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Model for π-π Scattering Satisfying Analyticity and Crossing Symmetry

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A crossing-symmetric model for π-π scattering is presented which satisfies the Mandelstam representation, has a finite number of resonances associated with an exchange-degenerate trajectory that turns over at high energies, and which has Regge asymptotic behavior in all channels. The Pomeranchukon amplitude is nonresonating and has background cuts. The total amplitude satisfies the Adler condition. Satellites are included that eliminate all the odd-daughter (ghost) resonances. The double-spectral functions are calculated and shown to have, except for the lack of curvature, the correct boundaries determined by elastic unitarity. The structure of the second-sheet singularities is briefly discussed. The π-π scattering lengths are calculated and found to be consistent with those obtained from current algebra, when terms of order m_π^2 and unitarity corrections are neglected.

I. INTRODUCTION

CONSIDERABLE effort has been devoted to establishing a dynamical theory of strong interactions which satisfies analyticity, crossing symmetry, and unitarity. The fundamental problem of strong interactions is to combine analyticity, the linear principle of crossing symmetry, and the nonlinear unitarity equation for the S matrix within a soluble scheme of equations. After a decade of strong-interaction physics the problem of unitarity remains unsolved because of the essentially many-body nature of the equations. The principle of analyticity in field theory is based on microscopic causality and is naturally related to the smoothness of the S matrix. We must discover an expression for the S matrix which can be continued in both the angle and energy variables in a way completely consistent with the postulate that the S matrix is a Lorentz-invariant function of all the momentum variables with only those singularities required by unitarity. A satisfactory n-body amplitude together with a consistent field-theory formalism is then required to solve the unitarity problem.

Early attempts at implementing the Chew-Mandelstam[1,2] program assumed elastic unitarity everywhere, and crossing symmetry was brought into the scheme in a piecemeal fashion.[3] It is clear that the linear principle of crossing symmetry plays a dominant role in the problem, since this principle permits us to analytically continue the scattering amplitude into the crossed channels corresponding to antiparticle scattering. Recent attempts to construct simple models of strong interactions, such as the Veneziano model,[4] give up unitarity altogether (narrow-resonance approximation), and crossing symmetry is explicitly built into the models. Because the Veneziano model is based on indefinitely rising Regge trajectories, it loses the Mandelstam double representation by invoking essential singularities at infinity, although retaining the fixed-t dispersion relations. It has long been felt that a satisfactory theory of strong interactions requires a logically consistent method for analytically continuing the two-body scattering amplitude in the smoothest possible way in both energy and momentum variables, and this should be combined with a consistent iterative scheme based on the unitarity equation.

In the following, we present a simple, few-parameter model which is explicitly crossing symmetric, has resonances in all nonexotic channels, and satisfies the Mandelstam representation. The Regge trajectories rise linearly up to high energies and then turn over and tend to finite, constant values at infinity; the model has the correct Regge asymptotic behavior in all channels. We shall concern ourselves mainly with π-π scattering and include the Pomeranchukon amplitude explicitly, since no model of π-π scattering is complete without it. The Adler condition for π-π scattering is satisfied by the amplitude off-the-mass shell, and when terms of order m_π^2 are neglected this leads to the current-algebra results for the π-π scattering lengths at threshold. The calculated double-spectral functions have approximately the correct boundaries prescribed by elastic unitarity and, therefore, it is anticipated that the violations of unitarity are small at low energies.

In an earlier model for π-π scattering[5] the analyticity properties of the amplitude were not satisfactory in that unwanted singularities occurred in the physical sheet violating unitarity. The assumption of infinitely rising Regge trajectories generated essential singularities at infinity, and the amplitude did not satisfy dispersion relations. However, in spite of these defects the model, when applied to low-energy scattering and meson decays, yielded results in fair agreement with the experi-

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[1] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[2] G. F. Chew, *S-Matrix Theory of Strong Interactions* (Benjamin, New York, 1962); G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[3] J. W. Moffat, Phys. Rev. 121, 926 (1961); B. H. Bransden and J. W. Moffat, Nuovo Cimento 21, 505 (1961); 23, 598 (1962); G. F. Chew and S. Mandelstam, *ibid.* 19, 752 (1961).

[4] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[5] J. W. Moffat, Nuovo Cimento 64A, 485 (1969).

mental data.[6] This served to show that the applications considered to date do not provide an exacting test between such dynamical philosophies as "duality" and "generalized interference" models.[7] This has been shown independently for the predictions of the threshold π-π parameters by Graham and Johnson.[8]

The paper is organized as follows. Section II establishes the notation and kinematical properties. In Sec. III, we set forth the fundamental requirements of the model and discuss the isospin amplitudes, and in Sec. IV introduce the model for the non-Pomeranchukon amplitude. In Sec. V, the residues and satellite terms corresponding to this amplitude are studied in detail. Then, in Sec. VI, the Regge asymptotic properties are studied, and, in Sec. VII, a model for the Pomeranchukon amplitude is introduced and its asymptotic properties are discussed. In Sec. VIII, the analyticity properties of the model are considered and the double-spectral functions are calculated. A brief discussion is given of the singularity structure of the amplitude in the second sheet, and also its threshold behavior. In Sec. IX, we present an approximate calculation of the π-π scattering lengths, and end the paper in Sec. X with concluding remarks.

II. SCATTERING AMPLITUDE, KINEMATICS, AND NOTATION

The invariant T matrix is connected to the S matrix by

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta(p_f-p_i)T_{fi} \tag{2.1}$$

and for two-particle scattering $1+2\to\bar{3}+\bar{4}$ (Fig. 1) in the center-of-mass (c.m.) system, the differential cross section is

$$d\sigma/d\Omega=(q'/q)\,|f(q,\theta)|^2, \tag{2.2}$$

where

$$f(q,\theta)=T/(8\pi\sqrt{s}) \tag{2.3}$$

and q and q' are the initial and final c.m. momenta. The familiar Mandelstam variables are

$$\begin{aligned} s&=(p_1+p_2)^2=(p_3+p_4)^2,\\ t&=(p_1+p_3)^2=(p_2+p_4)^2,\\ u&=(p_1+p_4)^2=(p_2+p_3)^2. \end{aligned} \tag{2.4}$$

The variables s, t, and u satisfy the relation

$$s+t+u=\sum_i m_i^2. \tag{2.5}$$

The unitarity of the S matrix $S^\dagger S=1$ leads to the equation for the T matrix,

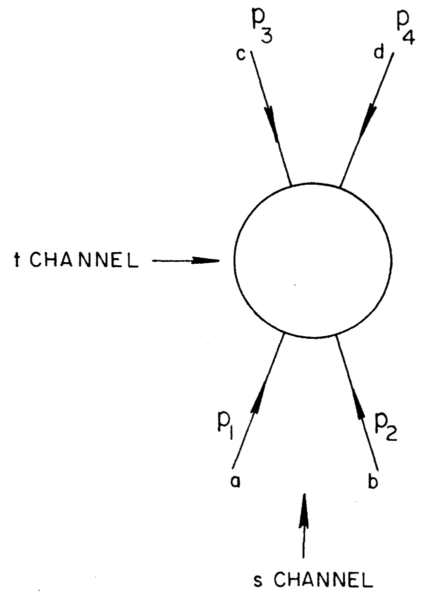

FIG. 1. π-π scattering process $p_1+p_2\to\bar{p}_3+\bar{p}_4$. Suffixes a, b, c, and d denote the isotopic-spin labels.

$$T_{fi}-T_{fi}^\dagger=i(2\pi)^4\sum_n\delta(p_i-p_n)T_{fn}T_{ni}^\dagger. \tag{2.6}$$

For a two-particle intermediate state and elastic scattering,

$$\mathrm{Im}f(s,\theta)=q\int\frac{d\Omega}{4\pi}f(s,\theta')f^*(s,\theta''), \tag{2.7}$$

where

$$\cos\theta''=\cos\theta\cos\theta'+\cos\phi\sin\theta'\sin\theta. \tag{2.8}$$

The partial waves for π-π scattering are defined by

$$f^I(s,\theta)=2\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)f_l^I(s), \tag{2.9}$$

where I denotes the isospin and

$$f_l^I(s)=\frac{1}{2}\int_{-1}^{1}d\cos\theta\, f^I(s,\theta)P_l(\cos\theta). \tag{2.10}$$

For $f_l^I(s)$, the elastic unitarity (2.7) becomes

$$\mathrm{Im}f_l^I(s)=qf_l^I(s)f_l^{I*}(s). \tag{2.11}$$

In π-π scattering, the most general amplitude satisfying crossing symmetry, isospin conservation, and Bose statistics is[2]

$$T\equiv M_{abcd}(s,t,u)=B_1(s,t,u)\delta_{ab}\delta_{cd}+B_2\delta_{ac}\delta_{bd}+B_3\delta_{ad}\delta_{bc}, \tag{2.12}$$

where B_1, B_2, and B_3 are the invariant amplitudes and a, b, c, and d are Cartesian basis vectors for the isotopic

[6] R. C. Johnson and J. W. Moffat, Toronto report, 1969 (unpublished); I. O. Moen and J. W. Moffat, Nuovo Cimento Letters **3**, 473 (1970).

[7] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968). There are indications that the strong "Veneziano" form of "duality" is violated in some two-body reactions and, therefore, probably strongly violated in n-body reactions. See Kwan Wu Lai and J. Louie, Nucl. Phys. **B19**, 205 (1970).

[8] R. H. Graham and R. C. Johnson, Phys. Rev. **188**, 2362 (1969); Phys. Rev. D **2**, 2114(E) (1970).

spin of each particle. We have

$$B_1(s,t,u)=B_1(s,u,t)\,,\quad B_2(s,t,u)=B_3(s,u,t)\,. \tag{2.13}$$

For interchange of u and s, we get

$$B_2(s,t,u)=B_2(u,t,s)\,,\quad B_1(s,t,u)=B_3(u,t,s)\,, \tag{2.14}$$

and for s and t interchange,

$$B_3(s,t,u)=B_3(t,s,u)\,,\quad B_1(s,t,u)=B_2(t,s,u)\,. \tag{2.15}$$

The isospin amplitudes in the s channel are

$$A_s^{I=0}=3B_1+B_2+B_3\,,\quad A_s^{I=1}=B_2-B_3\,,\quad A_s^{I=2}=B_2+B_3\,. \tag{2.16}$$

For identical particles (π-π scattering), the exchange of two particles in the final state gives a factor $(-1)^I$ (Bose statistics) and this accounts for the extra factor of 2 in (2.9); we define

$$f^I(s,\theta)=(1/16\pi\sqrt{s})A_s^I(s,t,u)\,. \tag{2.17}$$

The phase shifts are determined by

$$f_l^I(s)=\frac{1}{2iq}(\eta e^{2i\delta_l^I}-1)\,, \tag{2.18}$$

where η is the inelasticity parameter, $0\leqslant\eta\leqslant 1$, and for elastic scattering in the region $4m_\pi^2\leqslant s\leqslant 16m_\pi^2$ the inelasticity parameter η is equal to unity. The scattering lengths are defined by

$$a_I=\lim_{s\to 4m_\pi^2}\delta_0^I(s)/q=f_0^I(4m_\pi^2)=A_s^I(4m_\pi^2,0,0)/32\pi m_\pi\,. \tag{2.19}$$

III. FUNDAMENTAL PROPERTIES OF MODEL

Our model for π-π scattering is described by the amplitude

$$A^I(s,t,u)=F^I(s,t,u)+P^I(s,t,u)\,, \tag{3.1}$$

where F^I is the amplitude containing only Regge trajectories ρ, ω, f, etc., and P^I describes the Pomeranchukon amplitude. In the s channel, the $F^I(s,t,u)$ are given by[9]

$$F_s^{I=0}=\tfrac{3}{2}[F(s,t)+F(s,u)]-\tfrac{1}{2}F(t,u)\,,\quad F_s^{I=1}=F(s,t)-F(s,u)\,,\quad F_s^{I=2}=F(t,u)\,, \tag{3.2}$$

where the amplitude $F(s,t)=F(t,s)$. These isospin amplitudes are crossing-symmetric and there are no exotic resonances in the $I=2$ channel; the symmetry $(-1)^I$ under interchange of u and t is explicitly displayed. The crossing constraints and the requirement that the Pomeranchukon has $I=0$ determines the isotopic spin Pomeranchukon amplitudes[10]

$$\begin{aligned}P_s^{I=0}&=A_P(t,s)+A_P(t,u)+A_P(u,s)+A_P(u,t)+3A_P(s,t)+3A_P(s,u)\,,\\ P_s^{I=1}&=A_P(t,s)+A_P(t,u)-A_P(u,s)-A_P(u,t)\,,\\ P_s^{I=2}&=A_P(t,s)+A_P(t,u)+A_P(u,s)+A_P(u,t)\,.\end{aligned} \tag{3.3}$$

Here an unknown constant has been absorbed in $A_P(s,t)$ and we have assumed that the Pomeranchukon amplitude can be written as a sum of terms like $A_P(s,t)$, where A_P has only an s-channel Pomeranchukon and a t-channel nonresonant cut (background).

We shall demand that our model for the amplitude satisfies the following fundamental properties:

(1) It is a real analytic function of its arguments and only has the singularities corresponding to the unitarity equation.

(2) The Mandelstam representation. This means that the amplitude satisfies the correct fixed-t dispersion relation (axiomatic field theory) and partial-wave dispersion relations.

(3) Crossing symmetry.

(4) Resonances in all nonexotic channels.

(5) A self-consistent scheme for unitarizing the model.

(6) Regge behavior in all channels.

These six requirements more or less embody the basic properties that a microcausal and Lorentz-invariant theory of strong interactions should possess.

IV. MODEL FOR NON-POMERANCHUKON AMPLITUDE $F(s,t)$

The amplitude $F(s,t)$ is given by

$$F(s,t)=-[\gamma(s)\Gamma(1-\alpha(s))w(t)^{\alpha(s)}+\gamma(t)\times\Gamma(1-\alpha(t))w(s)^{\alpha(t)}]+\sum(\text{satellites})\,. \tag{4.1}$$

The trajectory $\alpha(s)$ describes the exchange-degenerate ρ,ω,f, and A_2 mesons, and for $\gamma(m_\pi^2)=0$ when $s=t=m_\pi^2$ the amplitude satisfies the Adler[11] condition $F(m_\pi^2,m_\pi^2)=0$ (including the satellite contributions). We consider the nonlinear trajectory[12]

$$\alpha(s)=a+\frac{bs-c(4m_\pi^2-s)^{1/2}}{\{1+[(4m_\pi^2-s)/\Delta]^{1/2}\}^2}, \tag{4.2}$$

where Δ is a constant. This trajectory is real analytic, has poles only in the second (unphysical) sheet, and has the elastic unitarity cut in the region $4m_\pi^2\leqslant s\leqslant\infty$.

[9] J. Shapiro and J. Yellin, Yadern. Fiz. 11, 443 (1970) [Soviet J. Nucl. Phys. 11, 247 (1970)]; J. Shapiro, Phys. Rev. 179, 1345 (1969); A. Yahil, *ibid.* 185, 1786 (1969); C. Lovelace, Phys. Letters 28B, 265 (1968).

[10] E. Del Giudice and G. Veneziano, Nuovo Cimento Letters 3, 363 (1970).

[11] S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B163 (1965).

[12] J. W. Moffat, Toronto report, 1970 (unpublished).

It is a Herglotz function that satisfies the once-subtracted dispersion relation

$$\alpha(s)=\alpha(0)+\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\mathrm{Im}\alpha(s')}{s'(s'-s-i\epsilon)}. \qquad (4.3)$$

We observe that by continuation $(4m_\pi^2-s)^{1/2}\to -i(s-4m_\pi^2)^{1/2}$ and for large Δ and intermediate values of s, we have

$$\frac{\mathrm{Im}\alpha(s)}{\mathrm{Re}\alpha'}\approx(s-4m_\pi^2)^{1/2}\left(\frac{2s}{\sqrt{\Delta}}+\frac{c}{b}\right). \qquad (4.4)$$

Because at intermediate values of s the $\mathrm{Re}\alpha(s)$ is approximately linear for large Δ, we find from the Adler condition $\alpha(m_\pi^2)=\frac{1}{2}$ and $\mathrm{Re}\alpha(m_\rho^2)=1$ that $a=0.503$, $b=0.848$ GeV^{-2}, and $\alpha(0)=0.480$ for $c=0.083$ GeV^{-1}. Then, the relation

$$\frac{\mathrm{Im}\alpha(m_R^2)}{\mathrm{Re}\alpha'}=m_R\Gamma_R \qquad (4.5)$$

leads to

$$\Gamma_R=\left(\frac{m_R^2-4m_\pi^2}{m_R^2}\right)^{1/2}\left(\frac{2m_R^2}{\sqrt{\Delta}}+\frac{c}{b}\right). \qquad (4.6)$$

For $\Delta^{1/2}=100$ GeV, this gives $\Gamma_\rho=102$ MeV, $\Gamma_f=120$ MeV, and $\Gamma_g=145$ MeV, in reasonable agreement with the experimental values of these widths.[13] The maximum value of the spin for a resonance on this trajectory will occur at $E\approx1500$ GeV and will have the value $J_{\max}=10^3$. It can be shown by solving the equation $\alpha(s)=n$ using the trajectory (4.2) that the maximum number of resonances on the leading trajectory will be $N=a+b\Delta$, which for $\Delta^{1/2}=100$ GeV is $N\approx10^4$. In the asymptotic region $\alpha(\pm\infty)=a-b\Delta\approx-b\Delta$. The function $\gamma(s)$ is real analytic, and of the form

$$\gamma(s)=\frac{\gamma[\alpha(s)-\frac{1}{2}]\exp[-g\alpha(s)^2]}{[1+x(s)]^{2q}}, \qquad (4.7)$$

where $x(s)=(4m_\pi^2-s)^{1/2}/\Lambda$ and Λ is a constant. This function has poles only on the second sheet and has the elastic unitarity cut; moreover, $q=b\Delta$ and $\gamma(s)\to0$ as $s\to\pm\infty$. The scale constant Λ is chosen large and the positive constant g sufficiently small so that $\gamma(s)\approx\gamma[\alpha(s)-\frac{1}{2}]$ for low energies.

The function $w(t)$ is defined by

$$w(t)=A+Bt+C(16m_\pi^2-t)^{1/2}. \qquad (4.8)$$

It possesses the inelastic unitarity cut in the region $16m_\pi^2\leqslant t\leqslant\infty$ and $w(t)^{\alpha(s)}$ generates additional cuts in t for fixed s only in the second sheet, provided the constants A, B, and C satisfy

$$A>0,\quad B<0,\quad C>0,\qquad C^2+4B(A+16m_\pi^2B)<0. \qquad (4.9)$$

[13] Particle Data Group, Rev. Mod. Phys. **41**, 1 (1969).

The last condition only holds if $A+16m_\pi^2B>0$.

Proof. Let $z=(16m_\pi^2-t)^{1/2}$ to give $w(z)=A+16m_\pi^2B-Bz^2+Cz=0$. This has the solution

$$z=\frac{-C\pm[C^2+4B(A+16m_\pi^2B)]^{1/2}}{-2B}.$$

For $\mathrm{Re}z<0$, the cuts generated by $w(z)=0$ occur only in the second sheet. This is true for the choice of conditions on A, B, and C described above.

V. RESIDUES AT POLES AND SATELLITES

The leading term in $F(s,t)$ has the residues at the poles $\alpha(t)=n$:

$$R_n(s)=\frac{\gamma(m_n^2)(-1)^{n-1}}{\alpha'\Gamma(n)}w(s)^n, \qquad (5.1)$$

where we have treated $\mathrm{Im}\alpha(m_n^2)$ as a small quantity. In the t channel, the isospin amplitudes are

$$\begin{aligned}F_t^{I=0}&=\tfrac{3}{2}[F(t,s)+F(t,u)]-\tfrac{1}{2}F(s,u),\\ F_t^{I=1}&=F(t,s)-F(t,u),\\ F_t^{I=2}&=F(s,u).\end{aligned} \qquad (5.2)$$

For the ρ resonance,

$$\begin{aligned}R_1^{I=1}&=\frac{\gamma(m_\rho^2)}{\alpha'}[w(s)-w(u)]\\ &\approx\frac{\gamma(m_\rho^2)}{\alpha'}B\cos\theta_t\,(m_\rho^2-4m_\pi^2),\end{aligned} \qquad (5.3)$$

where we have neglected for the moment the term $C(16m_\pi^2-s)^{1/2}$ in $w(s)$. Since we require that the constant B be negative,

$$R_1^{I=1}=\frac{(-1)\gamma(m_\rho^2)}{\alpha'}|B|\cos\theta_t\,(m_\rho^2-4m_\pi^2). \qquad (5.4)$$

This should be compared with the field-theory or dispersion-theory residue for the exchange of a ρ meson,

$$-2\gamma_{\rho\pi\pi}^2\cos\theta_t\,(m_\rho^2-4m_\pi^2), \qquad (5.5)$$

and gives for $|B|=\alpha'$

$$\gamma(m_\rho^2)=2\gamma_{\rho\pi\pi}^2. \qquad (5.6)$$

For the f^0 meson, we get

$$R_2^{I=0}=\frac{3}{2}\frac{(-1)\gamma(m_\rho^2)}{\alpha'}[w(s)^2+w(u)^2]. \qquad (5.7)$$

The leading term in $\cos\theta$ is

$$\frac{3}{4}\frac{(-1)\gamma(m_f^2)}{\alpha'}B^2(m_f^2-4m_\pi^2)^2\cos^2\theta, \qquad (5.8)$$

and the ratio of the ρ to the f^0 residues is

$$\frac{4}{3}\frac{\gamma(m_\rho^2)}{\gamma(m_f^2)|B|}\frac{(m_\rho^2-4m_\pi^2)}{(m_f^2-4m_\pi^2)^2}. \tag{5.9}$$

The Breit-Wigner amplitude for a definite spin J is

$$F_t^J=16\pi\left(\frac{\sqrt{s}}{k}\right)\frac{\Gamma_R m_R(2J+1)P_J(\cos\theta)}{m_R^2-t-i\Gamma_R m_R}. \tag{5.10}$$

Near the resonance the residue is

$$-16\pi\left(\frac{\Gamma_R m_R^2}{k_R}\right)(2J+1)P_J(\cos\theta). \tag{5.11}$$

Thus we find

$$\frac{\mathrm{Res}F_t^{J=1}}{\mathrm{Res}F_t^{J=2}}=\frac{2}{5}\frac{\Gamma_\rho}{\Gamma_f}\frac{(m_\rho^2/k_\rho)}{(m_f^2/k_f)}. \tag{5.12}$$

This leads to the result

$$\frac{\Gamma_\rho}{\Gamma_f}=\frac{10}{3}\frac{\gamma(m_\rho^2)}{\gamma(m_f^2)\alpha'}\frac{(m_f^2/k_f)}{(m_\rho^2/k_\rho)}\frac{(m_\rho^2-4m_\pi^2)}{(m_f^2-4m_\pi^2)^2}, \tag{5.13}$$

where we have set $|B|=\alpha'$, and

$$\gamma(m_n^2)\approx(n-\tfrac{1}{2})\gamma e^{-gn^2}. \tag{5.14}$$

For the daughter resonance of ρ with $I=0$ and $\alpha=1$, called the ϵ resonance, the residue of the leading term is

$$R_1^{I=0}=\frac{3}{2}\frac{\gamma(m_\rho^2)}{\alpha'}[w(s)+w(u)]$$

$$\approx\frac{3}{2}\frac{\gamma(m_\rho^2)}{\alpha'}[2A+|B|(m_\rho^2-4m_\pi^2)]. \tag{5.15}$$

Because A is positive, the residue $R_1^{I=0}$ is positive and the ratio of the ρ and ϵ residues is negative. This will be true for all the residues of the poles corresponding to the odd-daughter trajectory one unit of spin below the leading trajectory. Thus, all the resonances on this trajectory are "ghosts." In order to eliminate all these ghosts, we consider the satellite terms.

We write Eq. (4.1) as

$$F(s,t)=-\gamma(s)[\Gamma(1-\alpha(s))w(t)^{\alpha(s)}+\sum_{m=1}^{N}d_m\Gamma(m-\alpha(s))w(t)^{\alpha(s)-1}]+(s\leftrightarrow t). \tag{5.16}$$

We can then choose the coefficients d_m such that all the odd-daughter ghost resonances are eliminated. If we consider only the first three satellites, the model takes the form

$$F(s,t)=-\gamma(s)\Gamma(1-\alpha(s))\{w(t)^{\alpha(s)}+[d_1+d_2(1-\alpha(s))+d_3(2-\alpha(s))\times(1-\alpha(s))]w(t)^{\alpha(s)-1}\}+(s\leftrightarrow t). \tag{5.17}$$

The residue $R_1^{I=1}$ will be the same as Eq. (5.4), whereas the residue corresponding to the ϵ will now become

$$R_1^{I=0}=\frac{3}{2}\frac{\gamma(m_\rho^2)}{\alpha'}[w(s)+w(u)+2d_1]$$

$$\approx\frac{3}{2}\frac{\gamma(m_\rho^2)}{\alpha'}[2A+|B|(m_\rho^2-4m_\pi^2)+2d_1]. \tag{5.18}$$

For $\alpha=2$ and $I=1$, the residue of the ρ' is

$$R_2^{I=1}=\frac{(-1)\gamma(m_f^2)}{\alpha'}\{w(s)^2-w(u)^2+(d_1-d_2)[w(s)-w(u)]\}$$

$$\approx\frac{\gamma(m_f^2)}{\alpha'}|B|\cos\theta\,(m_f^2-4m_\pi^2)\times[2A+|B|(m_f^2-4m_\pi^2)+d_1-d_2]. \tag{5.19}$$

We can guarantee that the ϵ is not a ghost by demanding that

$$2A+|B|(m_\rho^2-4m_\pi^2)+2d_1\leqslant 0, \tag{5.20}$$

and we can eliminate the ρ' daughter resonance by requiring that $R_2^{I=1}=0$ and solving for d_2:

$$d_2=2A+|B|(m_f^2-4m_\pi^2)+d_1, \tag{5.21}$$

where $|B|=\alpha'$. The residue of the first daughter of the g meson is

$$R_3^{I=0}=\frac{3}{4}\frac{\gamma(m_g^2)}{\alpha'}\{w(s)^3+w(u)^3+(d_1-2d_2+2d_3)[w(s)^2+w(u)^2]\}. \tag{5.22}$$

Solving for the coefficient of $\cos^2\theta$ in (5.22), we can eliminate the first daughter of the g meson by requiring that

$$d_3=\tfrac{1}{2}[2d_2-3A-d_1-\tfrac{3}{4}|B|(m_g^2-4m_\pi^2)]. \tag{5.23}$$

There is no experimental evidence for a ρ' meson (first daughter of the f^0 meson) or a daughter of the g meson; therefore, we have removed them from the scheme by demanding Eqs. (5.21) and (5.23). Further satellites may have to be added to remove other daughter ghosts in the model lying on the trajectories more than one unit of spin below the parent trajectory. Because we have only a finite number of resonances N on the leading trajectory, there will be no problems of convergence of the sum over satellite terms.

Because the function $w(s)$ has an imaginary part for $s\geqslant 16m_\pi^2$, the requirement that the residue of the pole in t must be a polynomial in s is not satisfied. There is no basic physical principle underlying this requirement. However, if it is not satisfied we must concern ourselves with the high-spin ancestors that are generated in the partial waves. The projection of the residues of the amplitude at the poles $\alpha(t)=n$ is

$$\frac{1}{2}\int_{-1}^{+1} d\cos\theta\, R_n(s)P_l(\cos\theta). \tag{5.24}$$

At the ρ pole $\alpha(t)=1$, the term coming from (5.23) for $l>1$ will be nonvanishing. But an analysis[14] shows that for $C/|B|=0.1$ GeV, the contribution of the $l=2$ ancestor is less than 5% at the ρ pole and the contribution of the ancestors for $l>2$ at the ρ pole is smaller by a factor of $1/l$. Thus for small C the ancestors will have a negligible effect on the higher partial waves and can be ignored. We note that the physical residues at the poles are real in value.

VI. REGGE ASYMPTOTIC BEHAVIOR OF AMPLITUDE $F(s,t)$

The leading term of the amplitudes $F(s,t)$ for fixed t and $s\to\infty$ has the asymptotic behavior

$$F(s,t)\approx -\frac{\pi\gamma(s)\exp[\alpha(s)\ln|w(t)|]}{\Gamma(\alpha(s))\sin\pi\alpha(s)} - \frac{\pi\gamma(t)w(s)^{\alpha(t)}}{\Gamma(\alpha(t))\sin\pi\alpha(t)} \sim -\frac{\pi\gamma(t)}{\Gamma(\alpha(t))}\frac{(-s/s_0)^{\alpha(t)}}{\sin\pi\alpha(t)}, \tag{6.1}$$

where the scale $s_0=1/|B|=1/\alpha'$, and we have used the identity

$$\Gamma(1-z)=\pi/[\Gamma(z)\sin\pi z]. \tag{6.2}$$

The first term on the left-hand side of (6.1) for $s\to\infty$ and fixed t has the behavior

$$\frac{\text{const}}{(-s)^q}\exp[-b\Delta\ln|w(t)|]\to 0, \tag{6.3}$$

where we have used $\alpha(\infty)\approx -b\Delta$. If we choose $q=b\Delta$, then (6.3) vanishes faster than the expression on the right-hand side of (6.1) for any fixed value of t. For large-fixed-angle (fixed u) scattering corresponding to $s\to\infty$ and $t\to-\infty$, it follows that $F(s,t)\to 0$ without violating the Cerulus-Martin bound[15] $e^{-(\sqrt{s})C(t)}$ [$C(t)$ a slowly varying function of t]. From (5.2), (6.1), and (6.3), we get in the t channel for large s, recalling that $u\sim -s$, the Regge form

$$\begin{aligned} F_t^{I=0} &\sim -\frac{3}{2}\frac{\pi\gamma(t)}{\Gamma(\alpha(t))}\left(\frac{s}{s_0}\right)^{\alpha(t)}\frac{e^{-i\pi\alpha(t)}+1}{\sin\pi\alpha(t)},\\ F_t^{I=1} &\sim -\frac{\pi\gamma(t)}{\Gamma(\alpha(t))}\left(\frac{s}{s_0}\right)^{\alpha(t)}\frac{e^{-i\pi\alpha(t)}-1}{\sin\pi\alpha(t)},\\ F_t^{I=2} &\to 0. \end{aligned} \tag{6.4}$$

Thus the model has the correct Regge behavior and signature factors. The $\pi^+\pi^-$ and $\pi^+\pi^+$ amplitudes in the s channel are

$$\begin{aligned} F_s^{\pi^+\pi^-} &= F(s,t),\\ F_s^{\pi^+\pi^+} &= F(t,u). \end{aligned} \tag{6.5}$$

It then follows from Eq. (6.1) that for $s\to\infty$ and fixed t we get

$$\begin{aligned} \text{Im}F_s^{\pi^+\pi^-} &\sim \beta(t)(s/s_0)^{\alpha(t)},\\ \text{Im}F_s^{\pi^+\pi^+} &\to 0, \end{aligned} \tag{6.6}$$

where the residue is

$$\beta(t)=\pi\gamma(t)/\Gamma(\alpha(t)). \tag{6.7}$$

We observe that the model has a ghost-eliminating mechanism since the residue $\beta(t)$ vanishes for $\alpha(t)=0$. Result (6.6) is the behavior expected for the absorptive parts of the $\pi^+\pi^-$ and $\pi^+\pi^+$ amplitudes in the absence of the Pomeranchukon-exchange contribution, and is consistent with the absence of exotic resonances in the $\pi^+\pi^+$ channel. The charge-exchange process $\pi^+\pi^-\to\pi^0\pi^0$ in the t channel is dominated by the $I=1$ amplitude for $s\to\infty$. From (6.4) for $s\to\infty$ and fixed t we have

$$\frac{d\sigma}{dt}(\pi^+\pi^-\to\pi^0\pi^0)=\frac{\beta(t)^2(s/s_0)^{2\alpha(t)-2}}{\cos^2[\frac{1}{2}\pi\alpha(t)]}, \tag{6.8}$$

which is the generally accepted Regge form for $d\sigma/dt$ for this process as $s\to\infty$. Just when the asymptotic behavior (6.6) and (6.8) sets in is determined by the constants Λ and Δ. Recent measurements of πN and KN cross sections at 30–70 GeV indicate that the asymptotic region may occur at very high energies.[16]

VII. POMERANCHUKON CONTRIBUTION TO MODEL

In view of the lack of high-energy data in meson-meson scattering, the conditions that can be imposed on the Pomeranchukon amplitude $A_P(s,t)$ are much less restrictive than those for the Regge trajectory amplitude $F(s,t)$. The nature of the Pomeranchukon in high-energy scattering has long been a mystery. The knowledge that no resonances have been established on the Pomeranchukon trajectory, and the purely diffractive nature of the Pomeranchukon contribution, has led to the postulate that it is diffractive scattering, built up from nonresonating backgrounds of the crossed channels.[17]

We shall seek an amplitude $A_P(s,t)$ which satisfies the following requirements:

(a) The amplitude satisfies the asymptotic behavior $A_P(s,t)\to\beta_P(s)(-t/t_0)^{\alpha_P(s)}$ (within logarithmic factors) for $t\to\infty$ and fixed s, where $\alpha_P(s)$ describes the Pomer-

[14] I. O. Moen (private communication).

[15] F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).

[16] G. G. Beznogikh *et al.*, Phys. Letters **30B**, 274 (1969).

[17] H. Harari, Phys. Rev. Letters **20**, 1395 (1968); P. G. O. Freund, *ibid.* **20**, 235 (1968).

anchukon trajectory. In particular, the total cross section for π-π scattering should satisfy $\sigma_T \to$ const at asymptotic energies. This corresponds to the exchange of an $I=0$ Regge Pomeranchukon in the t-channel.

(b) $A_P(s,t)$ has a nonresonating cut in the t channel and no resonating poles in the s channel.

Let us consider the following expression:

$$A_P(t,s)=\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(s))}\times\ln\left[1+\left(\frac{4m_\pi^2-s}{s_0}\right)^{1/2}\right]w_P(s)^{\alpha_P(t)}, \quad (7.1)$$

where $\lambda=2\alpha_P(m_\pi^2)$, $s_0=t_0=u_0=1$ GeV^2, and

$$w_P(s)=A_P+Bs+C(16m_\pi^2-s)^{1/2}. \quad (7.2)$$

As before, the constants A_P, B, and C satisfy the following conditions $(1/|B|=s_0)$:

$$A_P>0,\quad B<0,\quad C>0,$$
$$C^2+4B(A_P+16m_\pi^2B)<0. \quad (7.3)$$

The Pomeranchukon trajectory is real analytic with a right cut starting at the inelastic threshold $s_T=16m_\pi^2$:

$$\alpha_P(s)=1+b_Ps\left[1+\left(\frac{16m_\pi^2-s}{\Delta_P}\right)^{1/2}\right]^{-2}, \quad (7.4)$$

where b_P and Δ_P are constants. This trajectory also satisfies a once-subtracted dispersion relation

$$\alpha_P(s)=1+\frac{s}{\pi}\int_{16m_\pi^2}^{\infty}\frac{ds'\,\mathrm{Im}\alpha_P(s')}{s'(s'-s-i\epsilon)}, \quad (7.5)$$

and for asymptotic energies,

$$\alpha_P(\pm\infty)=1-b_P\Delta_P. \quad (7.6)$$

The function $\gamma_P(s)$ is a suitable real analytic function (with possible unitarity cuts and poles on the second sheet) and $\gamma(s)\to 0$ as $s\to\pm\infty$. The amplitude (7.1) satisfies the Adler condition for $\lambda=2\alpha_P(m_\pi^2)$:

$$A_P(m_\pi^2,m_\pi^2)=0. \quad (7.7)$$

We observe from (3.3) that in contrast to the amplitude $F_s^{I=2}$, the Pomeranchukon amplitude $P_s^{I=2}$ has a nonvanishing imaginary part in the s channel.

Let us consider the high-energy limits of $A_P(s,t)$. For fixed s and $t\to\infty$, we have

$$A_P(s,t)\sim\frac{\gamma_P(s)}{2\Gamma(\lambda-\alpha_P(s)-\alpha_P(\infty))}\times\left(-\frac{t}{t_0}\right)^{\alpha_P(s)}\left(\ln\frac{t}{t_0}-i\pi\right). \quad (7.8)$$

The additional logarithmic factor $\ln(t/t_0)$ does not violate the Froissart bound for the amplitude and (7.8) is the required Regge asymptotic behavior. For fixed t and $s\to\infty$, we get

$$A_P(t,s)\sim\frac{\gamma_P(t)}{2\Gamma(\lambda-\alpha_P(t)-\alpha_P(\infty))}\times\left(-\frac{s}{s_0}\right)^{\alpha_P(t)}\left[\ln\left(\frac{s}{s_0}\right)-i\pi\right]. \quad (7.9)$$

On the other hand, for t fixed and $s\to\infty$, we have

$$A_P(s,t)\sim\frac{\text{const}}{(-s/\Lambda)^r}\ln\left[1+\left(\frac{4m_\pi^2-t}{t_0}\right)^{1/2}\right]\times\exp[-b_P\Delta_P\ln|w(t)|]\to 0, \quad (7.10)$$

where we have assumed that $\gamma_P(s)\sim 1/(-s/\Lambda)^r$ as $s\to\infty$ for $r>0$. If $r=b_P\Delta_P$, then (7.10) will vanish faster than (7.9) for any fixed value of negative t. Now consider the amplitude $A_P(s,u)$ in the limit $s\to\infty$ and t fixed $(u\to-\infty)$:

$$A_P(s,u)\sim\frac{\text{const}}{(-s/\Lambda)^r}\ln\left|\frac{u}{u_0}\right|^{1/2}\times\exp\left(-b_P\Delta_P\ln\left|\frac{u}{u_0}\right|\right)\to 0. \quad (7.11)$$

For all isospin amplitudes determined by (3.3), the asymptotic behavior for fixed t and $s\to\infty$ is

$$P_s^I\to\frac{1}{2}\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(\infty))}\left(\frac{s}{s_0}\right)^{\alpha_P(t)}\times\left[\ln\left(\frac{s}{s_0}\right)(1+e^{-i\pi\alpha_P(t)})-i\pi e^{-i\pi\alpha_P(t)}\right]. \quad (7.12)$$

Then for $t=0$ and $\alpha_P(0)=1$, we have for $s\to\infty$ the result

$$P_s^I\sim\frac{i}{2}\frac{\gamma_P(0)}{\Gamma(\lambda-1-\alpha_P(\infty))}\frac{s}{s_0}. \quad (7.13)$$

Thus, the Pomeranchukon amplitude at $t=0$ becomes pure imaginary as $s\to\infty$ and the total π-π cross section tends to a constant. This is the correct Regge behavior corresponding to the exchange of an $I=0$ Pomeranchukon in the t channel.

In the intermediate energy region the parameters in the model must be chosen so that unitarity is not violated; this would ensure that the elastic cross section is less than the total cross section in this energy region. Only a detailed calculation can reveal whether this can be satisfied. In the asymptotic region a calculation of the elastic cross section σ_{el} from Eq. (7.12) shows that $\sigma_{el}\sim 1/\ln(s/s_0)$, whereas the total cross section tends to a constant and, therefore, unitarity is not violated at high energy.

For large-angle scattering corresponding to $s\to\infty$ and $t\to-\infty$ (u fixed), we get

$$A_P(s,t)\sim \frac{\text{const}}{(-s/\Lambda)^r}\ln|t/t_0|^{1/2} \times\exp(-b_P\Delta_P\ln|t/t_0|)\to 0. \quad (7.14)$$

This does not violate the Cerulus-Martin bound for large-angle scattering.[15]

VIII. ANALYTICITY PROPERTIES OF TOTAL AMPLITUDE; MANDELSTAM REPRESENTATION

Let us now discuss the analyticity properties of the total amplitude $A^I(s,t,u)$. By inspection we see that the only cuts in $A^I(s,t,u)$ are those generated by the unitarity equation in the regions $4m_\pi^2\leqslant s\leqslant\infty$ and $16m_\pi^2\leqslant s\leqslant\infty$. Thus, $A^I(s,t,u)$ is a real analytic function with only proper threshold singularities; the poles corresponding to the resonances occur in the second sheet. Because $\alpha(s)$ is bounded by a constant in all directions in the s plane, the amplitude $A(s,t,u)$ is bounded by a polynomial everywhere at infinity and no essential singularities are present. This means that $A(s,t,u)$ satisfies the Mandelstam representation (with N subtractions where $N=J_{\max}$)[18]:

$$A^I(s,t,u)=\frac{1}{\pi^2}\int\int\frac{ds'dt'\rho_{st}{}^I(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int\int\frac{ds'du'\rho_{su}{}^I(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int\int\frac{dt'du'\rho_{ut}{}^I(u',t')}{(u'-u)(t'-t)}. \quad (8.1)$$

The subtraction constants occurring in the Mandelstam representation for the model, although large in number, are all determined by the spectral functions. For fixed t the amplitude satisfies the single dispersion relation of axiomatic field theory (with subtractions):

$$A^I(s,t,u)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'D_s{}^I(s',t)}{s'-s-i\epsilon}+\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{du'D_u{}^I(u',t)}{u'-u-i\epsilon}, \quad (8.2)$$

where $D_s(s,t)$ and $D_u(u,t)$ are the discontinuities of A across the positive s and u axes, respectively. The partial waves defined by

$$A_l{}^I(s)=\frac{1}{2}\int_{-1}^{+1}d\cos\theta\, A^I(s,t)P_l(\cos\theta) \quad (8.3)$$

satisfy the dispersion relation proved to all orders in perturbation theory[19]:

$$A_l{}^I(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\mathrm{Im}A_l{}^I(s')}{s'-s-i\epsilon}+\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds'\,\mathrm{Im}A_l{}^I(s')}{s'-s-i\epsilon}. \quad (8.4)$$

The discontinuity $\Delta_sA(s,t)$ is defined by

$$\Delta_sA(s,t)=\frac{1}{2i}\lim_{\epsilon\to 0}[A(s+i\epsilon,t)-A(s-i\epsilon,t)]. \quad (8.5)$$

A calculation of the discontinuity of $F(s,t)$ across the positive s axis gives

$$\Delta_sF(s,t)=\frac{1}{2i}[\gamma(s)\Gamma(1-\alpha(s))w(t)^{\alpha(s)}-\gamma^*(s)\Gamma(1-\alpha^*(s))w(t)^{\alpha^*(s)}]\theta(s-4m_\pi^2)-[\gamma(t)\Gamma(1-\alpha(t))|w(s)|^{\alpha(t)}\sin(\phi(s)\alpha(t))]\times\theta(s-16m_\pi^2), \quad (8.6)$$

[18] For equal-mass scattering, a double dispersion relation of the form

$$A=\frac{1}{\pi^2}\int\int\frac{ds'dt'\rho(s',t')}{(s'-s)(t'-t)}$$

is probably valid in all orders of perturbation theory. See, e.g., R. J. Eden, Phys. Rev. **121**, 1567 (1961); P. Landshoff, J. C. Polkinghorne, and J. G. Taylor, Nuovo Cimento **19**, 939 (1961). However, for certain higher-order diagrams of unequal masses there are indications of complex singularities: R. J. Eden, P. Landshoff, J. C. Polkinghorne, and J. G. Taylor, Phys. Rev. **122**, 307 (1961). For existence proofs of scattering amplitudes satisfying the Mandelstam representation, see D. Atkinson, Nucl. Phys. **B7**, 375 (1968); **B13**, 415 (1969).

[19] J. G. Taylor, Nuovo Cimento **22**, 92 (1961).

and the discontinuity of $A_P(t,s)$ is

$$\Delta_s A_P(t,s) = -\frac{1}{2}\left\{\left[\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(s))}+\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P^*(s))}\right]\ln\left[1+\left(\frac{s-4m_\pi^2}{s_0}\right)^{1/2}\right]\right.$$
$$\left.+\left[\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(s))}-\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P^*(s))}\right]\tan^{-1}\left(\frac{s-4m_\pi^2}{s_0}\right)^{1/2}\right\}$$
$$\times\sin[\phi_P\alpha_P(t)]\,|w_P(s)|^{\alpha_P(t)}\theta(s-16m_\pi^2)+\frac{1}{2}\left\{\left[\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(s))}-\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P^*(s))}\right]\right.$$
$$\left.\times\ln\left[1+\left(\frac{s-4m_\pi^2}{s_0}\right)^{1/2}\right]-\left[\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P(s))}+\frac{\gamma_P(t)}{\Gamma(\lambda-\alpha_P(t)-\alpha_P^*(s))}\right]\tan^{-1}\left(\frac{s-4m_\pi^2}{s_0}\right)^{1/2}\right\}$$
$$\times\cos[\phi_P\alpha_P(t)]\,|w_P(s)|^{\alpha_P(t)}\theta(s-4m_\pi^2)\,, \quad (8.7)$$

where

$$w(s\pm i\epsilon)=A+Bs\mp iC(s-16m_\pi^2)^{1/2}=|w(s)|\,e^{\mp i\phi(s)}\,,\qquad \phi(s)=\tan^{-1}\left[\frac{C(s-16m_\pi^2)^{1/2}}{A+Bs}\right], \quad (8.8)$$

and

$$w_P(s\pm i\epsilon)=A_P+Bs\mp iC(s-16m_\pi^2)^{1/2}=|w_P(s)|\,e^{\mp i\phi_P(s)}\,,\qquad \phi_P(s)=\tan^{-1}\left[\frac{C(s-16m_\pi^2)^{1/2}}{A_P+Bs}\right]. \quad (8.9)$$

Let us decompose the double-spectral function in the form

$$\rho(s,t)=\rho_F(s,t)+\rho_P(s,t)\,, \quad (8.10)$$

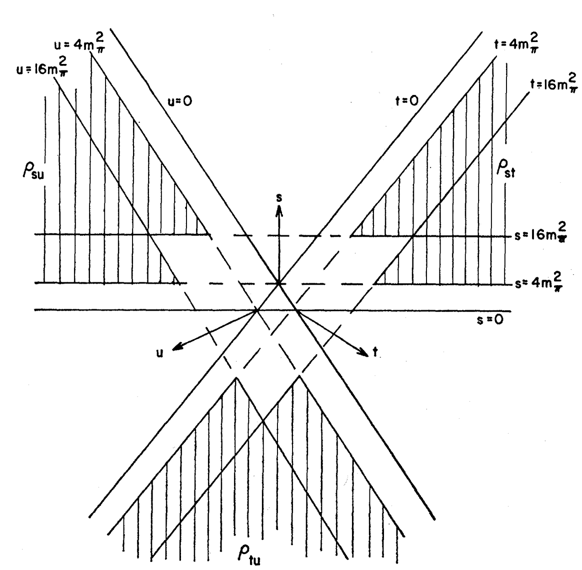

FIG. 2. Boundaries of the double-spectral functions calculated from the model.

where ρ_F and ρ_P are obtained from the discontinuities of (8.6) and (8.7), respectively, across the positive t axis. Then ρ_F is given by

$$\rho_F(s,t)=\tfrac{1}{2}i\{[\gamma(s)\Gamma(1-\alpha(s))\,|w(t)|^{\alpha(s)}\sin(\phi(t)\alpha(s))-\gamma^*(s)\Gamma(1-\alpha^*(s))\,|w(t)|^{\alpha^*(s)}\sin(\phi(t)\alpha^*(s))]$$
$$\times\theta(s-4m_\pi^2)\theta(t-16m_\pi^2)+[\gamma(t)\Gamma(1-\alpha(t))\,|w(s)|^{\alpha(t)}\times\sin(\phi(s)\alpha(t))-\gamma^*(t)\Gamma(1-\alpha^*(t))\,|w(s)|^{\alpha^*(t)}$$
$$\times\sin(\phi(s)\alpha^*(t))]\theta(s-16m_\pi^2)\theta(t-4m_\pi^2)\}\,. \quad (8.11)$$

The $\rho_P(s,t)$ can be obtained in a similar way from (8.7) and we assume that the cut in $\gamma_P(t)$ begins at $t_T=16m_\pi^2$, as in the case of $\alpha_P(t)$.

The double-spectral functions possess the correct boundaries determined by the elastic unitarity equation (2.7), except that these boundaries will not be curved as they should be according to the equations

$$t=16m_\pi^2\left(\frac{s}{s-4m_\pi^2}\right)\quad\text{for}\quad t>s\,,\qquad s=16m_\pi^2\left(\frac{t}{t-4m_\pi^2}\right)\quad\text{for}\quad s>t \quad (8.12)$$

obtained from (2.7). The shapes of the double-spectral functions obtained in our model are shown in Fig. 2.

The model possesses the correct threshold behavior. In order to see this, consider[20]

$$A_l^I(s)=\left(\frac{4}{s-4m_\pi^2}\right)\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}dt\,A_a^I(s,\,t,\,4m_\pi^2-s-t)\times Q_l\left(1+\frac{2t}{s-4m_\pi^2}\right), \quad (8.13)$$

where A_a^I is the complete absorptive part. For small

[20] See, e.g., A. O. Barut, *The Theory of the Scattering Matrix* (MacMillan, London, 1967), p. 214.

q^2, we have

$$Q_l\left(1+\frac{2t}{s-4m_\pi^2}\right)\approx\left(\frac{2t}{s-4m_\pi^2}\right)^{-l-1} \tag{8.14}$$

and it follows that

$$A_l^I(s) \underset{q^2\to 0}{\sim} (s-4m_\pi^2)^l \frac{4}{\pi 2^{l+1}} \times\int_{4m_\pi^2}^{\infty}\frac{dt\, A_a{}^I(s,t,4m_\pi^2-s-t)}{t^{l+1}}, \tag{8.15}$$

which is the correct threshold behavior as $q^2\to 0$.

Let us now consider our scattering amplitude in the second sheet. Consider the two-body elastic unitarity equation (2.11) for the partial waves. This can be written in terms of first- and second-sheet amplitudes

$$(A_l(s))_{\rm II}-(A_l(s))_{\rm I}=-i\rho(s)(A_l(s))_{\rm I}(A_l(s))_{\rm II}, \tag{8.16}$$

where $(A_l)_{\rm I}$ and $(A_l)_{\rm II}$ denote the partial-wave amplitudes on the first and second sheets, respectively. Also,

$$\rho(s)=\frac{1}{16\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}. \tag{8.17}$$

If we solve (8.16) for $(A_l)_{\rm II}$, we get

$$(A_l(s))_{\rm II}=\frac{(A_l(s))_{\rm I}}{1+i\rho(s)(A_l(s))_{\rm I}} \tag{8.18}$$

or, conversely, for $(A_l(s))_{\rm I}$, we have

$$(A_l(s))_{\rm I}=\frac{(A_l(s))_{\rm II}}{1-i\rho(s)(A_l(s))_{\rm II}}. \tag{8.19}$$

We see that if we continue $(A_l(s))_{\rm II}$ into the second sheet for complex s and encounter any cuts, then we should expect these cuts to occur at the corresponding value of s in the first sheet. Because unitarity and the Mandelstam representation do not allow any complex singularities in the first sheet for equal-mass two-particle scattering, the continued second-sheet amplitude must not possess such singularities either if these singularities can be reached by passing through the elastic cut $4m_\pi^2\leqslant s\leqslant 16m_\pi^2$.

The amplitude $(A(s))_{\rm II}$ has a cut generated by the vanishing of the function $w(s)=A+Bs+C(16m_\pi^2-s)^{1/2}$ in the second sheet [subject to the conditions (4.9) on A, B, and C]. However, this cut in the second sheet can only be reached by passing through the unitarity cut above the first inelastic threshold at $s_T=16m_\pi^2$ (Fig. 3). But this is a different sheet than the one defined by the continuation of $(A_l)_{\rm II}$ in (8.18) or (8.19), since these equations are only valid for purely elastic scattering and cannot be continued beyond the inelastic threshold at $s_T=16m_\pi^2$. Therefore, our second-sheet cut structure is not in conflict with elastic unitarity.

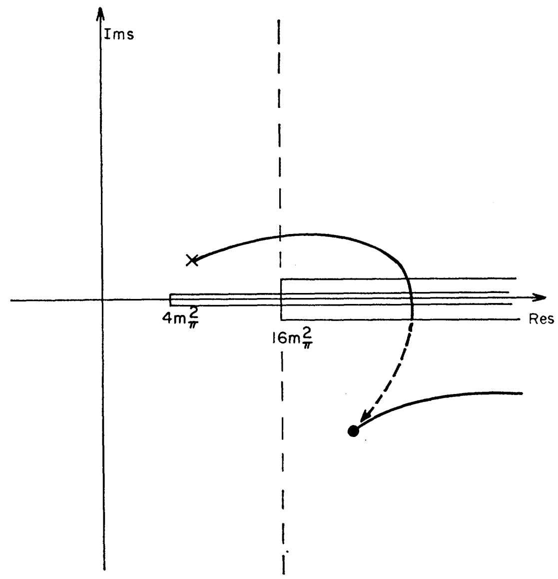

FIG. 3. Two-sheeted structure of the amplitude in the s plane.

IX. CALCULATION OF SCATTERING LENGTHS

The amplitude in our model satisfies the Adler condition[11]

$$A^I(m_\pi^2,m_\pi^2,m_\pi^2)=0 \tag{9.1}$$

for one of the external pions off the mass shell. This condition follows from the partial conservation of the axial-vector current (PCAC).[21] Let us assume that the contribution of the Pomeranchukon amplitude is small near threshold and consider the amplitude $F(s,t)$ including the first satellite. The trajectory $\alpha(s)$ is approximated at low energies by

$$\alpha(s)=\tfrac{1}{2}+\alpha'(s-m_\pi^2). \tag{9.2}$$

If we expand $F(s,t)$ around the point $s=t=u=m_\pi^2$ and consider only the linear approximation, we find

$$F(t,u)=-(\sqrt{\pi})\alpha'\gamma(t+u-2m_\pi^2)\left[\frac{A+Bm_\pi^2+d_1}{(A+Bm_\pi^2)^{1/2}}\right], \tag{9.3}$$

where we have assumed that C is small and the term $C(16m_\pi^2-s)^{1/2}$ in $w(s)$ can be neglected at low energies. The Weinberg amplitude takes the form[22]

$$M_{bdac}=(1/F_\pi^2)[\delta_{ac}\delta_{bd}(s-m_\pi^2)+\delta_{ab}\delta_{cd}(t-m_\pi^2)+\delta_{ad}\delta_{bc}(u-m_\pi^2)] \tag{9.4}$$

and

$$\begin{aligned}F^{I=0}&=(1/F_\pi^2)(3s+t+u-5m_\pi^2),\\ F^{I=1}&=(1/F_\pi^2)(t-u),\\ F^{I=2}&=(1/F_\pi^2)(t+u-2m_\pi^2),\end{aligned} \tag{9.5}$$

[21] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960).

[22] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

where F_π is the pion decay constant. From (9.3) and (9.5), we get

$$(\sqrt{\pi})\gamma\frac{(A+Bm_\pi^2+d_1)}{(A+Bm_\pi^2)^{1/2}}=-\frac{1}{\alpha' F_\pi^2}. \qquad (9.6)$$

This means that $A+Bm_\pi^2+d_1<0$. From (9.2) we see that $\alpha(0)\approx\alpha(4m_\pi^2)\approx\frac{1}{2}$ is a reasonable approximation where $\alpha'\approx 1/2m_\rho^2$. Using this in Eq. (5.16), we find

$$F(0,0)=2m_\pi^2\alpha'\gamma(\sqrt{\pi})\left(\frac{A+d_1}{A^{1/2}}\right) \qquad (9.7)$$

and

$$F(4m_\pi^2,0)=m_\pi^2\alpha'\gamma(\sqrt{\pi})\left[(A+4m_\pi^2B)^{1/2}+\frac{d_1}{(A+4m_\pi^2B)^{1/2}}-3\left(A^{1/2}+\frac{d_1}{A^{1/2}}\right)\right]. \qquad (9.8)$$

Let us now assume that $4m_\pi^2B$ is small compared to both A and d_1. Then by neglecting this quantity, we get[23]

$$F(4m_\pi^2,0)\approx -2m_\pi^2\alpha'\gamma(\sqrt{\pi})\left(\frac{A+d_1}{A^{1/2}}\right). \qquad (9.9)$$

From (2.19) and (3.2), we have

$$32\pi m_\pi a_0=A^{I=0}(4m_\pi^2,0,0)=3F(4m_\pi^2,0)-\tfrac{1}{2}F(0,0),$$
$$32\pi m_\pi a_2=A^{I=2}(4m_\pi^2,0,0)=F(0,0), \qquad (9.10)$$

and from (9.5) within the approximation of neglecting m_π^2B, we get

$$(\sqrt{\pi})\gamma\left(\frac{A+d_1}{A^{1/2}}\right)=-\frac{1}{\alpha' F_\pi^2}. \qquad (9.11)$$

Therefore substituting (9.7) and (9.9) into (9.10) and using (9.11), we find

$$a_0=\frac{7}{32\pi}\frac{m_\pi}{F_\pi^2},\quad a_2=-\frac{1}{16\pi}\frac{m_\pi}{F_\pi^2}. \qquad (9.12)$$

If we use the Goldberger-Treiman value $F_\pi=0.087M_p=0.58m_\pi$, the scattering lengths are

$$a_0=0.20m_\pi^{-1},\quad a_2=-0.06m_\pi^{-1}, \qquad (9.13)$$

and

$$a_0/a_2=-\tfrac{7}{2},\quad 2a_0-5a_2=0.70m_\pi^{-1}. \qquad (9.14)$$

These scattering lengths are exactly the same as those calculated by Weinberg[22] from current algebra. Apart from the assumption that $4m_\pi^2B$ is small and can be neglected compared to A and d_1 (this is consistent with our requirement that $A-4m_\pi^2|B|>0$),[23] results (9.13) and (9.14) follow from the identification of (9.3) with (9.5).

This means that a model that explicitly satisfies the Mandelstam representation and crossing symmetry is consistent with the current-algebra results at threshold. However, we should stress that we have imposed the current-algebra constraints on the model; in a complete calculation of π-π scattering only PCAC will be imposed on the model.

By ensuring that the lower partial waves do not violate unitarity in the low-energy region the parameters will be determined and the lower partial waves can be calculated. Provided crossing symmetry is maintained, the rigorous consistency conditions near threshold based on crossing symmetry and positivity[24] will be satisfied.

X. DISCUSSION OF RESULTS

We have shown that there exists a simple, few-parameter model for π-π scattering which satisfies the Mandelstam representation and crossing symmetry. The model has certain defects common to all models of its kind, e.g., there are daughter resonances which do not seem to correspond to observed particles, and some of these resonances are ghosts. This necessitated the introduction of a finite number of satellites. The most serious problem to solve is the discovery of a satisfactory method of unitarizing the model. One approach that has been used in the past is to iterate the equation for the elastic double-spectral function given by[1]

$$\rho_{el}(s,t)=\frac{1}{32\pi^2q_s\sqrt{s}}\int\int\frac{dt'dt''D^*(t',s)D(t'',s)}{K^{1/2}(s;t,t',t'')}, \qquad (10.1)$$

where

$$K(s;t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-(tt't''/q^2). \qquad (10.2)$$

Here the discontinuity $D(t,s)$ is determined by

$$D(t,s)=V(t,s)+\frac{1}{\pi}\int\frac{ds'\rho_{el}(s',t)}{s'-s}. \qquad (10.3)$$

Because for $s\to\infty$ the discontinuity $D_{el}(t,s)\approx\beta(t)s^{\alpha(t)}$, the function ρ_{el} will diverge like $s^{2n\alpha(t)-n}$ after n iterations for large enough t. This demands that (10.1) be modified by a cutoff function[25] which simulates the damping effect of the many-body intermediate states in the unitarity equation. This is not satisfactory and it is clear that a basic treatment of the unitarity equation is required.

A more fundamental approach is to consider a model

[23] If we use the value $|B|=\alpha'\approx 1/2m_\rho^2$, then we are in effect neglecting terms of order m_π^2/m_ρ^2 compared to A and d_1. Terms of this order and unitarity corrections are neglected within the Weinberg current-algebra calculations of the scattering lengths (see Ref. 22).

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of an n-point production amplitude of the kind proposed in Ref. 26. Now the constant C, in (4.8), is set equal to zero and we also choose $\alpha(s)$ to be real from the outset, corresponding to a narrow-resonance approximation; this guarantees that the n-point model can be factorized, and that all parameters in the n-point amplitude can be calculated from a knowledge of the 4-point residue. In this sense, the model constitutes a "bootstrap" system with a degeneracy of states corresponding to n^2. Initially, the model only contains "tree" graphs, but by calculating the loop diagrams, cuts in $w(t)$ and $\alpha(s)$ will be generated, hopefully reinstating the correct analyticity properties for the 4-point function, and by summing up all the loop diagrams a complete unitarization of the scattering amplitude can be accomplished. This kind of program has been attempted for the n-point generalization of the Veneziano model,[27] but the simplest, single-planar dual loop constructed gives rise to an integral with an essential endpoint singularity due to the large degeneracy of the states which circulate in the loop diagram. A somewhat complicated and arbitrary procedure of renormalization has been introduced to deal with this problem.[28] In the model proposed in Ref. 26, the treatment of the diagrams follows familiar methods of Feynman graphs, and since the degeneracy of states in the model is only n^2, there are no divergence problems in constructing loop diagrams. Whether such a program can succeed in analogy with quantum electrodynamics is still an open question, but it is clear that a basic solution to the inelasticity problem appears to be essential in strong interactions. One interesting question arising in connection with such a method for unitarizing the model is whether the corrected trajectory will turn over at large energies, in the way assumed for the $\alpha(s)$ in our model for the 4-point amplitude, or whether it will rise indefinitely to infinity. This is the kind of fundamental question that could be answered by a basic treatment of the unitarity problem.

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Application of a Crossing-Symmetric Model Satisfying the Mandelstam Representation to π-π and K-π Scattering*

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A model for meson-meson scattering satisfying the Mandelstam representation, crossing symmetry, and Regge behavior and including a Pomeranchukon amplitude is applied to π-π and K-π scattering. Solutions are found for the K-π partial waves that satisfy unitarity approximately at low energies, give a satisfactory fit to the on-mass-shell data, and predict scattering lengths consistent with current algebra. Apart from a change in the coupling constant, effectively the same parameters are then used to predict the low-energy π-π scattering, and the solutions are found to satisfy unitarity approximately up to 900 MeV. The predicted on-mass-shell results agree well with the available data. The general conditions below threshold for π-π scattering that follow from crossing symmetry and positivity are well satisfied. The extrapolated π-π and K-π amplitudes off the mass shell are found to agree satisfactorily with the data for $\pi N \to \pi\pi N$ and $KN \to K\pi N$ when a phenomenological form factor is used in the extrapolation. The total and differential cross sections at high energy are found to have characteristic Regge behavior. The Pomeranchukon amplitude produces total π-π and K-π cross sections consistent with factorization in the asymptotic region.

I. INTRODUCTION

A MODEL for π-π scattering has been developed by one of us[1] in which the scattering amplitude satisfies the following properties: (a) Mandelstam representation; (b) crossing symmetry; (c) resonances in all nonexotic channels; (d) Regge behavior in all channels; and (e) the Adler condition.

The Pomeranchukon is incorporated in the model as a nonresonant, diffractive background satisfying crossing symmetry. The Regge trajectory corresponding to the exchange-degenerate ρ-f^0 mesons is assumed to rise linearly to high energies and then turn over and tend

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