Branch Cuts in the Balázs Method. II. Application to Pion-Pion Scattering*

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The modified Balázs method, which retains the near part of the left-hand cut, is studied with regard to its ability to reproduce known, resonant functions and is applied to the pion-pion scattering problem. The known functions are reproduced reliably, with little dependence on the matching points and the number of poles, provided that sufficiently good estimates of the gap and the cut are used, and that the pole positions are chosen to optimize the agreement in the gap. The pion-pion results show a systematic improvement as crossed-channel partial waves are added to the input, and four partial waves are sufficient to produce the ρ and f^0 in rough agreement with experiment. These calculations do not use cutoffs, other arbitrary adjustable parameters, Castillejo-Dalitz-Dyson poles, coupling to other channels, or inelasticity.

I. INTRODUCTION

SOME time ago Balázs[1] proposed a method of estimating the effects of distant singularities in the solution of partial-wave dispersion relations in an attempt to overcome certain difficulties in calculating these singularities from crossed-channel processes. In the usual procedure, a partial-wave amplitude is given by the Froissart-Gribov expression,[2] which, in the equal-unit-mass case and with $\pi\pi$ crossing, is

$$A_l^I(s)=\frac{4}{\pi(s-4)}\sum_{I'}\beta^{II'}\int_4^\infty dt\,A_l^{I'}(t,s)Q_l\left(1+\frac{2t}{s-4}\right), \quad (1)$$

where $I+l$ is assumed to be even and the crossing matrix is

$$\beta^{II'}=\begin{pmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}. \quad (2)$$

On the left-hand cut, which begins at $s=0$, the discontinuity can be obtained by taking the imaginary part of (1). The result, out to $s=-32$, which is the beginning of the double-spectral function A_{tu}, is

$$\mathrm{Im}A_l^I(s)=\frac{2}{s-4}\sum_{I'}\beta^{II'}\int_4^{4-s}dt \times A_l^{I'}(t,s)P_l\left(1+\frac{2t}{s-4}\right), \quad (3)$$

where the path of integration is shown in Fig. 1.

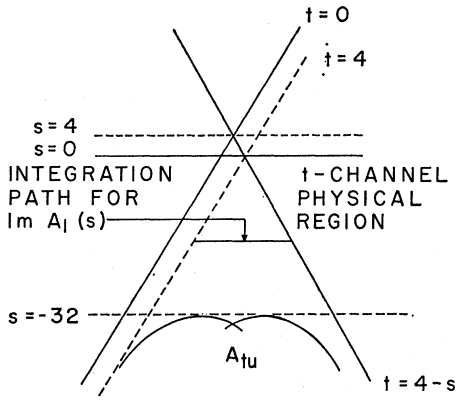

FIG. 1. Region of integration for $\mathrm{Im}A_l(s)$.

A_t is obtained from the imaginary part of the t-channel amplitude and is normally given by an expansion in t-channel partial waves,

$$A^{I'}(t,s)=\sum_{l'}(2l'+1)A_{l'}^{I'}(t)P_{l'}(\cos\theta_t), \quad (4)$$

$$A_t^{I'}(t,s)=\sum_{l'}(2l'+1)\,\mathrm{Im}A_{l'}^{I'}(t)P_{l'}(\cos\theta_t), \quad (5)$$

where $\cos\theta_t$ is the t-channel center-of-mass scattering angle.

The expansion (5) diverges in the region of the double-spectral function A_{tu}, which is shown in Fig. 1, and therefore should not be used for $s<-32$. Thus only the near part of the left-hand cut, $-32<s\leq 0$, is obtainable in this way. The Balázs method makes use of the fact that (5) is also convergent for $0\leq s\leq 4$, which is the gap between the left- and right-hand cuts of $A_l(s)$, so that (1) and (5) may be combined in this region to find the (real) form of $A_l(s)$. The direct-channel amplitude is represented by the N/D method[3] with the left-hand cut replaced by a series of poles, and the residues are determined by matching the N/D form with that obtained from (1) for $0\leq s\leq 4$. In principle, only the part of the cut for $s<-32$ has to be represented in this way, but Balázs actually replaced the entire cut by poles to simplify the practical calculations. In addition, he gave a prescription for finding the pole positions from an examination of the kernel of the integral equation for N.

This method was applied to the ρ-bootstrap problem by Balázs,[1] and he apparently obtained the mass and width of the ρ in approximate agreement with experiment. It was subsequently used in a variety of other

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[1] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962); **129**, 872 (1963); **132**, 867 (1963); **134**, AB1(E) (1964).

[2] M. Froissart, in Proceedings of the La Jolla Conference on Theoretical Physics, 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1961) [Soviet Phys. JETP **14**, 478 (1962)].

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

problems.[4] However, the validity of the matching procedure was questioned by several authors,[5–8] and it was demonstrated, particularly by Bond,[6] that the results were extremely unstable with respect to changes in the matching points and in n, the number of poles. These problems were discussed in recent papers by Williamson and Everett[7] and by Antippa and Everett,[8] in which the matching points were chosen by means of a criterion which demands, in effect, the closest possible agreement between the two forms of the amplitude in the gap. Here the variation of the solutions with the choice of matching points remains, but there is a method of choosing an optimum matching point, and this choice also seems to reduce the variation with n.

Recently one of us discussed[9] another method of stabilizing these solutions. In this paper, which we refer to as I, the near part of the left-hand cut was retained explicitly, as in the original proposal, and only the distant part was replaced by poles. The N/D integral equations were decoupled by a method similar to Pagels's approximation,[10] and closed-form solutions containing the cut were written for both N and D. The problem of scattering by a Yukawa potential was used as a model to show that the stability is improved by use of the cut, and the N/D output was in good agreement with the solutions obtained from the Schrödinger equation. In this paper, the modified Balázs method is applied to pion-pion scattering and its stability is discussed further.

The question of stability is first studied by using the method to reproduce known functions which have the desired analytic properties, satisfy elastic unitarity, and exhibit resonant behavior in the region $s>4$. For a given function, the gap and cut are known exactly and are used as the input to an N/D calculation. The positions of the poles replacing the distant parts of the cut are then adjusted, by a method similar to that used by Williamson and Everett,[7] to choose the matching points, so that the output leads to a best match in the gap. We find that when an optimal set of pole positions is used, the known function is reproduced with great accuracy and there is virtually no dependence on the choice of matching points nor on the number of poles.

We find, however, that the input on the cut and gap must be accurately specified and that modifying these quantities can lead to unpredictable changes in the output. This fact is relevant to all realistic calculations because here the input must be approximated from the crossed channels. Our findings for pion-pion scattering indicate that an accurate treatment of crossed channels is very important in obtaining stable and reliable results. In particular, we find that inputs of only one or two partial waves in the crossed channels do not lead to a reliable output; however, as additional partial waves are included, both the stability and accuracy of the output improve. In fact, if crossed-channel terms through $l=3$ are used, both the ρ and f_0 resonances appear automatically in approximate agreement with experiment. These solutions are achieved without the use of cutoffs or other adjustable parameters.

Section II outlines the general method, Sec. III gives the numerical results for the reproduction of the known functions and describes the search for the optimum pole positions, Sec. IV gives the numerical details for the $\pi\pi$ calculations, and the summary and conclusions are given in Sec. V.

II. GENERAL METHOD

To outline the method, we assume that the $N(s)$ and $D(s)$ both satisfy once-subtracted dispersion relations with s_0 as the common subtraction point, so that $D(s_0)=1$ and $N(s_0)=A(s_0)$. In what follows, the subscript l will be omitted. The amplitudes are normalized so that elastic unitarity is written as

$$\mathrm{Im}A(s)=\rho(s)|A(s)|^2, \tag{6}$$

where $\rho(s)$ is the usual phase-space factor

$$\rho(s)=\left(\frac{s-4}{s}\right)^{1/2}. \tag{7}$$

We replace the portion of the left-hand cut $s<-32$ with a series of poles at $s=a_j$ and residues α_j so that the equations for N and D become

$$N(s)=A(S_0)-(s-s_0)\sum_j \frac{\alpha_j}{(s-a_j)(s_0-a_j)}+\frac{s-s_0}{\pi} \times\int_{-32}^{0}\frac{D(s')\,\mathrm{Im}A(s')}{(s'-s)(s'-s_0)}ds', \tag{8}$$

$$D(s)=1-\frac{s-s_0}{\pi}\int_4^\infty \frac{\rho(s'')N(s'')}{(s''-s)(s''-s_0)}ds'', \tag{9}$$

where $\rho(s)$ is given in (7).

The details of decoupling these equations are given in I, and the procedure will only be outlined here. When (8) is substituted into (9), there arises after some

[4] For example, S. K. Bose and M. Der Sarkissian, Nuovo Cimento **30**, 878 (1963); V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963); P. Narayanaswamy and L. K. Pande, *ibid.* **136**, B1760 (1964); M. Der Sarkissian, Nuovo Cimento **30**, 894 (1963); J. C. Pati and K. V. Vasavada, Phys. Rev. **144**, 1270 (1966); K. C. Gupta, R. P. Saxena, and V. S. Mathur, *ibid.* **141**, 1479 (1966).

[5] M. L. Mehta and P. K. Srivastava, Phys. Rev. **137**, B423 (1965).

[6] A. H. Bond, Phys. Rev. **147**, 1058 (1966).

[7] M. R. Williamson and A. E. Everett, Phys. Rev. **147**, 1074 (1966).

[8] A. F. Antippa and A. E. Everett, Phys. Rev. **178**, 2443 (1969); **186**, 1571 (1969).

[9] J. Dilley, Phys. Rev. **186**, 1678 (1969).

[10] H. Pagels, Phys. Rev. **140**, B1599 (1965).

integration

$$D(s)=1+\frac{G(s)N(s)}{\pi}-\frac{G(s_0)N(s_0)}{\pi}-\frac{s-s_0}{\pi^2}\times\int_{-32}^{0}\frac{G(s'')\,\mathrm{Im}N(s'')ds''}{(s''-s)(s''-s_0)}+\frac{s-s_0}{\pi}\times\sum_j\frac{\alpha_jG(a_j)}{(s-a_j)(s_0-a_j)},\quad(10)$$

where the function $G(s)$ arises in the integration and is given by

$$G(s)=\rho(s)\ln\frac{\rho(s)+1}{\rho(s)-1}.\quad(11)$$

It is real for $s\leq 4$ and has a branch cut for $4\leq s<\infty$. Explicit forms for various ranges of real s are given in Eq. (13) of I.[11] The integral in (10) could be done exactly if it were meromorphic in s; thus the essential approximation is to replace $G(s)$ by a meromorphic function $\bar{G}(s)$, such that $G(s)\simeq\bar{G}(s)$ for $-32\leq s\leq 0$, and $\bar{G}(s)$ has poles at $s=s_n$ with residues β_n. Then the integral in (10) can be evaluated by means of a contour about the cut giving, after a substitution of the final, approximate form of D back into (8),

$$N(s)\simeq A(s_0)-(s-s_0)\times\sum_j\frac{\alpha_j}{s-a_j}+(s-s_0)\left[1-\frac{A(s_0)\Delta G(s_0)}{\pi}\right]I_0(s)+\frac{s-s_0}{\pi}\sum_j\alpha_j\Delta G(a_j)I_j(s)-\frac{s-s_0}{\pi}\times\sum_n\frac{\beta_nN(s_n)I_n(s)}{s_0-s_n},\quad(12)$$

$$D(s)\simeq 1+\frac{N(s)\Delta G(s)}{\pi}-\frac{A(s_0)\Delta G(s_0)}{\pi}+\frac{s-s_0}{\pi}\times\sum_j\frac{\alpha_j\Delta G(a_j)}{s-a_j}-\frac{s-s_0}{\pi}\sum_n\frac{\beta_nN(s_n)}{(s-s_n)(s_0-s_n)},\quad(13)$$

where

$$\Delta G(s)=G(s)-\bar{G}(s)\quad(14)$$

and

$$I_0(s)=\frac{1}{\pi}\int_{-32}^{0}\frac{\mathrm{Im}A(s')}{(s'-s)(s'-s_0)}ds',\quad(15)$$

$$I_n(s)=\frac{1}{\pi}\int_{-32}^{0}\frac{\mathrm{Im}A(s')}{(s'-s)(s'-s_n)}ds',\quad(16)$$

$$I_j(s)=\frac{1}{\pi}\int_{-32}^{0}\frac{\mathrm{Im}A(s')}{(s'-s)(s'-a_j)}ds',\quad(17)$$

and the substitution $\alpha_j/(s_0-a_j)\rightarrow\alpha_j$ has been made.

We do not think that the approximation involved in introducing $\bar{G}(s)$ has any serious effect on the results, since it is introduced in an integral over a finite range. As can be seen in (13), this approximation appears to introduce the cut of N into D, but the discontinuity is multiplied by $\Delta G(s)$, which is quite small on the cut. The function used for these calculations was

$$\bar{G}(s)=5.414104+\frac{3.103397}{s-5.5}+\frac{115.4189}{s-40.5},\quad(18)$$

which differs from $G(s)$ by less than 0.1% everywhere on the cut. Perturbations in $\bar{G}$ of up to 1% were found to leave the output phase shifts stable within a degree, and the use of other forms of $\bar{G}$ that matched G with equal accuracy left the output completely stable. In I, a check of the N/D output in the exact integral equations indicated that the error was negligible.

The input, which is given either by (3) or by the imaginary part of the known function, is $\mathrm{Im}A(s)$ in the integrals (15)–(17). The pole positions are determined by the optimization procedure mentioned previously, which will be described in Sec. III. For the n-pole case, there are $n+2$ constants left to be determined, assuming two poles in $\bar{G}(s)$ as in (18). They are the n residues α_j and the values of $N(s)$ at the poles of $\bar{G}$, $N(s_n)$. The $N(s_n)$ are found from setting $s=s_n$ in (12), and the α_j from setting $N(s_m)=A(s_m)D(s_m)$ at n points s_m in the gap. Thus the problem is reduced to solving $n+2$ simultaneous equations. In addition the subtraction point s_0 is chosen in the gap and becomes, in effect, another matching point. For these calculations, $s_0=0$ was usually used although it was varied within the gap in some cases to verify directly that the results were independent of it.

III. STUDIES OF KNOWN FUNCTIONS

Part of the error in calculations such as these comes from the pole approximation itself. This error was studied by an attempt to reproduce certain known functions. The results of this study indicate that the error of the pole approximation can be made extremely small when the amplitude is known on the cut and in the gap, and provided that enough care is used in searching for the pole positions which produce the best match between the two forms in the gap.

The functions that were used for this purpose satisfied elastic unitarity and the proper analytic and threshold properties for an $l=1$, equal-mass partial wave. In addition they contained a resonance at the approximate ρ mass. It was also required that ghost poles on the physical sheet be absent and that the high-energy phase shifts be of a form corresponding to no Castillejo-

[11] The notation in this paper is slightly different from that which was used in I. What we call G is called $-i\pi G$ in I. This produces a corresponding difference in the residues of β_n of $\bar{G}$. Also the pole positions a_j are called $-a_j^2$ in I. For further discussion of $G(s)$ see J. Dilley, J. Math. Phys. **8**, 2022 (1967).

Dalitz-Dyson (CDD) poles, that is, $\delta(\infty)\to 0$. This seems necessary because the present form of the N/D equations assumes the absence of CDD effects, and it was recently shown by Coulter[12] that one should not expect to reproduce a function with CDD phase shifts $[\delta(\infty)\to\pi]$ by a non-CDD form of the N/D method. In addition we have verified directly that the present form of the N/D equations will not reproduce a function which satisfies all of the above properties except for the non-CDD requirement.[13]

We have recently discussed the construction of such functions.[13] A suitable form is

$$A(s)=\frac{1}{\rho(s)}\frac{1}{\cot\delta(s)-i},$$
$$\cot\delta(s)=\frac{a+bs^{1/2}+cs+ds^{3/2}+es^2}{(s-4)^{3/2}}, \tag{19}$$

where the constants must be adjusted to give the required resonance, the non-CDD form of the phase shifts, and the absence of ghost poles. The known form of (19) on the near part of the cut and in the gap was used as the input for several N/D calculations, and the accuracy of the results was checked by a comparison of the phase shifts as given by the N/D equations with the known phase shifts from (19). The integrals in (15)–(17) were performed numerically using a sufficient number of points, usually 65 including the two ends, so that an increase in this number left the output stable.

The following parameter proved useful in evaluating the match in the gap:

$$Z=\sum_{i=1}^{50}[A(s_i)-N(s_i)/D(s_i)]^2, \tag{20}$$

where the sum is taken over 50 equally spaced points s_i in the gap, and $A(s_i)$ is the known amplitude. In each case, as the pole positions were varied over a wide range, the value of Z changed considerably, and a definite minimum could always be found. The output phase shifts also varied considerably with the pole positions, but the output corresponding to the minimum in Z was always very close to the known function. The matching points were varied throughout the gap and, except for a few exceptional cases to be discussed below, the variation of the output with the matching points was negligible. This variation was small even away from the minimum, but with the optimum set of poles it was usually less than the width of the lines on our graphs.

We now present examples of an attempt to reproduce a function of the form of (19). A suitable set of constants, which satisfy all necessary requirements in (19),

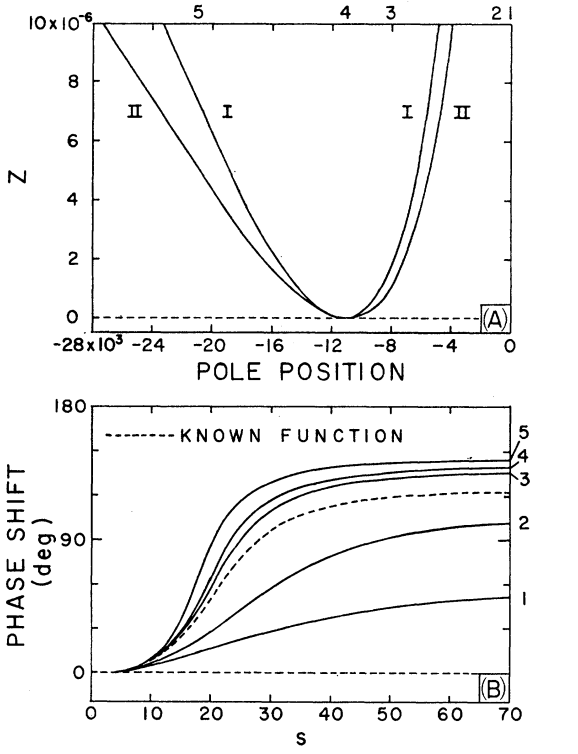

FIG. 2. (A) Variation of Z with the pole position in the one-pole case. I represents $s_m=2$, and II represents $s_m=4$, while the labels 1–5 give the pole positions used in the calculations of (B). (B) Comparison of the N/D output with the known function, represented by the dashed line, in the one-pole case.

is

$$a=19.875,\quad b=75.0875,\quad c=-15.5375,$$
$$d=0.05875,\quad e=0.0125. \tag{21}$$

The resonance occurs at $s=28.09$ in this case.

The results of an attempt to reproduce (21) in the one-pole case are given in Fig. 2. Figure 2(A) shows the variation of Z with the pole position for two different matching points, $s_m=2$ for case I, and $s_m=4$ for case II. Z is small over a wide range, approximately $-20\,000<a_i<-4000$, with a clear minimum at $a_i=-11\,000$. Figure 2(B) shows the output phase shifts and the known function for a representative sample of pole positions, where the labels 1–5 give the pole positions and correspond in Fig. 2(A) and Fig. 2(B). Case I corresponds to $a_i=-544$. There is some error, as should be expected, but the best solutions correspond to the small values of Z. Studies of intermediate pole positions show that the variation of the phase shift with the pole position appears to be smooth, and it follows the trend shown in Fig. 2(B). It would, of course, be possible to choose a pole position which could make the agreement almost exact, but this procedure, if used in a practical calculation, would amount to adjusting an arbitrary parameter to bring about agreement with experiment.

[12] P. W. Coulter, Phys. Rev. 179, 1590 (1969).
[13] T. Gibbons and J. Dilley (unpublished).

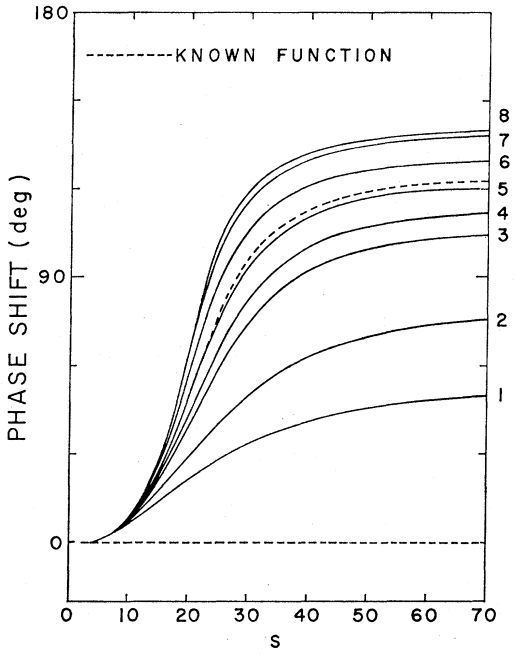

FIG. 3. Comparison of the N/D output with the known function in the two-pole case. The dashed line is the known function, and the labels correspond to Table I.

The idea here is to adjust all parameters so as to bring about agreement in a known region, and then to investigate the resulting error in the physical region. We next indicate that this error decreases in the two-pole case.

Table I shows the variation of Z with a representative sample of pole positions in the two-pole case, where (21) is still the model. There is a clear minimum for the set $(-70, -900)$. The corresponding phase shifts are compared to the known function in Fig. 3, and it can be seen that the error is very small for the optimum set. Although there is considerable agreement between the one- and two-pole cases, the error is substantially reduced with two poles. Table II and Fig. 4 give the same information for the three-pole case, and although the variation of Z is not as great, the smallest values still correspond to the most accurate solutions. The best solutions in Fig. 4, cases 4–6, are coincident with the known function on the scale used.

TABLE I. Two-pole fits to a known function in the gap.

Case	Pole positions	$10^7 Z$ Matching points 0.56, 4.0	1.6, 4.0
1	−32.43, −41.0	34.4	10.2
2	−33.0, −131.0	8.84	2.66
3	−45.0, −400.0	0.984	0.321
4	−50.0, −600.0	0.268	0.100
5	−70.0, −900.0	0.0066	0.0021
6	−150.0, −1250.0	0.362	0.073
7	−400.0, −4000.0	0.94	0.209
8	−1000, −10 000	1.06	0.236

The above discussion indicates that the position of the minimum in Z is independent of the matching points, and that the resulting variation of the output with the matching points is negligible. This was true over a wide range of points, but there were a few sets of matching points where, although the same optimum set of poles was found, the accuracy decreased. These cases tended to occur when most of the matching points were bunched near one end. An example is shown in Fig. 5, which is a three-pole case with pole positions and labels corresponding to Fig. 4. The values of Z in cases 1–6 were 558, 271, 223, 195, 200, and 204, all times 10^{-10} as in Table II, so that the matching was much worse. This poorer match occured in all such cases so that a choice could always be made.

Thus we have been able to find in each case a set of optimal pole positions where a best match in the sense described above exists, and for these sets there is a progressive improvement in the results with the number of poles. Even the one-pole case is surprisingly good, and the three-pole case reproduced the known function almost exactly. The output is stable with respect to the matching points except for a few exceptional cases that can be screened out *a priori* by means of their noticeably worse matches.

In most of these calculations, the threshold condition was enforced by the simple device of using $s=4$ as a matching point. However, this was not really necessary,

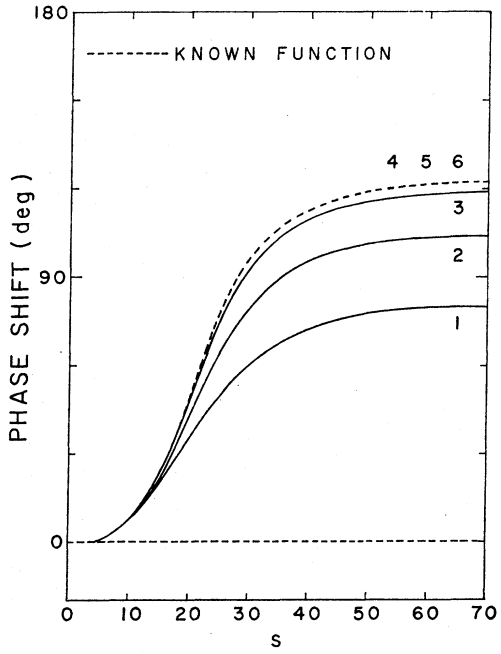

FIG. 4. Comparison of the N/D output with the known function in the three-pole case. The dashed line is the known function, and the best solutions, cases 4–6, are coincident with it. The labels correspond to Table II.

and in the one-pole case it was not always possible. With other matching points, the two forms matched at threshold as well as at any other point.

The pole positions chosen here do not always correspond to what would be expected from the procedure given by Balázs.[1] The pole approximation involves the integral

$$I(s)=\int_{-\infty}^{-32}\frac{\mathrm{Im}N(s')ds'}{(s'-s)(s'-s_0)}, \tag{22}$$

which would appear in the exact dispersion relation for $N(s)$. In the usual procedure, the substitution $s'=-x^{-1}$ is made, giving

$$I(s)=\int_0^{1/32}\frac{\mathrm{Im}N(x)}{1+s_0x}\left(\frac{1}{1+sx}\right)dx. \tag{23}$$

The approximation is to set

$$\frac{1}{1+sx}\simeq\sum_{i=1}^{n}\frac{G_i^n(x)}{1+sx_i}, \tag{24}$$

where the functions $G_i^n(x)$ can be chosen as the Lagrange interpolating polynomials,

$$G_i^n(x)=\prod_{j=1;j\neq i}^{n}\frac{x-x_j}{x_i-x_j}, \tag{25}$$

giving an n-pole form

$$I(s)\simeq\sum_{i=1}^{n}\frac{\alpha_i}{s-a_i}, \tag{26}$$

where

$$a_i=-x_i^{-1} \tag{27}$$

and

$$\alpha_i=\frac{1}{x_i}\int_0^{1/32}\frac{\mathrm{Im}N(x)G_i^n(x)dx}{1+s_0x}. \tag{28}$$

The error in the amplitude thus depends on the integral

$$\Delta I(s)=\int_0^{1/32}dx\frac{\mathrm{Im}N(x)}{1+s_0x}\left[\frac{1}{1+sx}-\sum_{i=1}^{n}\frac{G_i^n(x)}{1+sx_i}\right]. \tag{29}$$

Now a possible explanation for the discrepancy in the pole positions presents itself. The procedure outlined here consists of minimizing the error between the pole form and the exact form as a function of a_i, and, therefore, x_i. This probably minimizes the error as given by (29) as a function of x_i. The usual procedure involves optimizing the agreement in (24), in some sense, in the region $0\leq x\leq\frac{1}{32}$, which, because of the effect of $\mathrm{Im}N(x)$, would not necessarily give the same result.

The set of constants given in (21) is not unique, and calculations with other sets gave the same results as those quoted here. The same method also reproduced a nonresonant function with similar accuracy.

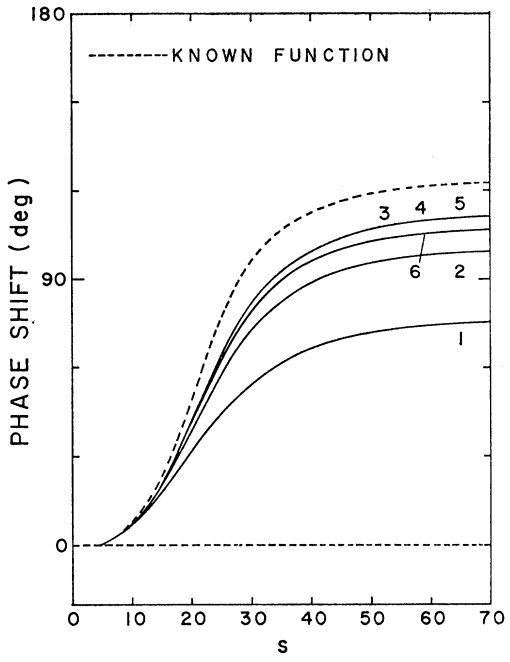

FIG. 5. The same calculation as in Fig. 4 with a set of matching points, $s_m=0.24$, 1.2, 4.0, which produced a poorer match in the gap. The labels correspond to Fig. 4 and Table II.

Some attempt was made to determine the effect of errors in the known function in the gap and on the cut. This question is relevant to the $\pi\pi$ problem, because these quantities must be approximated from crossed-channel processes. However, the results of perturbing the cut and the gap were not clear in a quantitative sense, and only a few general qualitative statements seem possible. Errors of up to 10% in the known regions produced unpredictable changes in the output. Sometimes the solutions were destroyed, and sometimes there was little difference. In addition, the agreement between the one- and two-pole cases was either weakened or destroyed. It seems clear that the method may fail in a practical calculation unless sufficient accuracy can be attained from the t channel. The results of Sec. IV indicate that this problem can be overcome.

TABLE II. Three-pole fits to a known function in the gap.

Case	Pole positions	$10^{10}Z$ Matching points 0.56, 2.0, 4.0	1.2, 2.8, 4.0
1	−32.43, −36.0, −51.0	44.8	17.4
2	−33.5, −41.0, −231.0	10.5	3.72
3	−40, −100, −600	5.05	2.13
4	−50, −300, −1000	3.55	1.73
5	−55, −500, −5000	3.76	1.75
6	−65, −900, −12 000	3.92	1.79

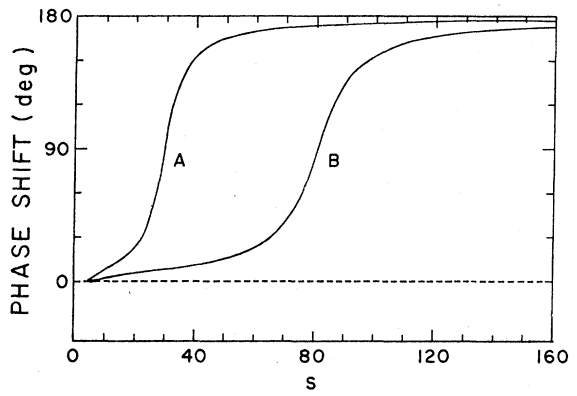

FIG. 6. Phase shifts for the crossed-channel resonances. A is the ρ, and B is the f^0.

IV. PION-PION SCATTERING

Several combinations of crossed-channel partial waves were used to calculate the $l=1$ and $l=2$ direct channel amplitudes for $\pi\pi$ scattering.[14] It is not clear in advance how many crossed-channel partial waves are necessary to attain sufficient input accuracy, but the results show, roughly speaking, a progressive improvement in stability and accuracy as the number is increased.

The usual narrow-width approximation, that is, the practice of representing crossed-channel resonances by δ functions, was not used in order to retain as much of the cut as possible. This approximation moves the beginning of the cut from $s=0$ back to $s=4-t_r$, where t_r gives the position of the t-channel resonance. Thus any resonance with $t_r \geq 36$ would not contribute to the cut, and there would be no cut over most of the range $-32 \leq s \leq 0$. The crossed-channel amplitudes were actually represented by the unitary Breit-Wigner form[15]

$$A_l(t) = \frac{\Gamma \rho^{2l}(t)}{t_r - t - i\Gamma\rho^{2l+1}(t)}, \tag{30}$$

where Γ is related to the width at half-maximum, $W_{1/2}$, by

$$\Gamma = \frac{W_{1/2}\sqrt{t_r}}{\rho^{2l+1}(t_r)} \tag{31}$$

and l is the partial-wave number. This form also satisfies the proper threshold conditions, and the parameters can be varied to give the desired partial wave and resonance. The imaginary part of (30) was used in (5) to obtain A_t, which was used to obtain the cut and gap by means of (3) and (1), respectively.

This was done numerically, and six-place accuracy was used to ensure that roundoff errors would be absent. Thus the cut and gap were known to a greater number of figures than was necessary even for the most accurate match with the N/D form, and such errors as were present in the inputs could be traced to errors in the crossed-channel model itself. In each case, i.e., each combination of crossed channel inputs, $A(s)$ was computed for all 50 points in the gap, and $\mathrm{Im}A(s)$ was computed on the cut at a sufficient number of points for the numerical integrations in Eqs. (15)–(17). Usually 65 points, including the two ends, were used, because this proved to be a sufficient number of points to keep the output stable as the number was increased. For purposes of comparison with the output, the input ρ and f^0 phase shifts are plotted in Fig. 6.

The N/D calculations were carried out in the elastic approximation, as given in Sec. II, for the one- and two-pole cases. The method was to minimize Z with respect to the pole positions, as in Sec. III. The phase shifts again changed with the pole positions, but clear minima were obtained in Z, which gave optimal sets of poles. In each case, the subtraction point was set as $s_0=0$ and served as a matching point, while the other matching points were varied within the gap to check the stability of the solutions with respect to them. It was easy, in the one-pole case, to move the single matching point from one end of the gap to the other; and in each two-pole case, one matching point was fixed at threshold while the other was moved through the gap. In some two-pole cases both matching points were also chosen within the gap so that there was no forced matching at threshold. This made little difference, and the agreement at threshold turned out to be as good as at any other point. With one exception to be discussed shortly, the solutions were completely stable with respect to the matching points.

For inputs, various combinations of crossed-channel resonances were used: case I, the ρ only; case II, ρ and f^0; case IV, ρ, f^0, and g. The corresponding resonance parameters were taken from experimental results and are given in Table III. In addition, an s-wave, $I=0$ resonance (σ) was introduced in two cases: III, σ, ρ, and f^0; and V, σ, ρ, f^0, and g. Some such resonance is increasingly favored by experiment,[16] and the exact

TABLE III. Crossed-channel resonance parameters.

Resonance	l	t_r	Γ
σ	0	30	5.25
ρ	1	30	6.0
f^0	2	81	10.6
g	3	138.9	11.2

[14] For $l=0$, the use of (1) is an inadequate method for calculating the gap because of substantial high-t contributions and because of the possibility of subtractions. We find, for example, that the resulting amplitudes badly violate the Martin constraints: A. Martin, Nuovo Cimento **47A**, 265 (1967). More pragmatically, it is impossible to obtain good fits in the gap and a stable output.

[15] This form was used in a different kind of N/D calculation, involving a cutoff, by D. Atkinson and K. M. Ong, Phys. Rev. **168**, 1692 (1968).

[16] For example, L. Gutay, in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, Argonne National Laboratory, 1969, edited by F. Loeffler and E. Malamud (unpublished).

TABLE IV. Summary of $\pi\pi$ output.

Case		I	II	III	IV	V		
Inputs		ρ	ρ, f^0	σ, ρ, f^0	ρ, f^0, g	σ, ρ, f^0, g		
							Resonance	
Out	Input case	No. poles	Optimum pole positions		Z'		Mass (MeV)	Width (MeV)
$I=1, l=1, (\rho)$	I	1	-160		3.56×10^{-7}			
		2	$-39.6, -56$		3.69×10^{-9}			
	II	1	-9394		2.56×10^{-6}		1075	...
		2	$-38.9, -115.2$		1.36×10^{-8}			
	III	1	-5298		5.43×10^{-7}		1188	...
		2	$-63.9, -128$		3.16×10^{-7}			
	IV	1	-7.34×10^{5}		5.10×10^{-7}		754	171
		2	$-193.6, -880$		7.22×10^{-8}		907	...
	V	1	-5.243×10^{5}		1.06×10^{-6}		786	210
		2	$-1056, -6176$		4.41×10^{-7}		798	244
$I=0, l=2, (f^0)$	I	1	-32.1		0.697			
		2	$-58.1, -128$		4.36×10^{-8}			
	II	1	-32.1		0.754			
		2	$-64, -246.4$		7.81×10^{-8}			
	III	1	-32.1		0.594			
		2	$-40, -88$		6.09×10^{-5}			
	IV	1	-2.147×10^{9}		6.30×10^{-4}		1465	218
		2	$-1.448\times10^{5}, -7.895\times10^{5}$		1.50×10^{-3}		1043	75
	V	1	-1.933×10^{9}		1.12×10^{-4}		1328	193
		2	$-1.316\times10^{5}, -7.895\times10^{5}$		2.36×10^{-4}		1154	122
$I=2, l=2$	I	1	-32.1		0.705			
		2	$-544, -3414.4$		1.06×10^{-7}			
	II	1	$-52\,450$		0.516			
		2	$-34, -44$		2.51×10^{-7}			
	III	1	-32.1		1.58			
		2	$-283.4, -800$		8.14×10^{-3}			
	IV	1	-38.4		3.32×10^{-4}			
		2	$-994.9, -13\,552$		5.41×10^{-6}			
	V	1	-32.1		2.61×10^{-2}			
		2	$-994.9, -12\,320$		8.23×10^{-4}			

parameters are not at all crucial in this calculation. The values given in Table III correspond to a σ having the same mass t_r and width $W_{1/2}$ as the ρ.

For output, three direct-channel partial waves were calculated, the p wave with $I=1$, and the d wave with both $I=0$ and $I=2$. A summary of various pertinent parameters is given in Table IV, which includes the optimal pole positions for each case, typical values of the matching parameters, and where a resonance exists in the output, the mass, and width. A slightly different matching parameter Z' is used, where

$$Z'=Z/\sum_{i=1}^{50}[A(s_i)]^2, \tag{32}$$

and the notation is the same as in (20). This provides a standard for comparing different partial waves with different values of $A(s)$ in the gap, and the smallest values of Z' given here are of the order of magnitude of those obtained for the known functions in Sec. III.

The phase shifts for the $I=1$, $l=1$ output are shown in Fig. 7, A giving the results for one pole and B those for two poles. Case I, where only the ρ is used as input, fails to produce any resonance in the direct channel and does not even qualitatively resemble the physical amplitude. Cases II and III show some improvement, but the results are still not satisfactory. Though a ρ of sorts is produced in the one-pole case, the resonance disappears in the more accurate two-pole calculations, indicating that the input is still inadequate. On the other hand, cases IV and V produce quite satisfactory results. A ρ agreeing roughly with experiment is now produced, and the resonance appears when either one or two poles are used. The one-pole cases actually produce a better output, but this is probably accidental since the two-pole match in the gap is better. As can be seen by comparing IV and V, the σ does not play a crucial role in producing the ρ, but its introduction does lead to a slight improvement in the results.

Figure 8 shows the phase shifts in the $l=2$, $I=0$ case where, again, A gives the one-pole results and B gives the two-pole case. Cases IV and V, as before, show satisfactory results, and the f^0 is produced in approximate agreement with experiment. As in the previous case, the σ is not essential in producing the resonance, but its introduction improves the results somewhat. However there is some deterioration of the results here as compared to the $l=1$, $I=1$ case. There is now no sign of a resonance in cases I–III, and a check of Table IV shows extremely poor matches for cases I–III with one pole. This is probably reflected in the poorer results of these cases. In addition the matching of cases IV and V

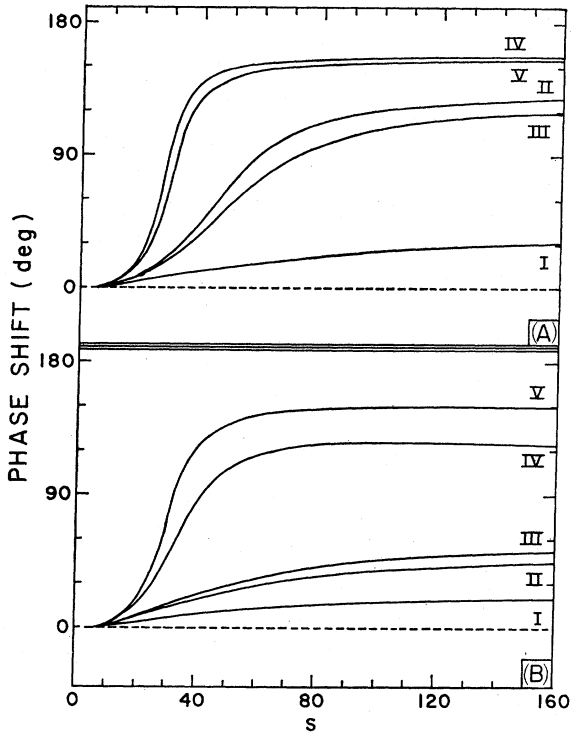

FIG. 7. N/D output for the ρ in A, the one-pole case, and B, the two pole case. The labels give the inputs and correspond to Table IV.

has become worse, and these are also the only cases which contain any significant variation of the output with the matching points. Figure 9 shows this variation for case V of Fig. 8(B), that is, the two-pole calculation with all four resonances in the crossed channel. One matching point was fixed at threshold while the other was varied throughout the gap, and the effect on the results can be seen. However, Z' also varied somewhat, and in these cases the matching points producing the lowest Z' were used in Fig. 8, even though these did not produce the best resonances. In the region of the f^0, one would expect that inelastic effects would start to become important, and without them, these results are probably as good as can reasonably be expected.

To check that the resonances produced here are not simply fortuitous, we placed a hypothetical $l=4$ Regge recurrence of the f^0 in the crossed channel to see if this could change the results in any significant way. However, the resulting change was very small, indicating that the partial waves used are indeed adequate in describing the main forces driving the low-energy region. In addition, we looked at the output in the $l=2$, $I=2$ case, a case which presumably should not contain any resonance at all. These results are shown in Fig. 10, where the scale is different from Figs. 6–9, and it can be seen that there is no sign of any resonance. The first three one-pole cases showed similar matching troubles as in the calculation of the f^0, and these cases show small, positive phase shifts. However, all the other cases, where the matching was good, produced small, negative phase shifts in approximate agreement with some other recent estimates.[17]

These calculations could undoubtedly be improved. First, one could improve the low-energy t-channel amplitude by means of, for example, keeping $I=2$ amplitudes or other possible resonances.[18] In addition, further information on the $I=0$, $l=0$ amplitude is becoming available and a better treatment should soon be possible. Second, we have made no attempt to estimate realistically the contributions to the gap from very large t and it is possible that some Reggeized form would do better here. Finally, inelasticity, either by means of a phenomenological inelasticity factor or by coupling to other channels, can be introduced. This is probably necessary to reduce the ρ width further and to stabilize the behavior of the f^0. It is, however, impressive that, except for s waves,[14] the basic features of the low-energy region emerge even without these refinements.

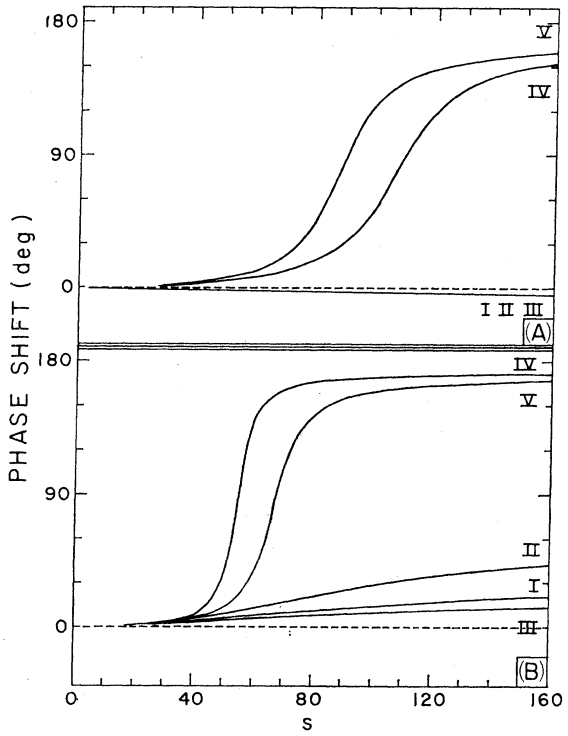

FIG. 8. N/D output for the f^0 in A, the one-pole case, and B, the two pole case. The labels give the inputs and correspond to Table IV.

[17] B. Y. Oh, W. D. Walker, J. T. Carroll, M. Firebaugh, A. Garfinkel, R. Morse, J. D. Prentice, N. R. Steenberg, and E. West, Phys. Rev. Letters **23**, 331 (1969); B. Y. Oh, A. Garfinkel, R. Morse, W. D. Walker, J. D. Prentice, E. West, and T. S. Yoon, Phys. Rev. D 1, 2494 (1970).

[18] For example, S. L. Kramer, H. R. Barton, Jr., L. J. Gutay, S. Lichtman, D. H. Miller, and J. H. Scharenguivel, Phys. Rev. Letters **25**, 396 (1970).

V. SUMMARY AND CONCLUSIONS

In this paper we have studied the accuracy and stability of the modified Balázs method, which retains part of the left-hand cut, by means of reproducing resonance models, and we have applied it to low-energy $\pi\pi$ scattering. The chief conclusion is that when good estimates of the cut and gap are available, and when the pole positions are chosen to optimize the agreement in the gap, this method is able to give very accurate results. There is a well-defined method for finding the optimum set of poles, and once this is done, there is little ambiguity in the output with respect to the matching points and the number of poles.

However, errors in the inputs produce unpredictable changes in the output, so that it is necessary to treat the crossed-channel processes very carefully in the $\pi\pi$ problem. It was found here that when realistic crossed-channel resonance forms rather than narrow-width approximations are used in order to avoid losing the cut, four crossed-channel partial waves are sufficient to produce the ρ and f^0 in approximate agreement with experiment. Three partial waves, the ρ, the f^0, and the g, seem to be necessary for reasonably reliable results. These calculations were done without the use of arbitrary adjustable parameters, such as cutoffs, and there were no CDD poles or inelastic effects.

In recent years, there has been some doubt concerning the general reliability of the N/D method, and a suspicion that the approximations involved are too crude to allow reliable conclusions.[19] There have also been hints that CDD poles are necessary to produce the ρ.[20] In these calculations, all of the basic features of low-energy $\pi\pi$ scattering emerged in an apparently natural way without the use of CDD poles or adjustable parameters. The numerical details are not quite in agreement with experiment, but in view of the accuracy of the known function calculations and the unpredictability of the effects of input errors, it is natural to attribute this to the remaining approximations. It should be noted in this connection that an attempt to repeat the calculations of this paper without keeping the cut, that is, in the pure pole case, resulted in failure. Even the known functions of Sec. III were not reproduced. Thus we are inclined to the view that the troubles of the N/D method are due primarily to poor approximations and inadequate input.

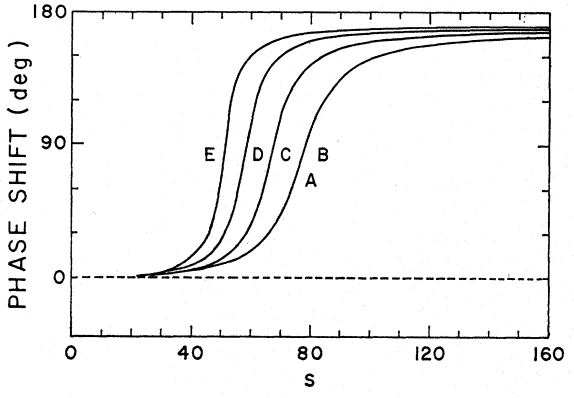

FIG. 9. Variation of the calculation of the f^0, case V of Fig. 8(B), with the matching points. One matching point is at $s_m=4$, while the other is, for cases A-E, 0.72, 1.44, 2.16, 2.88, and 3.6. The values of Z' are, for the same cases, 2.36, 2.34, 1.19, 2.69, and 7.44, all times 10^{-4}. C is the best solution and is plotted in Fig. 8(B).

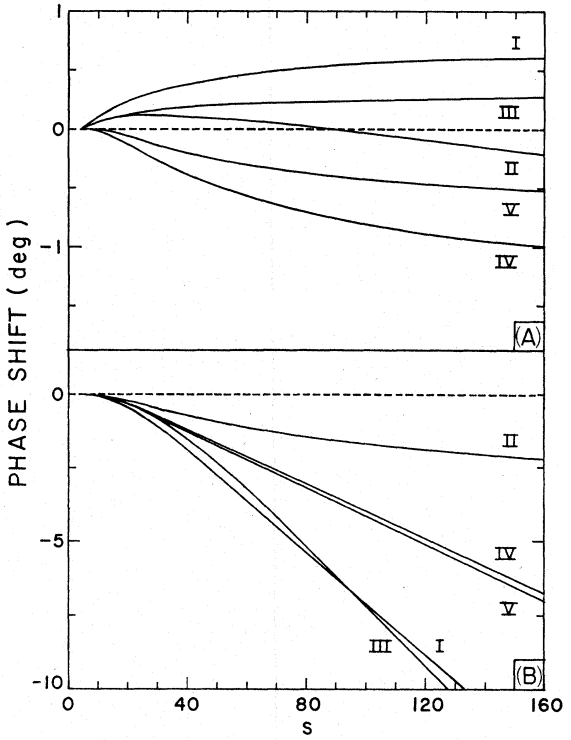

FIG. 10. N/D output for the $I=2$, $l=2$ partial wave in A, the one-pole case and B, the two-pole case; the labels give the inputs and correspond to Table IV. Note the change in scale from what was used in Figs. 6–9.

[19] For a review of some of these objections, see F. Zachariasen, in *Recent Developments in Particle Physics*, edited by M. J. Moravcsik (Gordon and Breach, New York, 1966).

[20] See, for example, R. Majumdar and V. S. Varma, Nucl. Phys. **B16**, 364 (1970).