(\mu/2)+(1/2)}L_n^{-\mu+1}e^{i(\mu-1)\varphi} \\ b\rho^{-\mu/2}L_n^{-\mu}e^{i\mu\varphi} \\ c\rho^{-(\mu/2)-(1/2)}L_n^{-\mu-1}e^{i(\mu+1)\varphi} \end{bmatrix} e^{-(\rho/2)+ip_z z}$$
$$=\begin{bmatrix} [p_z a+i(eB)^{1/2}b]\rho^{-(\mu/2)+(1/2)}L_n^{-\mu+1}e^{i(\mu-1)\varphi} \\ [-i(eB)^{1/2}(n-\mu+1)a+i(eB)^{1/2}c]\rho^{-\mu/2}L_n^{-\mu}e^{i\mu\varphi} \\ [-i(eB)^{1/2}(n-\mu)b-p_z c]\rho^{-(\mu/2)-(1/2)}L_n^{-\mu-1}e^{i(\mu+1)\varphi} \end{bmatrix} \times e^{-(\rho/2)+ip_z z}. \quad (12)$$

Also, it is seen that

$$\boldsymbol{\pi}\cdot\boldsymbol{\pi}(e^{-\rho/2}\rho^{-\mu/2}L_n^{-\mu}e^{i\mu\varphi})$$
$$=[p_z^2+eB(2n-2\mu+1)](e^{-\rho/2}\rho^{-\mu/2}L_n^{-\mu}e^{i\mu\varphi}). \quad (13)$$

Consequently it is found that Eq. (4) is verified provided that the c's satisfy the algebraic equations

$$\begin{bmatrix}
eB(2n-2\mu-\lambda+3)+p_z^2-E^2+2m^2 & 0 & 0 & -(p_z+E)^2 & -2i(eB)^{1/2}(p_z+E) & 2eB \\
0 & eB(2n-2\mu+1)+p_z^2-E^2+2m^2 & 0 & 2\imath(eB)^{1/2}(n-\mu+1)\times(p_z+E) & -eB(2n-2\mu+1)+p_z^2-E^2 & 2i(eB)^{1/2}(p_z-E) \\
0 & 0 & eB(2n-2\mu+\lambda-1)+p_z^2-E^2+2m^2 & 2eB(n-\mu)\times(n-\mu+1) & -2i(eB)^{1/2}(n-\mu)\times(p_z-E) & -(p_z-E)^2 \\
-(p_z-E)^2 & -2i(eB)^{1/2}(p_z-E) & 2eB & eB(2n-2\mu-\lambda+3)+p_z^2-E^2+2m^2 & 0 & 0 \\
2i(eB)^{1/2}(n-\mu+1)\times(p_z-E) & -eB(2n-2\mu+1)+p_z^2-E^2 & 2i(eB)^{1/2}(p_z+E) & 0 & eB(2n-2\mu+1)+p_z^2-E^2+2m^2 & 0 \\
2eB(n-\mu)\times(n-\mu+1) & -2i(eB)^{1/2}(n-\mu)\times(p_z+E) & -(p_z+E)^2 & 0 & 0 & eB(2n-2\mu+\lambda-1)+p_z^2-E^2+2m^2
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}=0. \quad (14)$$

The allowed values of the energy are obtained by setting the determinant of the coefficients equal to zero. It turns out that one of the following equations must apply:

$$E^2=p_z^2+[m^4-\tfrac{1}{4}(eB)^2(1-\lambda)^2]^{-1}[m^6+m^4eB \times(2n-2\mu+1)-\tfrac{1}{4}m^2(eB)^2(1-\lambda)(3-\lambda)], \quad (15a)$$

$$E^2=p_z^2+m^2+eB(2n-2\mu+1)+\tfrac{1}{4}(eB/m)^2(3-\lambda) \pm\tfrac{1}{4}(eB/m)(3-\lambda)\{4[m^2+eB(2n-2\mu+1)] +(eB/m)^2\}^{1/2}. \quad (15b)$$

Either sign may be used in the second equation. Either sign is allowable in taking the square root of E^2, so there are altogether six solutions of the problem (for fixed quantum numbers p_z, μ, and n) corresponding to three spin orientations for the particle and three for the antiparticle. For a choice of E, one can in principle solve Eq. (14) for the c's except for a normalizing factor. The solution is complicated, but the problem of finding the stationary-state functions is solved in the sense that it is reduced to a set of linear algebraic equations.

V. DISCUSSION

There is a degeneracy in the coordinate dependence of the states; the energies depend on n and μ only in

[5] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966).

the combination $(2n-2\mu+1)$. This is the quantum-mechanical generalization of the classical result that the energy E is $(m^2+p_z^2+\pi^{-1}e^2B^2A)^{1/2}$, where A is the area of the orbit in the projection perpendicular to the z axis. Thus one finds from the definitions of R_0 and J_z that

$$\pi[(x-x_0)^2+(y-y_0)^2]=\pi[R_0^2-2(eB)^{-1}(J_z-s_z)].$$

Therefore, the operator

$$A=\pi[R_0^2-2(eB)^{-1}J_z]$$

is the area of the orbit, with an additional spin contribution. Eigenvalues of A are $\pi(eB)^{-1}(2n+1-2\mu)$. Except for spin complexities, the energy depends only on p_z and the area of the orbit projection.

The nonrelativistic approximation applies when the energies p_z^2/m and eB/m are small compared to the rest energy m. On taking the positive roots of E^2 to first order only, one finds, as approximations to Eqs. (15),

$$E=m+\frac{1}{2m}p_z^2+\frac{eB}{2m}(2n+1-2\mu)\,, \tag{16a}$$

$$E=m+\frac{1}{2m}p_z^2+\frac{eB}{2m}(2n+1-2\mu) -(g-2)\frac{eB}{2m}(\pm1)\,. \tag{16b}$$

The $\pm$ sign in Eq. (16b) is the same $\pm$ as in Eq. (15b) and λ has been replaced by $2g-1$. By comparison with the nonrelativistic problem, Appendix A, one sees that the level of Eq. (16a) is for $m_s=0$ and those of Eq. (16b) are for $m_s=\pm1$.

For a definite coordinate dependence, as specified by the quantum numbers p_z, μ, and n, and a definite energy sign, one can ask whether further degeneracy may occur in this problem. This does happen in the non-relativistic and spin-$\frac{1}{2}$ problems, as reviewed in Appendices A and C, in which cases the further degeneracy is complete at $g=2$. In the spin-1 problem there is a partial further degeneracy at $g=2$, $\lambda=3$; Eqs. (15) then become

$$E^2=p_z^2+[1-(eB/m^2)^2]^{-1}[m^2+eB(2n-2\mu+1)]\,, \tag{17a}$$

$$E^2=p_z^2+m^2+eB(2n-2\mu+1)\,, \tag{17b}$$

and the $\pm$ sign drops out of the second equation. The states that in the nonrelativistic limit are spin-up and -down, $m_s=\pm1$, are the ones that are degenerate at $g=2$. At $g=2$ in the spin-$\frac{1}{2}$ problem the degeneracy is resolved by the operator $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$ as reviewed in Appendix C. In the spin-1 problem the operator $\mathbf{s}\cdot\boldsymbol{\pi}$ commutes with p_z, J_z, and R_0^2, so that simultaneous eigenfunctions of these four operators may be found. However, in contrast to the spin-$\frac{1}{2}$ problem, these simultaneous eigenfunctions do not satisfy the wave equation (4) and are not stationary states of the problem at any g value.

APPENDIX A: NONRELATIVISTIC PROBLEM

Here the wave equation is

$$H\psi=i\frac{\partial\psi}{\partial t}\,,$$

where the Hamiltonian is

$$H=(2m)^{-1}\boldsymbol{\pi}\cdot\boldsymbol{\pi}-g(e/2m)\mathbf{s}\cdot\mathbf{B}\,.$$

One can use the operators p_z, $L_z=xp_y-yp_x$, and R_0^2 [still defined by Eqs. (6) and (7)] to settle the coordinate dependences. The simultaneous eigenfunctions are

$$\psi_{p_zm_ln}=c\rho^{-m_l/2}L_n^{-m_l}(\rho)e^{im_l\varphi}e^{-(\rho/2)+ip_zz}\,,$$

and the eigenvalue properties are

$$-i\frac{\partial}{\partial z}\psi_{p_zm_ln}=p_z\psi_{p_zm_ln}\,,$$

$$L_z\psi_{p_zm_ln}=m_l\psi_{p_zm_ln}\,,$$

$$eBR_0^2\psi_{p_zm_ln}=(2n+1)\psi_{p_zm_ln}\,.$$

In this case $n=0, 1, 2, \ldots$, if $m_l<0$ and $n=m_l, m_l+1, m_l+2, \ldots$, if $m_l\geq0$. The Hamiltonian can also be expressed as

$$H=\frac{1}{2m}p_z^2+\frac{e^2B^2}{2m}R_0^2-\frac{eB}{m}L_z-\frac{ge}{2m}\mathbf{s}\cdot\mathbf{B}\,.$$

Accordingly, the stationary-state solutions are

$$\psi=\psi_{p_zm_ln}\chi_{m_s}e^{-iE't}\,,$$

where χ_{m_s} are the eigenfunctions of s_z with eigenvalues $m_s=-s$ to $+s$, and where the allowed values of the energy are

$$E'=\frac{1}{2m}p_z^2+\frac{eB}{2m}(2n+1-2m_l-2m_s) -(g-2)\frac{eB}{2m}m_s\,. \tag{A1}$$

When $g=2$ the energy depends only on the quantum number $2n+1-2\mu$, where $\mu=m_l+m_s$.

APPENDIX B: SPIN-0 PROBLEM

The wave equation is

$$(\pi_\alpha\pi_\alpha+m^2)\psi=0\,.$$

Stationary-state solutions for the constant-magnetic-field problem are

$$\psi=\psi_{p_zm_ln}e^{-iEt}\,,$$

where the allowed energy values are given by

$$E^2=p_z^2+m^2+eB(2n+1-2m_l)\,. \qquad \text{(B1)}$$

APPENDIX C: SPIN-$\frac{1}{2}$ PROBLEM

The wave equation for the constant-magnetic-field problem, with Pauli anomalous moment term included, is

$$H\psi=i\partial\psi/\partial t\,,$$

where

$$H=\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta m-(4m)^{-1}e\kappa\beta\boldsymbol{\sigma}\cdot\mathbf{B}\,,$$

and κ is $g-2$. A convenient choice of matrices is

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \boldsymbol{\sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},$$

where $\boldsymbol{\sigma}$ on the right-hand side are the Pauli matrices. The stationary states for this problem were found by Ternov, Bagrov, and Zhukovskii[2]; their results are reviewed here for comparison with the spin-1 case.

The operators that settle the coordinate dependence are p_z and R_0^2, as defined by Eqs. (6) and (7), and

$$J_z=xp_y-yp_x+\tfrac{1}{2}\sigma_z\,.$$

The simultaneous eigenfunctions are

$$\psi_{p_z\mu n}=\begin{bmatrix}c_1\rho^{-(\mu/2)+(1/4)}L_n^{-\mu+1/2}(\rho)e^{i(\mu-1/2)\varphi}\\ c_2\rho^{-(\mu/2)-(1/4)}L_n^{-\mu-1/2}(\rho)e^{i(\mu+1/2)\varphi}\\ c_3\rho^{-(\mu/2)+(1/4)}L_n^{-\mu+1/2}(\rho)e^{i(\mu-1/2)\varphi}\\ c_4\rho^{-(\mu/2)-(1/4)}L_n^{-\mu-1/2}(\rho)e^{i(\mu+1/2)\varphi}\end{bmatrix}e^{-(\rho/2)+ip_zz}\,. \qquad \text{(C1)}$$

Their eigenvalue properties are

$$-i\frac{\partial}{\partial z}\psi_{p_z\mu n}=p_z\psi_{p_z\mu n}\,,$$

$$J_z\psi_{p_z\mu n}=\mu\psi_{p_z\mu n}\,,$$

$$eBR_0^2\psi_{p_z\mu n}=(2n+1)\psi_{p_z\mu n}\,.$$

Now the following ranges of the quantum numbers apply: $\mu=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\ldots$; $n=0,1,2,\ldots$ if $\mu\leq-\frac{1}{2}$; $n=\mu+\frac{1}{2},\mu+\frac{3}{2},\ldots$ if $\mu\geq\frac{1}{2}$.

The substitution

$$\psi=\psi_{p_z\mu n}e^{-iEt}$$

reduces the problem of finding stationary-state solutions to a system of linear equations,

$$\begin{bmatrix} m-(\kappa eB/4m)-E & 0 & p_z & i(2eB)^{1/2}\\ 0 & m+(\kappa eB/4m)-E & -i(2eB)^{1/2}(n-\mu+\frac{1}{2}) & -p_z\\ p_z & i(2eB)^{1/2} & -m+(\kappa eB/4m)-E & 0\\ -i(2eB)^{1/2}(n-\mu+\frac{1}{2}) & -p_z & 0 & -m-(\kappa eB/4m)-E\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{bmatrix}=0\,. \qquad \text{(C2)}$$

Then the determinant of the coefficients gives the allowed energy values. The result is

$$E^2=p_z^2+\{[m^2+eB(2n-2\mu+1)]^{1/2}\pm\kappa eB/4m\}^2\,, \qquad \text{(C3)}$$

where either sign on the right-hand side is allowed and where independently either sign of square root of E^2 may be chosen. Thus there are four energy values for each set of allowed values of p_z, μ, and n.

The nonrelativistic limit of the energy formula is

$$E=m+\frac{1}{2m}p_z^2+\frac{eB}{2m}(2n-2\mu+1)-\kappa\frac{eB}{2m}(\mp\tfrac{1}{2})\,.$$

Thus the $\pm$ sign in Eq. (C3) corresponds to $m_s=\mp\frac{1}{2}$ in the nonrelativistic limit.

When $\kappa=0$, $g=2$, there is the further degeneracy and the energy formula simplifies to just

$$E^2=p_z^2+m^2+eB(2n-2\mu+1)\,.$$

In this case the operator $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$ provides a convenient additional label.[1] It commutes with p_z, J_z, R_0^2, and H (when $g=2$), so it must also be possible to arrange the solutions of the problem as eigenfunctions of $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$. One introduces the functions $\psi_{p_z\mu n\eta}$ as the special case of the functions $\psi_{p_z\mu n}$ when the c's are restricted by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_{p_z\mu n\eta}=\eta\psi_{p_z\mu n\eta}\,.$$

The eigenvalues are easily found to be

$$\eta=\pm[p_z^2+eB(2n-2\mu+1)]^{1/2}$$

and the constants to be related by

$$c_2/c_1=c_4/c_3=i(2eB)^{-1/2}(p_z-\eta)\,.$$

Stationary-state solutions of the problem when $g=2$ are then determined up to a normalization constant by the quantum numbers p_z, μ, n, sign of η, and sign of E. Equations (C2) simplify in this case to give simply

$$c_3/c_1=\eta^{-1}(E-m)\,.$$