Radiative Corrections to Decays with a Dalitz Pair*†

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The radiative corrections to the process $A\to B\gamma\to Be^+e^-$ (where A and B are hadrons) can be separated into a part which is purely dependent upon quantum electrodynamics (the one-photon-exchange diagrams) and a part dependent upon the hadronic structure (the two-photon-exchange diagrams). Neglecting the latter because such graphs are not singular when the lepton mass is set equal to zero, we concentrate on the one-photon-exchange terms, and calculate both the radiative corrections to the total decay rate including hard photons and the radiative corrections to the differential decay rate in the soft-photon approximation. Our results are directly applicable to the decay $\pi^0\to e^+e^-\gamma$, where our assumption that the two-photon-exchange graphs can be neglected is justified because $\pi^0 \not\to 3\gamma$ and the amplitude for $\pi^0\to 2\gamma\to e^+e^-\gamma$ is proportional to the lepton mass.

I. INTRODUCTION

IN competition with the process

$$A\to B\gamma\,,$$

one will always find the process

$$A\to Be^+e^-\,,$$

where the γ virtually converts into an electron pair (a Dalitz pair). Characteristic examples of such decays are

$$\pi^0\to\gamma\gamma\,,\quad \omega\to\pi^0\gamma\,,\quad \Sigma^0\to\Lambda\gamma\,.$$

Over the past two decades considerable interest has been focused on Dalitz pair production in the first decay ($\pi^0\to e^+e^-\gamma$). The branching ratio for this decay mode was calculated by Dalitz[1] in 1951. It is essentially independent of the strong interactions, the result being[2]

$$\frac{\Gamma(\pi^0\to e^+e^-\gamma)}{\Gamma(\pi^0\to\gamma\gamma)}=\frac{\alpha}{\pi}\left\{\tfrac{4}{3}\ln\frac{m_\pi}{m_e}-\frac{7}{3}+\frac{a}{3}+O\left(\left(\frac{m_e}{m_\pi}\right)^2\right)\right\}.\quad(1.1)$$

Here the first two terms[3] are of purely quantum-electrodynamical origin, while the third, involving the parameter a, is a contribution from strong interactions.

Experimental determination of the strong-interaction parameter has been made by Samios,[4] by Kobrak,[5] and by Némethy[6] using the dependence on a of the differential decay rate. The results are shown in Table I. As can be seen, the first two determinations of a gave negative results, while the last one yielded a value consistent with zero. Theoretically it is very difficult to obtain values of a which differ much from the intuitive expectation of $a\simeq+(m_\pi/m_\rho)^2<0.033$, although the failure of the first two experimental results to comply with this value triggered a deluge of papers on the subject.[7]

In view of the expected smallness of a [the value above makes a contribution to Eq. (1.1) of the order of 0.2%], it is clear that more precise experiments must take into account the radiative corrections to the Dalitz pair decay. As the experiments use the differential decay rate, it will be necessary to evaluate the differential radiative corrections. Joseph[8] has evaluated the radiative correction to the total decay rate numerically, with the result

$$\Gamma^{\rm rad}(\pi^0\to e^+e^-\gamma)/\Gamma(\pi^0\to\gamma\gamma)=1.05\times10^{-4}.\quad(1.2)$$

In this paper we rederive this result analytically. We furthermore evaluate the differential radiative correc-

TABLE I. Comparison between the theoretical and experimental determination of the strong-interaction parameter a.

	a	Error
Theory, ρ exchange.	0.032	
Theory, unsubtracted dispersion relations.	0.046	
Samios, bubble chamber (Ref. 4)	-0.24	±0.16
Kobrak, bubble chamber (Ref. 5)	-0.15	±0.10
Némethy, spark chamber (Ref. 6)	$+0.01$	±0.11

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[1] R. H. Dalitz, Proc. Phys. Soc. (London) **A64**, 667 (1951).

[2] This equation is derived in detail in Sec. III, together with the terms in α^2.

[3] Numerically we have $(\alpha/\pi)[\frac{4}{3}\ln(m_\pi/m_e)-7/3]=0.01185$.

[4] N. P. Samios, Phys. Rev. **121**, 275 (1961).

[5] H. Kobrak, Nuovo Cimento **20**, 1115 (1961).

[6] P. Némethy, Nevis Laboratories Report No. 165, 1968 (unpublished); S. Devons, P. Némethy, C. Nissim-Sabat, E. Di-Capua, and A. Lanzara, Phys. Rev. **184**, 1356 (1969).

[7] See, for example, S. M. Berman and D. A. Geffen, Nuovo Cimento **18**, 1192 (1960); H. S. Wong, Phys. Rev. **121**, 289 (1961); M. Gell-Mann and F. Zachariasen, *ibid.* **124**, 953 (1961); D. A. Geffen, *ibid.* **128**, 374 (1962); G. Barton and B. G. Smith, Nuovo Cimento **36**, 436 (1965).

[8] D. Joseph, Nuovo Cimento **16**, 997 (1960).

tions (i.e., the radiative correction to the Dalitz plot) in the soft-photon limit. Although the general formula is very complicated, the correction to the Dalitz plot in the case where the mass of the virtual photon is large (compared to the mass of the electron) reduces to a surprisingly simple analytical form involving only one dilogarithmic function. This correction is valid over almost all of the Dalitz plot and covers the region where experiments have been done. We can then find the radiative correction to the spectrum in the virtual-photon mass. As a check on our result we get the correct formula for the radiative correction to elastic electron scattering in the extreme relativistic region by taking the appropriate limit of our equation. Our final result that the radiative corrections are large and *negative* has not been used in experiments and helps to reduce the discrepancy between the theoretical and experimental values of a.

The outline of the paper is as follows. We first discuss the radiative corrections to the decay $A \rightarrow B\gamma \rightarrow Be^+e^-$, where A and B are hadrons (see Fig. 1). In general, the radiative corrections to this process (see Fig. 2) will fall into two different parts. The major contribution comes from the one-photon-exchange diagrams and subsequent radiative corrections to the lepton line. Such diagrams as (1), (2), (5), and (6) of Fig. 2 are essentially independent of the hadronic structure of A and B as we only need to know the amplitude for $A \rightarrow B$ together with one off-mass-shell photon. Another contribution comes from the two-photon exchange diagram (4) of Fig. 2, where we need to know the hadron structure because there is an integration over a virtual loop involving $A \rightarrow B$ together with two off-mass-shell photons. Such diagrams are very complicated to evaluate and here we draw on the results of a paper by Brown.[9]

In this paper the interference between one-photon- and two-photon-exchange diagrams in electron-positron scattering was examined in the limit when the electron mass is set equal to zero. Brown showed that, due to gauge invariance, the resulting limit is finite, so there are no lepton mass singularities. Fortunately, we can immediately use the result because it is precisely the interference between one-photon- and two-photon-exchange graphs which gives us the radiative correction to our decay rate. As the radiative corrections from the one-photon-exchange graphs are divergent in the limit when the mass is set equal to zero (there are both terms involving logarithms and the square of logarithms), we expect these terms to dominate in our final answer. We therefore concentrate entirely on the purely quantum-electrodynamical corrections to the lepton line in the one-photon-exchange graphs (1), (2), (5), and (6) of Fig. 2. Diagrams (3) and (7) will be discussed later.

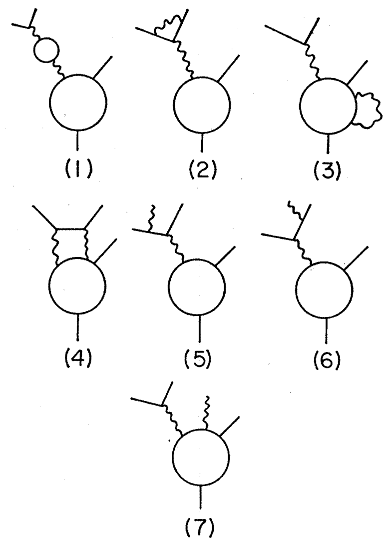

FIG. 2. The radiative corrections to the differential decay rate. The notation is the same as in Fig. 1.

In Sec. II we write the general rate for $A \rightarrow B\gamma$ where the photon is off the mass shell, and then in Sec. III we calculate the radiative corrections to the total decay rate for this process. Our calculation only needs the values of the second- and fourth-order spectral functions for the photon propagator. Section IV contains the derivation of the differential decay rate for $A \rightarrow Be^+e^-$, i.e., the Dalitz plot. We then calculate in Sec. V the radiative corrections to the Dalitz plot in the soft-photon approximation. Finally in Sec. VI we apply our results to the π^0-decay Dalitz plot. This decay is a case where we are completely justified in neglecting two-photon-exchange diagrams, because the amplitude for $\pi^0 \rightarrow 3\gamma$ is identically zero and the other class of diagrams (omitted in Fig. 2), where $\pi^0 \rightarrow 2\gamma \rightarrow e^+e^-\gamma$, have amplitudes proportional to the lepton mass.[8]

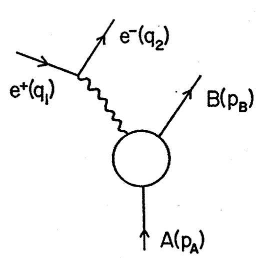

FIG. 1. The basic diagram for the differential decay rate. The wavy line denotes the photon while the solid lines denote either electrons or hadrons.

II. PROCESS $A \rightarrow B\gamma$

The amplitude for the process $A \rightarrow B\gamma$ is[10]

$$\langle B\gamma | T | A \rangle = -\epsilon^{\mu} \langle B | J_{\mu} | A \rangle ,$$

where ϵ_{μ} is the polarization vector of the γ and J_{μ} is the electromagnetic current. We are interested in the tensor

$$M_{\mu\nu}(p_A,p_B) = \sum_{\text{spins } A,B} \langle B | J_{\mu} | A \rangle \langle B | J_{\nu} | A \rangle^{*} , \qquad (2.1)$$

[9] R. W. Brown, Phys. Rev. D 1, 1432 (1970).

[10] The negative sign in front of ϵ_{μ} is consistent with the usual definition $\delta \mathcal{L}_I / \delta A_{\mu}$ even in the case when B is a photon.

which satisfies the relations

$$M_{\mu\nu}^* = M_{\nu\mu},$$

$$k^\mu M_{\mu\nu} = M_{\nu\mu} k^\mu = 0,$$

where $k = p_A - p_B$ is the photon momentum. Using these conditions[11] we find the most general form of $M_{\mu\nu}$:

$$M_{\mu\nu}(p_A,p_B) = (k_\mu k_\nu - g_{\mu\nu}k^2)M_1(k^2) - [g_{\mu\nu}(p_A\cdot k)^2 - (k_\mu p_{A\nu} + p_{A\mu}k_\nu)p_A]\cdot k + [p_{A\mu}p_{A\nu}k^2]M_2(k^2), \quad (2.2)$$

where $M_1(k^2)$ and $M_2(k^2)$ are real functions of k^2 free from kinematical singularities. In particular, we emphasize that they are regular as $k^2 \to 0$. It is often convenient to represent $M_{\mu\nu}$ in terms of a transverse and a longitudinal part:

$$M_{\mu\nu}(p_A,p_B) = -T_{\mu\nu}M_T(k^2) - L_{\mu\nu}M_L(k^2), \quad (2.3)$$

where

$$M_T(k^2) = k^2 M_1(k^2) + (p_A\cdot k)^2 M_2(k^2)$$

and (2.4)

$$M_L(k^2) = k^2[M_1(k^2) + m_A^2 M_2(k^2)]$$

are the transverse and longitudinal spectral functions. These functions are positive definite. The longitudinal projection operator is given by

$$L_{\mu\nu} = L_\mu L_\nu / L^2,$$

where

$$L_\mu = k_\mu(p_A\cdot k) - p_{A\mu}k^2,$$

and the transverse projection operator is given by

$$T_{\mu\nu} = P_{\mu\nu} - L_{\mu\nu},$$

where

$$P_{\mu\nu} = g_{\mu\nu} - k_\mu k_\nu / k^2.$$

The total decay rate into a massive photon is therefore

$$\Gamma_0(k^2) = \frac{\lambda^{1/2}(m_A^2, m_B^2, k^2)}{16\pi m_A^3}[2M_T(k^2) + M_L(k^2)], \quad (2.5)$$

where we use the notation of Källén[12]

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx.$$

The kinematical region for k^2 is

$$0 \le k^2 \le \Delta m^2 = (m_A - m_B)^2.$$

We shall assume $m_e \ll \Delta m$. Let us now calculate the two functions M_T and M_L in the case of $\pi^0 \to 2\gamma$. Here we have

$$\langle\gamma|J_\mu|\pi^0\rangle = (F/m_\pi)f(k^2)\epsilon_{\mu\nu\rho\sigma}\epsilon'^\nu k'^\rho k^\sigma,$$

where F is a dimensionless constant, $f(k^2)$ is the so-called form factor of the neutral pion satisfying

$$f(0) = 1,$$

and ϵ'_μ is the polarization vector for the photon of momentum k'_μ $(= p_B)$. A simple calculation yields

$$M_1 = -|Ff(k^2)|^2,$$

$$M_2 = -M_1/m_\pi^2,$$

so that

$$M_T = (1/4m_\pi^2)|Ff(k^2)|^2(m_\pi^2 - k^2)^2, \quad (2.6)$$

$$M_L = 0. \quad (2.7)$$

The result $M_L = 0$ could have been anticipated because $p_\pi^\mu\langle\gamma|J_\mu|\pi\rangle = 0$. Hence we get (remembering the factor of $\frac{1}{2}$ due to statistics)

$$\Gamma_0^{\pi^0\to 2\gamma}(k^2) = \frac{m_\pi}{64\pi}|Ff(k^2)|^2\left(1 - \frac{k^2}{m_\pi^2}\right)^3. \quad (2.8)$$

III. CALCULATION OF TOTAL DECAY RATE

We now consider the process

$$A \to (B\gamma) \to B + \text{e.m.},$$

where e.m. stands for any electromagnetic state containing electrons, positrons, and photons but excluding the one-photon state.

It can readily be derived that

$$\Gamma(A \to B\ \text{e.m.}) = \int_0^{\Delta m^2} \frac{ds}{s}\frac{1}{\pi}\operatorname{Im}\Pi(s)\Gamma_0(s), \quad (3.1)$$

where $\Pi(s)$ is the spectral function of the photon propagator, the imaginary part of which is given by

$$\theta(p)\operatorname{Im}\Pi(p^2) = -\frac{1}{6p^2}\sum_{\text{e.m.}}(2\pi)^4\delta(p - p_{\text{e.m.}}) \times\langle 0|J_\mu|\text{e.m.}\rangle\langle 0|J^\mu|\text{e.m.}\rangle^*.$$

Equation (3.1) can be interpreted as a probability

$$\frac{ds}{s}\frac{1}{\pi}\operatorname{Im}\Pi(s)$$

for a heavy photon to convert into any electromagnetic state of invariant mass between s and $s+ds$, times the decay rate into a heavy photon, summed over all mass values. This equation is exact in the one-photon-exchange approximation. Because $\operatorname{Im}\Pi(s)$ only depends on the electron mass and $\Gamma_0(s)$ essentially only depends on masses of the order of Δm, it is clear that the integral in Eq. (3.1) splits into two disjoint parts, one where $\operatorname{Im}\Pi(s)$ can be considered to be asymptotic and one where $s \simeq 4m_e^2$, where $\Gamma_0(s)$ can be approximated by

[11] To be more specific, we use Lorentz invariance, parity conservation, and current conservation. If the decay does not conserve parity, we must add a term $\epsilon_{\mu\nu\rho\sigma}p_A{}^\rho k^\sigma M_3(k^2)$.

[12] G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, Mass., 1964).

$\Gamma_0(0)$. To be precise, we write

$$\Gamma(A\to B\text{ e.m.})=\Gamma_0-\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi(s)+\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi^{\text{as}}(s)[\Gamma_0(s)-\Gamma_0(0)]+R,$$

where R is the overlap between the regions, i.e.,

$$R=\frac{1}{\pi}\int_0^{\Delta m^2}[\operatorname{Im}\Pi(s)-\operatorname{Im}\Pi^{\text{as}}(s)][\Gamma_0(s)-\Gamma_0(0)]\frac{ds}{s}.$$

We have here assumed that $\operatorname{Im}\Pi^{\text{as}}(s)$ is chosen so that

$$\operatorname{Im}\Pi(s)-\operatorname{Im}\Pi^{\text{as}}(s)=O(m_e^2/s)\quad\text{for } s\gg 4m_e^2.$$

It can be shown that[13] $R\approx O(m_e/\Delta m)$, and we shall therefore disregard it. It should also be noticed that

$$\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi(s)$$

is the cutoff quantum-electrodynamical charge renormalization constant. Defining

$$K(s)=\Gamma_0(s)/\Gamma_0(0),$$

we obtain [with $\Gamma(A\to B\gamma)=\Gamma_0(0)$]

$$\frac{\Gamma(A\to B\text{ e.m.})}{\Gamma(A\to B\gamma)}=\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi(s)+\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\times\operatorname{Im}\Pi^{\text{as}}(s)[K(s)-1]+O(m_e/\Delta m).\quad(3.2)$$

A. Lowest-Order Branching Ratio

In lowest order of α/π, we have[14]

$$\frac{1}{\pi}\operatorname{Im}\Pi^{(2)}(s)=\frac{\alpha}{\pi}\frac{1}{3}\left(1+\frac{2m_e^2}{s}\right)\left(1-\frac{4m_e^2}{s}\right)^{1/2}\theta(s-4m_e^2),$$

from which we obtain[15]

$$\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi^{(2)}(s)=\frac{\alpha}{\pi}\left[\tfrac{2}{3}\ln\left(\frac{\Delta m}{m_e}\right)-\frac{5}{9}\right]+O\left(\left(\frac{m_e}{\Delta m}\right)^4\right).$$

The asymptotic behavior is

$$\frac{1}{\pi}\operatorname{Im}\Pi^{(2)\text{as}}(s)=\frac{1}{3}\frac{\alpha}{\pi},$$

so that if we define

$$I_1=\int_0^{\Delta m^2}\frac{ds}{s}[K(s)-1],\quad(3.3)$$

which must be of order unity, we get

$$\frac{\Gamma^{(2)}(A\to Be^+e^-)}{\Gamma(A\to B\gamma)}=\frac{\alpha}{\pi}\left[\tfrac{2}{3}\ln\left(\frac{\Delta m}{m_e}\right)-\frac{5}{9}+\tfrac{1}{3}I_1+O\left(\left(\frac{m_e}{\Delta m}\right)^2\right)\right].\quad(3.4)$$

The reason that the correction term is of order $(m_e/\Delta m)^2$ follows from the inequality

$$\frac{1}{\pi}[\operatorname{Im}\Pi(s)-\operatorname{Im}\Pi^{\text{as}}(s)]\leq-\frac{\alpha}{3\pi}\frac{4m_e^4}{s^2},\quad s\geq 4m_e^2.$$

B. Fourth-Order Branching Ratio

The same method can be applied in fourth order, where the e.m. state can be either e^+e^- or $e^+e^-\gamma$. Here we have[16]

$$\frac{1}{\pi}\int_0^{\Delta m^2}\frac{ds}{s}\operatorname{Im}\Pi^{(4)}(s)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{4}{9}\ln^2\left(\frac{\Delta m}{m_e}\right)-\frac{13}{54}\ln\left(\frac{\Delta m}{m_e}\right)+\zeta(3)-\frac{\pi^2}{27}+\frac{65}{648}+O\left(\left(\frac{m_e}{\Delta m}\right)^2\right)\right].\quad(3.5)$$

The asymptotic form is[16]

$$\frac{1}{\pi}\operatorname{Im}\Pi^{(4)\text{as}}(s)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{4}{9}\ln\left(\frac{\Delta m}{m_e}\right)-\frac{13}{108}-\frac{2}{9}\ln\frac{\Delta m^2}{s}\right].\quad(3.6)$$

Thus we get

$$\frac{\Gamma^{(4)}(A\to B\text{ e.m.})}{\Gamma(A\to B\gamma)}=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{4}{9}\ln^2\left(\frac{\Delta m}{m_e}\right)+\left(\frac{4}{9}I_1-\frac{13}{54}\right)\ln\left(\frac{\Delta m}{m_e}\right)+\zeta(3)-\frac{\pi^2}{27}+\frac{65}{648}-\frac{13}{108}I_1-\frac{2}{9}I_2+O\left(\frac{m_e}{\Delta m}\right)\right],\quad(3.7)$$

[13] An analogous calculation has been given by B. E. Lautrup and E. de Rafael, Phys. Rev. **174**, 1835 (1968).

[14] G. Källén, in *Handbuch der Physik*, edited by S. Flugge (Springer, Berlin, 1958), Vol. 5, Part 1, p. 283.

[15] See Ref. 14, p. 284, where this is the subtraction constant.

[16] G. Källén and A. Sabry, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, No. 17 (1955), evaluate $\operatorname{Im}\Pi^4(s)$. The integral (3.5) is evaluated in Ref. 9.

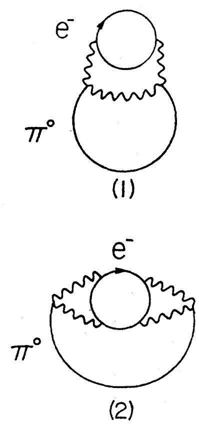

FIG. 3. The square of the $\pi^0\to e^+e^-\gamma\gamma$ amplitude. Diagram (1) represents the direct square of a term such as that represented in diagram (5) of Fig. 1. Diagram (2) represents the interference term specific to the π^0 decay because the two photons are identical.

where

$$I_2=\int_0^{\Delta m^2}\frac{ds}{s}\ln\frac{\Delta m^2}{s}[K(s)-1]. \qquad (3.8)$$

C. π^0 Decay

For the case of $\pi^0\to 2\gamma$, Eqs. (3.4) and (3.7) have to be multiplied by a factor of 2 because the photons are indistinguishable. From Eq. (2.8) we have

$$K(s)=(1-s/m_\pi^2)^3|f(s)|^2,$$

so employing the usual expansion for the form factor

$$f(s)=1+as/m_\pi^2+O((s/m_\pi^2)^2) \qquad (3.9)$$

and retaining only the first-order terms in a, we find

$$I_1=-\frac{11}{6}+\frac{a}{2},$$

$$I_2=-\frac{85}{36}+\frac{25}{24}a.$$

Here the final branching ratio is

$$\frac{\Gamma(\pi^0\to e^+e^-\gamma)}{\Gamma(\pi^0\to 2\gamma)}=\frac{\alpha}{\pi}\left[\frac{4}{3}\ln\left(\frac{m_\pi}{m_e}\right)-\frac{7}{3}+\frac{a}{3}+O\left(\left(\frac{m_e}{m_\pi}\right)^2\right)\right] +\left(\frac{\alpha}{\pi}\right)^2\left[\frac{8}{9}\ln^2\left(\frac{m_\pi}{m_e}\right)-\left(\frac{19}{9}-\frac{4}{9}a\right)\ln\left(\frac{m_\pi}{m_e}\right)+2\zeta(3) -\frac{2}{27}\pi^2+\frac{137}{81}-\frac{63}{108}a+O\left(\frac{m_e}{m_\pi}\right)\right]. \qquad (3.10)$$

Numerically the second-order term in α has the value (for $a=0$) of 1.05×10^{-4}, in agreement with the result of Joseph.[8] The dominant term is obviously the one in $\ln^2(m_\pi/m_e)$, although it is partially canceled by the term in $\ln(m_\pi/m_e)$. All finite terms (in the limit when the electron mass tends to zero) are smaller by approximately a factor of 4. Hence it would not be a bad approximation to drop all terms which are finite in the limit when the lepton mass is set equal to zero. This is our justification in using Eq. (3.7) and dropping terms from diagrams where two photons are exchanged between the lepton and hadron lines.

It should be emphasized that for π^0 decay, the above calculation does not include interference terms between the radiative photon and the decay photon. If we represent the ordinary square of the $\pi^0\to e^+e^-\gamma$ radiative correction by diagram (1) in Fig. 3, then the interference term can be represented by diagram (2). This rate is thus proportional to part of the general fourth-order photon-photon scattering amplitude. However, because we have a two-photon exchange between the π^0 and the lepton loop, we will get two additional factors of m_e from diagram (2) relative to diagram (1). Hence there cannot be any logarithmic term from diagram (2) in the limit when the lepton mass is set equal to zero. We are therefore justified in neglecting such an interference term in Eq. (3.10).

The final radiative correction is positive and rather small. However, this does not mean that the radiative corrections to the Dalitz plot are small because there are usually large cancellations between positive and negative regions. Therefore, it is wrong to apply this correction to the whole of the Dalitz plot. We will now evaluate the radiative corrections to the plot and to the spectrum in the mass of the virtual photon.

IV. DIFFERENTIAL DECAY RATE IN LOWEST ORDER

The basic diagram for the differential decay rate in lowest order is shown in Fig. 1. The kinematics are determined by

$$p_A=p_B+q_1+q_2,$$

$$p_A^2=m_A^2,\quad p_B^2=m_B^2,\quad q_1^2=q_2^2=m_e^2.$$

We define the vectors $q=q_1+q_2$, $Q=q_1-q_2$, satisfying

$$q^2=Q^2+4m_e^2,\quad q\cdot Q=0$$

and introduce the kinematical variables

$$x=q^2/\Delta m^2,\quad 1\geq x\geq r^2=(2m_e/\Delta m)^2.$$

It is also necessary to introduce the energy partition between the electrons in the rest system of the A particle

$$y=\frac{E_1-E_2}{|\mathbf{q}_1+\mathbf{q}_2|}.$$

In invariants this quantity is given by

$$y=\frac{2p_A\cdot Q}{\lambda^{1/2}(q^2,m_A^2,m_B^2)}$$

and in the rest system of the pair we have

$$y=-\beta\cos\theta,$$

where $\beta=(1-4m_e^2/q^2)^{1/2}=(1-r^2/x)^{1/2}$ is the velocity of the electrons in this system and θ is the angle between the momentum of the positron and the A particle. Thus the kinematic range for y is

$$-\beta\leq y\leq\beta.$$

The decay rate is found to be

$$\frac{d^2\Gamma(A\to Be^+e^-)}{dxdy}=\frac{1}{512\pi^3}\frac{\Delta m^4}{m_A^3}[(1-x)(\rho^2-x)]^{1/2}\times\sum_{\text{spins}}|\langle e^+e^-B|T|A\rangle|^2, \quad (4.1)$$

where

$$\rho=\frac{m_A+m_B}{m_A-m_B}\geq 1.$$

Our matrix element is

$$\langle e^+e^-B|T|A\rangle=(e/q^2)\bar{u}\gamma^\mu v\langle B|J_\mu|A\rangle,$$

where u and v are electron and positron spinors. Thus we get

$$\sum_{\text{spins}}|\langle e^+e^-B|T|A\rangle|^2=\frac{e^2}{q^4}\mathrm{Tr}[\gamma_\mu(q_1-m_e)\gamma_\nu(q_2+m_e)]M^{\mu\nu}(p_A,p_B)=\frac{2e^2}{q^2}\left[\left(1+y^2+\frac{r^2}{x}\right)M_T+(1-y^2)M_L\right].$$

Finally we obtain the well-known result[17]

$$\frac{d^2\Gamma(A\to Be^+e)}{dxdy}=\frac{\alpha}{64\pi^2}\frac{\Delta m^2}{m_A^3}\frac{[(1-x)(\rho^2-x)]^{1/2}}{x}\times\left[\left(1+y^2+\frac{r^2}{x}\right)M_T+(1-y^2)M_L\right]. \quad (4.2)$$

Integration over dy yields

$$\frac{d\Gamma(A\to Be^+e^-)}{dx}=\frac{\alpha}{48\pi^2}\frac{\beta}{x}\left(1+\frac{r^2}{2x}\right)\frac{\Delta m^2}{m_A^3}\times[(1-x)(\rho^2-x)]^{1/2}(2M_T+M_L)=\frac{\alpha}{3\pi}\frac{\beta}{x}\left(1+\frac{r^2}{2x}\right)\Gamma_0(x), \quad (4.3)$$

with $\Gamma_0(x)$ defined by Eq. (2.5).

V. RADIATIVE CORRECTIONS TO DIFFERENTIAL DECAY RATE

The radiative corrections to the differential decay rate are determined by the diagrams shown in Fig. 2. Of these seven diagrams, (1)–(4) are the virtual corrections while diagrams (5)–(7) are the real corrections (or inner bremsstrahlung corrections). While diagrams (1), (2), and (5), (6) can be calculated explicitly in terms of M_L and M_T, diagrams (3), (4), and (7) depend on the nature of the strong interactions. In writing down these diagrams, we have furthermore assumed that the renormalization program has been carried out, so there are no corrections to the external lines. The case where particle B is a photon will be treated in Sec. VI.

We shall evaluate the correction to $d^2\Gamma/dxdy$, i.e., the correction to the Dalitz plot. We write it in the following way:

$$\frac{d^2\Gamma^{\text{rad}}}{dxdy}=\delta(x,y)\frac{d^2\Gamma}{dxdy}, \quad (5.1)$$

where $d^2\Gamma/dxdy$ is given by Eq. (4.2). Clearly we have

$$\delta=\delta_{\text{virtual}}+\delta_{\text{real}},$$

corresponding to the seven diagrams of Fig. 2. The quantity δ_{real} will depend on the experimental configuration. We shall assume that the experimental situation only allows inner bremsstrahlung photons in a certain energy interval:

$$0\leq\omega\leq\Delta E.$$

If ΔE is sufficiently small (how small will be determined later), we can use the soft-photon approximation which is a major calculational simplification.

Integrating Eq. (5.1) over y, we obtain

$$\frac{d\Gamma^{\text{rad}}}{dx}=\delta(x)\frac{d\Gamma}{dx}, \quad (5.2)$$

which defines $\delta(x)$ because we know that

$$\frac{d\Gamma}{dx}=\frac{\alpha}{3\pi}\frac{\beta}{x}\left(1+\frac{r^2}{2x}\right)\Gamma_0(x). \quad (5.3)$$

A. Virtual Corrections

The second-order virtual corrections arise from the interference between the virtual diagrams and the lowest-order graph. Using Eq. (4.1), we obtain

$$\frac{d^2\Gamma^{\text{virtual}}}{dxdy}=\frac{1}{512\pi^3}\frac{\Delta m^4}{m_A^3}[(1-x)(\rho^2-x)]^{1/2}\times\sum_{\text{spins}}2\,\mathrm{Re}(\langle e^+e^-B|T_0|A\rangle\times\langle e^+e^-B|T_{\text{virtual}}|A\rangle^*)$$

so that

$$\delta_{\text{virtual}}=\sum_{\text{spins}}2\,\mathrm{Re}\langle e^+e^-B|T_0|A\rangle\langle e^+e^-B|T_{\text{virtual}}|A\rangle^*/\sum_{\text{spins}}|\langle e^+e^-B|T_0|A\rangle|^2.$$

The amplitude for diagram (1) in Fig. 2 is evidently

[17] N. M. Kroll and W. Wada, Phys. Rev. 98, 1355 (1955).

given by

$$\langle e^+e^-B|T|A\rangle = -\Pi^{(2)}(q^2)\langle e^+e^-B|T_0|A\rangle,$$

where $\Pi^{(2)}(q^2)$ is the second-order approximation to the renormalized photon spectral function. Explicitly, we have[18] for $q^2 \leq 4m_e^2$

$$\Pi^{(2)}(q^2) = \frac{\alpha}{\pi}\left[\frac{8}{9} - \frac{\beta^2}{3} + \beta\left(\frac{1}{2} - \frac{\beta^2}{6}\right)\ln\gamma\right],$$

with

$$\gamma = (1-\beta)/(1+\beta).$$

We obtain

$$\delta_1(x,y) = -2\Pi^{(2)}(q^2),$$

which only depends on x.

In diagrams (2) of Fig. 2, we represent the electromagnetic vertex by

$$-ie\bar{u}\left[\gamma_\mu(F_1^{(2)}+F_2^{(2)}) - \frac{(q_2-q_1)_\mu}{2m_e}F_2^{(2)}\right]v, \quad (5.4)$$

where $F_1^{(2)}$ and $F_2^{(2)}$ are the second-order renormalized electromagnetic form factors of the electron. Explicitly we have[19]

$$F_1^{(2)} = \frac{\alpha}{\pi}\left\{-1 - \frac{1+2\beta^2}{4\beta}\ln\gamma - \frac{1+\beta^2}{2\beta}\right.$$
$$\times[\mathrm{Li}_2(1-\gamma) - \tfrac{1}{4}\pi^2 + \tfrac{1}{4}\ln^2\gamma]$$
$$\left. + \left(1 + \frac{1+\beta^2}{2\beta}\ln\gamma\right)\ln\left(\frac{m_e}{\lambda}\right)\right\}, \quad (5.5)$$

$$F_2^{(2)} = \frac{\alpha}{\pi}\frac{1-\beta^2}{4\beta}\ln\gamma, \quad (5.6)$$

where $\mathrm{Li}_2(x)$ is the dilogarithm[20] defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}dt$$

and λ is the photon mass (infrared cutoff). Corresponding to the two terms in Eq. (5.4), we have two contributions to δ_2:

$$\delta_2 = \delta_2' + \delta_2''.$$

It is obvious that

$$\delta_2' = 2(F_1^{(2)} + F_2^{(2)}), \quad (5.7)$$

and we find δ_2'' from

$$\sum_{\text{spins}} \mathrm{Re}[\langle e^+e^-B|T_0|A\rangle\langle e^+e^-B|T_2''|A\rangle^*]$$
$$= \frac{e^2}{q^4}\mathrm{Tr}\left[\gamma_\mu(\not{q}_1 - m_e)\frac{Q_\nu}{2m_e}F_2^{(2)}(\not{q}_2+m_e)\right]M^{\mu\nu}(p_A,p_B),$$

which reduces to

$$\frac{2e^2}{q^2}F_2^{(2)}(x)\left[\left(1-y^2-\frac{r^2}{x}\right)M_T + y^2M_L\right],$$

so that

$$\delta_2''(x,y) = F_2^{(2)}(x)$$
$$\times\frac{[(1-y^2-r^2/x)M_T+y^2M_L]}{[(1+y^2+r^2/x)M_T+(1-y^2)M_L]}. \quad (5.8)$$

This is the only virtual correction which is a function of y. We find upon integration

$$\delta_2''(x) = F_2^{(2)}(x)\frac{1-r^2/x}{1+r^2/2x}.$$

Finally, we expand the virtual corrections for $q^2 \gg 4m_e^2$, i.e., for $x \gg r^2$, and obtain to lowest order in r

$$\delta_1(x,y) = \frac{\alpha}{\pi}\left(-\tfrac{4}{3}\ln\frac{r}{2} - \frac{10}{9} + \tfrac{2}{3}\ln x\right),$$

$$\delta_2'(x,y) = \frac{\alpha}{\pi}\left[2\left(1+2\ln\frac{r}{2} - \ln x\right)\ln\left(\frac{m_e}{\lambda}\right) - 2\ln^2\frac{r}{2}\right.$$
$$\left. + 2\ln\frac{r}{2}\ln x - 3\ln\frac{r}{2} - \tfrac{1}{2}\ln^2 x + \tfrac{3}{2}\ln x - 2 + \frac{\pi^2}{6}\right],$$

$$\delta_2''(x,y) = 0.$$

Thus the total virtual corrections from diagrams (1) and (2) of Fig. 2 give, expressing our result in terms of the variable $R = 4x/r^2 = q^2/m_e^2$,

$$\delta_{1,2}(x,y) = \frac{\alpha}{\pi}\left[2(1-\ln R)\ln\frac{m_e}{\lambda} - \tfrac{1}{2}\ln^2 R\right.$$
$$\left. + \frac{13}{6}\ln R - \frac{28}{9} + \frac{\pi^2}{6}\right] \quad (5.9)$$

in this approximation.

B. Inner Bremsstrahlung Corrections

In this section we calculate the contributions from diagrams (5) and (6) of Fig. 2 in the soft-photon approximation. The amplitude for these diagrams is

$$\langle e^+e^-\gamma B|T_{5,6}|A\rangle = e^2\frac{\langle B|J_\mu|A\rangle}{l^2}$$
$$\times\bar{u}\left[\frac{\gamma^\mu(\gamma\cdot\epsilon\not{k} - 2q_1\cdot\epsilon)}{2q_1\cdot k} + \frac{(-\not{k}\gamma\cdot\epsilon + 2q_2\cdot\epsilon)\gamma^\mu}{2q_2\cdot k}\right]v,$$

[18] See Ref. 14, p. 284.

[19] See Ref. 14, pp. 304 and 305. To be explicit, $F_1^{(2)}+F_2^{(2)} = \bar{R}^{(0)}(-Q^2) - \bar{R}^{(0)}(0) - \bar{S}^{(0)}(-Q^2) + \bar{S}^{(0)}(0)$, and $F_2^{(2)} = \bar{S}^{(0)}(-Q^2)$.

[20] L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958).

where k is the momentum of the bremsstrahlung photon and ϵ_μ is its polarization vector. The virtual photon has momentum

$$l=q+k\,,\quad q=q_1+q_2\,.$$

We shall assume that the experimental resolution is such that only photons up to a certain energy ΔE go undetected. We shall also assume that both electron and positron are highly energetic (in A's rest system),

$$E_1,E_2\gg m_e\,,$$

and that the opening angle θ_{12} is nonvanishing[21] such that

$$q^2=2m_e^2+2E_1E_2(1-v_1v_2\cos\theta_{12})$$

will satisfy

$$q^2\gg 4m_e^2\,.$$

In the soft-photon limit we also take

$$\Delta E\ll E_1,E_2\,,$$

so that

$$l^2\simeq q^2\,,$$

and then

$$\langle e^+e^-\gamma B|T_{5,6}|A\rangle=-eb\cdot\epsilon\langle e^+e^-B|T_0|A\rangle\,,$$

where

$$b=\frac{q_1}{q_1\cdot k}-\frac{q_2}{q_2\cdot k}\,.$$

Integrating over all photon directions and all photon energies within the interval ΔE, we obtain the real contribution from diagrams (5) and (6):

$$\delta_{5,6}(x,y)=\int_{|\mathbf{k}|\leq\Delta E}\frac{d\mathbf{k}}{2\omega(2\pi)^3}\sum_{\text{pol}}|eb\cdot\epsilon|^2\,,\qquad(5.10)$$

where

$$\omega=(\mathbf{k}^2+\lambda^2)^{1/2}\,.$$

In the Appendix this integral is evaluated with the result

$$\delta_{5,6}(x,y)=(\alpha/\pi)[2G_1(x)\ln(2\Delta E/\lambda)+G_2(x,y)]\,,$$

where, for all x, we have

$$G_1(x)=-\frac{1+\beta^2}{2\beta}\ln\gamma-1\,,$$

which is just the negative of the coefficient of the infrared term in Eq. (5.5). For $x\gg r^2$, we have

$$G_1(x)=-1+\ln R\,,\qquad(5.11)$$

$$G_2(x,y)=-\tfrac{1}{2}\ln^2R-(\ln R-1)\left(-1+\ln\frac{\sigma^2-y^2}{\sigma^2-1}\right)+1-\frac{\pi^2}{3}+\text{Li}_2\left(\frac{1-y^2}{\sigma^2-y^2}\right)-\tfrac{1}{2}\ln^2\left(\frac{\sigma+y}{\sigma-y}\right),\qquad(5.12)$$

where

$$\sigma=\frac{\rho+x}{[(1-x)(\rho^2-x)]^{1/2}}$$

and

$$R=4x/r^2=q^2/m_e^2\,.$$

C. Total Correction

We now find the total correction $\delta(x,y)$ from diagrams (1), (2), (5), and (6):

$$\delta_{1256}(x,y)=\frac{\alpha}{\pi}\left[-\left(\ln\frac{q^2}{m_e^2}-1\right)\left(\ln\frac{q^2(\sigma^2-y^2)}{4\Delta E^2(\sigma^2-1)}-\frac{13}{6}\right)-\frac{17}{18}-\tfrac{1}{2}\ln^2\left(\frac{\sigma+y}{\sigma-y}\right)+\left(\text{Li}_2\left(\frac{1-y^2}{\sigma^2-y^2}\right)-\frac{\pi^2}{6}\right)\right].\qquad(5.13)$$

Note that

$$\tfrac{1}{4}q^2\frac{\sigma^2-y^2}{\sigma^2-1}=E_1E_2\,,$$

so that the condition for the photons being soft is obviously

$$\Delta E\ll(E_1E_2)^{1/2}\,.$$

Setting $y=0$, we obtain $(E=E_1=E_2)$

$$\delta_{1256}(x,0)=\frac{\alpha}{\pi}\left[-\left(\ln\frac{q^2}{m_e^2}-1\right)\left(\ln\frac{E^2}{\Delta E^2}-\frac{13}{6}\right)-\frac{17}{18}+\text{Li}_2\left(\frac{1}{\sigma^2}\right)-\frac{\pi^2}{6}\right].\qquad(5.14)$$

If we define θ as the angle between the electrons,

$$q^2=4E^2\sin^2(\tfrac{1}{2}\theta)\,,$$

we obtain

$$1/\sigma^2=\cos^2(\tfrac{1}{2}\theta)\,.$$

Using well-known relations between dilogarithmic functions, we can cast our result into the form

$$\delta_{1256}(x,0)=-\frac{e^2}{\pi^2}\left\{\left[\ln\left(\frac{2E}{m_e}\sin\tfrac{1}{2}\theta\right)-\frac{1}{2}\right]\left[\ln\frac{E}{\Delta E}-\frac{13}{12}\right]+\frac{17}{72}-\frac{1}{4}\left[\Phi(-\sin^2(\tfrac{1}{2}\theta))+\frac{\pi^2}{12}-\ln(\sin^2(\tfrac{1}{2}\theta))\ln(\cos^2(\tfrac{1}{2}\theta))\right]\right\},\qquad(5.15)$$

where $\Phi(x)=-\frac{1}{12}\pi^2-\text{Li}_2(-x)$, and this result is identical to that given by Källén[22] for potential scattering in

[21] That is, $\theta_{12}\gg m_e(E_1E_2)^{1/2}$.

[22] See Ref. 14, p. 309.

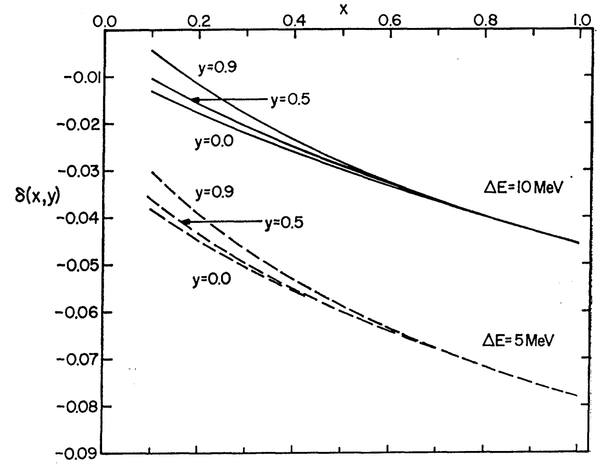

FIG. 4. $\delta(x,y)$ as a function of x for fixed values of y. The solid lines are for $\Delta E=10$ MeV and the dashed ones for $\Delta E=5$ MeV.

the extreme relativistic case (apart from the sign which stems from our definition of δ). It is interesting to note that all terms in $\delta_{1256}(x,y)$ are negative so that the total correction becomes negative.

By means of Eq. (5.2) we obtain the radiative corrections to the spectrum in x,

$$\delta(x)=\int_{-1}^{+1}dy\,\delta(x,y)[(1+y^2)M_T+(1-y^2)M_L]\Big/\int_{-1}^{+1}dy\,[(1+y^2)M_T+(1-y^2)M_L]. \quad (5.16)$$

After some algebra, this becomes

$$\delta_{1256}(x)=f_1(x)+\frac{M_L}{2M_T+M_L}f_2(x), \quad (5.17)$$

with

$$f_1(x)=\frac{\alpha}{\pi}\bigg[-\left(\ln\frac{q^2}{m_e^2}-1\right)\left(\ln\frac{q^2}{4\Delta E^2}-\frac{13}{6}+\tfrac{1}{3}f(\sigma)+\sigma\ln\frac{\sigma+1}{\sigma-1}-2\right)-\frac{17}{18}-\frac{\pi^2}{6}+f(\sigma)\bigg], \quad (5.18)$$

$$f_2(x)=\frac{\alpha}{\pi}\left(\ln\frac{q^2}{m_e^2}-4\right)f(\sigma), \quad (5.19)$$

where

$$f(\sigma)=1-\tfrac{3}{2}\sigma^2+\tfrac{3}{4}(\sigma^2-1)\sigma\ln\frac{\sigma+1}{\sigma-1}. \quad (5.20)$$

Note that as $x\to 1$, $\sigma\to\infty$, $f(\sigma)\to -1/5\sigma^2$; also note that $f_2(x)$ does not have a term in $\ln(q^2/4\Delta E^2)$ and the coefficient $\ln(q^2/m_e^2)-4$ is of order unity, so that $f_2(x)$ is numerically much smaller than $f_1(x)$.

VI. DECAY $\pi^0\to e^+e^-\gamma$

In the case of π^0 decay, diagrams (4) and (7) in Fig. 2 are identically zero. Diagram (3) is a radiative correction to the form factor and must therefore be of order (α/π) times the π^0 form factor. It will therefore not influence the determination of the *slope* of the form factor. Hence the formulas found above are applicable to the determination of the slope. Using $M_L=0$, we now have

$$\delta(x,y)=\frac{\alpha}{\pi}\bigg\{-\left(2\ln\frac{m_\pi}{m_e}-1+\ln x\right)\times\bigg[2\ln\frac{m_\pi}{2\Delta E}-\frac{13}{6}+\ln\left(\frac{(1+x)^2-y^2(1-x)^2}{4}\right)-\frac{17}{18}-\tfrac{1}{2}\ln^2\left(\frac{1+x+y(1-x)}{1+x-y(1-x)}\right)\bigg]\bigg\},$$

$$\delta(x)=\frac{\alpha}{\pi}\bigg[-\left(2\ln\frac{m_\pi}{m_e}-1+\ln x\right)\times\left(2\ln\frac{m_\pi}{2\Delta E}-\frac{13}{6}+\tfrac{1}{3}g(x)\right)-\frac{17}{18}-\frac{\pi^2}{6}+g(x)\bigg],$$

$$g(x)=-\frac{1}{(1-x)^2}\left[\tfrac{1}{2}+5x+\tfrac{1}{2}x^2+3x\frac{(1+x)}{(1-x)}\ln x\right].$$

In Fig. 4 we plot $\delta(x,y)$ for fixed values of y with $\Delta E=5$ and 10 MeV. In Fig. 5 we plot $\delta(x,y)$ for fixed values of x and $\Delta E=5$ and 10 MeV. Then we also plot $\delta(x)$ for various values of ΔE in Fig. 6. The figures show the following features: For constant x, $\delta(x,y)$ varies very

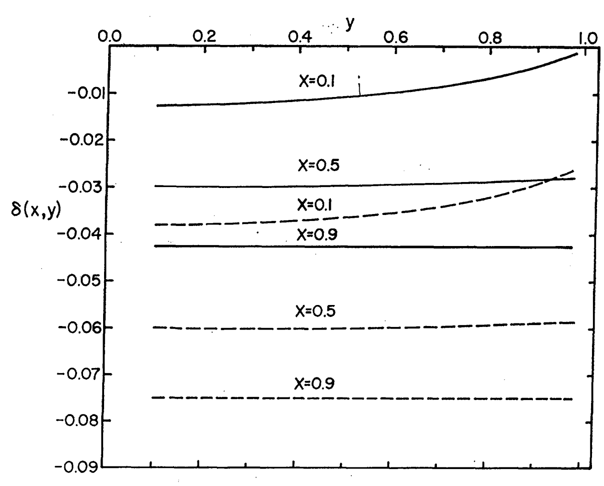

FIG. 5. $\delta(x,y)$ as a function of y for fixed values of x. The solid lines are for $\Delta E=10$ MeV and the dashed ones for $\Delta E=5$ MeV.

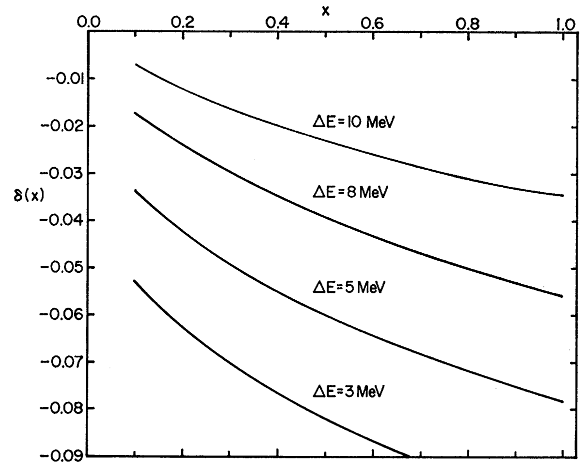

FIG. 6. $\delta(x)$ as a function of x for values of ΔE.

little with y but for constant y it changes by a factor of 4 for variation with x. The plot of $\delta(x)$ in Fig. 6 shows that the radiative corrections to the spectrum are *negative* and change appreciably as one varies the photon cutoff ΔE. The total correction is, as we have remarked before, positive, and this must be caused by two separate phenomena: first, that the small-x corrections, which we cannot calculate because of our approximations, are large and positive; second, that the hard-photon contributions, which are excluded experimentally, are significant. Both these extra contributions need to be calculated before one can integrate $\delta(x)$ to find the correction given in Eq. (3.10). As far as experiment is concerned the radiative corrections to the Dalitz plot and to $\Gamma(x)$ are large and negative in all of the region investigated. Hence the determination of a should probably yield positive results rather than the negative ones given previously, provided that a reliable ΔE can be estimated from the experimental setup.

APPENDIX

In this appendix we calculate the integral defined in Eq. (5.10),

$$\delta_{5,6}(x,y)=\int_{|\mathbf{k}|\leq\Delta E}\frac{d\mathbf{k}}{2\omega(2\pi)^3}\sum_{\text{pol}}|eb\cdot\epsilon|^2, \quad \text{(A1)}$$

where

$$\omega=(\mathbf{k}^2+\lambda^2)^{1/2}$$

and

$$b=\frac{q_1}{q_1\cdot k}-\frac{q_2}{q_2\cdot k}.$$

We define

$$\delta_{5,6}(x,y)=\frac{\alpha}{\pi}G(x,y),$$

where

$$G(x,y)=\frac{1}{4\pi}\int_{|\mathbf{k}|\leq\Delta E}\frac{d\mathbf{k}}{\omega}\left\{\frac{q^2-2m_e^2}{q_1\cdot kq_2\cdot k}-\frac{m_e^2}{(q_1\cdot k)^2}-\frac{m_e^2}{(q_2\cdot k)^2}\right\}. \quad \text{(A2)}$$

Defining

$$J(q_1,q_2)=\frac{1}{4\pi}\int_{|\mathbf{k}|\leq\Delta E}\frac{d\mathbf{k}}{\omega}\frac{1}{q_1\cdot kq_2\cdot k}, \quad \text{(A3)}$$

we obtain

$$G=J(q_1,q_2)(q^2-2m_e^2)-m_e^2[J(q_1,q_1)+J(q_2,q_2)]. \quad \text{(A4)}$$

Using well-known techniques for combination of denominators, we have

$$J(q_1,q_2)=\int_0^1 d\alpha\, K(q_1,q_2,\alpha), \quad \text{(A5)}$$

where

$$K(q_1,q_2,\alpha)=\frac{1}{4\pi}\int_0^{\Delta E}\frac{|\mathbf{k}|^2 d|\mathbf{k}|}{(\mathbf{k}^2+\lambda^2)^{1/2}}\times\int\frac{d\Omega}{\{[q_1\alpha+q_2(1-\alpha)]\cdot k\}^2}.$$

Putting

$$A=\alpha E_1+(1-\alpha)E_2$$

and (note that $A\geq B$)

$$B=|\alpha\mathbf{q}_1+(1-\alpha)\mathbf{q}_2|,$$

we have

$$(\alpha q_1+(1-\alpha)q_2)\cdot k=\omega A-|\mathbf{k}|Bz,$$

where z is the cosine of the angle between $\mathbf{k}$ and $\alpha\mathbf{q}_1+(1-\alpha)\mathbf{q}_2$. The angular integration is now trivial and using the integration variable ξ, where $|\mathbf{k}|=\xi\lambda$, we obtain

$$K(q_1,q_2,\alpha)=\int_0^{\Delta E/\lambda}\frac{\xi^2 d\xi}{(1+\xi^2)^{1/2}}\frac{1}{\xi^2(A^2-B^2)+A^2}. \quad \text{(A6)}$$

We now write

$$K=K_1+K_2, \quad \text{(A7)}$$

where

$$K_1=\frac{1}{A^2-B^2}\int_0^{\Delta E/\lambda}\frac{d\xi}{(1+\xi^2)^{1/2}} \quad \text{(A8)}$$

$$K_2=-\frac{A^2}{A^2-B^2}\int_0^{\Delta E/\lambda}\frac{d\xi}{(1+\xi^2)^{1/2}}\frac{1}{\xi^2(A^2-B^2)+A^2}. \quad \text{(A9)}$$

K_2 is finite in the limit of $\lambda\to 0$ so that we can replace the upper limit by infinity. The integration over ξ is straightforward with the result (for $\lambda\to 0$)

$$K_1=\frac{\ln(2\Delta E/\lambda)}{A^2-B^2},\tag{A10}$$

$$K_2=-\frac{1}{A^2-B^2}\frac{A}{2B}\ln\frac{A+B}{A-B}.\tag{A11}$$

It is now clear that G can be written as

$$G(x,y)=2G_1(x)\ln(2\Delta E/\lambda)+G_2(x,y),\tag{A12}$$

where, if

$$J_1(q_1,q_2)=\frac{1}{2}\int_0^1 d\alpha\frac{1}{A^2-B^2}\tag{A13}$$

and

$$J_2(q_1,q_2)=-\int_0^1 d\alpha\frac{1}{A^2-B^2}\frac{A}{2B}\ln\frac{A+B}{A-B},\tag{A14}$$

then G_1 and G_2 are related to J_1 and J_2 by Eq. (A4). For $q_1=q_2$, A and B are independent of x, and

$$A^2-B^2=m_e^2,\quad B/A=v,$$

the velocity of the electron. Hence

$$J_1(q_1,q_1)=1/2m_e^2,$$

$$J_2(q_1,q_1)=-\frac{1}{m_e^2}\frac{1}{2v}\ln\frac{1+v}{1-v}.$$

For $q_1\neq q_2$, we find by direct integration that

$$J_1(q_1,q_2)=\frac{1}{q^2}\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right),$$

where β is defined in Sec. IV. The integral $J_2(q_1,q_2)$ is vastly more complicated. Defining (x and ρ are used in Sec. IV)

$$x_2=\frac{1}{\rho+1}[(1-x)(\rho^2-x)]^{1/2}$$

and

$$x_1=(x+x_2^2)^{1/2},$$

we find

$$E_1+E_2=\Delta m x_1,$$
$$E_1-E_2=\Delta m x_2 y.$$

Hence

$$A=\tfrac{1}{2}\Delta m(x_1+x_2\xi y),$$

where $\xi=2\alpha-1$ is a new integration variable. Thus

$$A^2-B^2=\tfrac{1}{4}\Delta m^2 x(1-\beta^2\xi^2),$$

and, defining $\sigma=x_1/x_2>1$, we can solve for A and B, i.e.,

$$A=\tfrac{1}{2}\Delta m x_2(\sigma+\xi y),\tag{A15}$$

$$B=\tfrac{1}{2}\Delta m x_2\{1+2\xi y\sigma+[y^2+(\sigma^2-1)\beta^2]\xi^2\}^{1/2}.\tag{A16}$$

It is now evident from the form of Eq. (A13) that x_2 no longer appears explicitly in $J_2(q_1,q_2)$. Now we must change variable to remove the square root. Define the variable t by

$$\{1+2\xi y\sigma+[y^2+(\sigma^2-1)\beta^2]\xi^2\}^{1/2}=\xi t+1;$$

so solving for ξ we find

$$\xi=2(y\sigma-t)/(t^2-e^2),$$

where

$$e^2=y^2+(\sigma^2-1)\beta^2,$$

$$d\xi=2\frac{t^2+e^2-2y\sigma t}{(t^2-e^2)^2}dt.$$

The interval $-1\leq\xi\leq 1$ becomes $t_1\leq t\leq t_2$ with

$$t_1=1-[(\sigma-y)^2-(\sigma^2-1)(1-\beta^2)]^{1/2},\tag{A17}$$

$$t_2=[(\sigma+y)^2-(\sigma^2-1)(1-\beta^2)]^{1/2}-1.\tag{A18}$$

In terms of these variables,

$$A-B=\frac{\Delta m}{2}\frac{x_2}{t^2-e^2}(\sigma+1)[(t-y)^2-\beta^2(\sigma-1)^2],$$

$$A+B=\frac{\Delta m}{2}\frac{x_2}{t^2-e^2}(\sigma-1)[(t+y)^2-\beta^2(\sigma+1)^2],$$

and

$$B=\frac{\Delta m}{2}x_2\left(-\frac{d\xi}{dt}\right)\frac{t^2-e^2}{2}.$$

Thus we find

$$J_2=\frac{1}{q^2}\int_{t_1}^{t_2}dt\left[\frac{\sigma-1}{(t-y)^2-\beta^2(\sigma-1)^2}+\frac{\sigma+1}{(t+y)^2-\beta^2(\sigma+1)^2}\right]\times\ln\left[\frac{\sigma-1}{\sigma+1}\frac{(t+y)^2-\beta^2(\sigma+1)^2}{(t-y)^2-\beta^2(\sigma-1)^2}\right].\tag{A19}$$

Defining the denominators

$$N_1(t)=t+y+\beta(\sigma+1),$$
$$N_2(t)=t+y-\beta(\sigma+1),$$
$$N_3(t)=t-y+\beta(\sigma-1),$$
$$N_4(t)=t-y-\beta(\sigma-1),$$

then

$$J_2(q_1,q_2)=\frac{1}{q^2}\frac{1}{2\beta}\int_{t_1}^{t_2}dt\times\left[\frac{1}{N_4(t)}-\frac{1}{N_3(t)}+\frac{1}{N_2(t)}-\frac{1}{N_1(t)}\right]\times\ln\left[\frac{(\sigma-1)N_1(t)N_2(t)}{(\sigma+1)N_3(t)N_4(t)}\right].$$

We now put

$$J_2=\frac{1}{q^2}\frac{1}{2\beta}\left(M\ln\frac{\sigma-1}{\sigma+1}+L\right),$$

where

$$M=\int_{t_1}^{t_2} dt\left[\frac{1}{N_4(t)}-\frac{1}{N_3(t)}+\frac{1}{N_2(t)}-\frac{1}{N_1(t)}\right], \quad (A20)$$

$$L=\int_{t_1}^{t_2} dt\left[\frac{1}{N_4(t)}-\frac{1}{N_3(t)}+\frac{1}{N_2(t)}-\frac{1}{N_1(t)}\right] \times\ln\left[\frac{N_1(t)N_2(t)}{N_3(t)N_4(t)}\right]. \quad (A21)$$

Note that t_1 and t_2 are symmetric in β and if $\sigma\to-\sigma$ or $y\to-y$,

$$t_1\leftrightarrow -t_2.$$

This implies that as functions of β, σ, and y

$$M\to -M\,,\quad L\to -L$$

under

$$\beta\to-\beta\,,$$

and

$$M\to -M\,,\quad L\to +L$$

under

$$\sigma\to-\sigma\,,$$

and furthermore that

$$M\to M\,,\quad L\to L$$

under

$$y\to -y\,.$$

We will exploit these symmetry properties during the calculation to keep track of various terms.

It is possible to prove the following inequalities:

$$t_1\le t_2\,,$$
$$N_1(t)>0\,,\quad N_2(t)<0\,,$$
$$N_3(t)>0\,,\quad N_4(t)<0\,,$$

when x and y are in the interior of the physical region. This shows that the basic integral is well defined, as we would expect. We now define the basic integrals

$$M_i=\int_{t_1}^{t_2} dt\frac{1}{N_i(t)}=[\ln|N_i(t)|]_{t_1}^{t_2}, \quad (A22)$$

$$L_{ik}=\int_{t_1}^{t_2} dt\frac{1}{N_i(t)}\ln|N_k(t)|\,. \quad (A23)$$

We obviously have the symmetry relation

$$L_{ik}+L_{ki}=[\ln|N_i(t)|\ln|N_k(t)|]_{t_1}^{t_2} \quad (A24)$$

and

$$L_{ik}=[\ln|N_i(t)-N_k(t)|\ln N_k(t)]_{t_1}^{t_2} -\left[\mathrm{Li}_2\left(\frac{N_i(t)}{N_i(t)-N_k(t)}\right)\right]_{t_1}^{t_2}. \quad (A25)$$

Using these relations, we obtain

$$L=-\tfrac{1}{2}\left[\ln^2\left(\frac{N_1(t)}{N_4(t)}\right)\right]_{t_1}^{t_2}+\tfrac{1}{2}\left[\ln^2\left(\frac{N_3(t)}{N_4(t)}\right)\right]_{t_1}^{t_2}+L_{21}-L_{12} +L_{13}-L_{31}+L_{34}-L_{43}+L_{42}-L_{24}\,. \quad (A26)$$

We shall now show how to bring $L_{21}-L_{12}$ into a convenient form. If we omit the argument of the various N's where there is no ambiguity, then clearly

$$L_{21}-L_{12}=2L_{21}-[\ln|N_1|\ln|N_2|]_{t_1}^{t_2}$$
$$=[2\ln|N_1-N_2|\ln|N_2|-\ln|N_1|\ln|N_2|]_{t_1}^{t_2} -2\left[\mathrm{Li}_2\left(\frac{N_2}{N_2-N_1}\right)\right]_{t_1}^{t_2}.$$

In the last term we use the relation

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\mathrm{Li}_2(1)-\ln x\,\ln(1-x)\,,$$

so that

$$\mathrm{Li}_2\left(\frac{N_2}{N_2-N_1}\right)=-\mathrm{Li}_2\left(\frac{N_1}{N_1-N_2}\right)+\mathrm{Li}_2(1) -\ln\left(\frac{N_2}{N_2-N_1}\right)\ln\left(\frac{N_1}{N_1-N_2}\right).$$

Then we find

$$L_{12}-L_{21}=[2\ln|N_1-N_2|\ln|N_2|-\ln|N_1|\ln|N_2|]_{t_1}^{t_2} -2\,\mathrm{Li}_2\left(\frac{N_1(t_1)}{N_1(t_1)-N_2(t_1)}\right)-2\,\mathrm{Li}_2\left(\frac{N_2(t_2)}{N_2(t_1)-N_1(t_1)}\right) +2\,\mathrm{Li}_2(1)-2\ln\left|\frac{N_2(t_1)}{N_2(t_1)-N_1(t_1)}\right|\ln\left|\frac{N_1(t_1)}{N_1(t_1)-N_2(t_1)}\right|.$$

Remark that the dilogarithms explicitly exhibit the symmetry under $y\to-y$. In the same way we also obtain the other terms, using

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2(-1/x)=-\mathrm{Li}_2(1)-\tfrac{1}{2}\ln^2 x,$$

$$L_{13}-L_{31}=\left[\ln|N_1|\ln|N_3|-2\ln|N_1-N_3|\ln|N_3|\right]_{t_1}^{t_2}-2\,\mathrm{Li}_2\left(\frac{N_3(t_1)}{N_3(t_1)-N_1(t_1)}\right)-2\,\mathrm{Li}_2\left(\frac{N_3(t_2)-N_1(t_2)}{N_3(t_2)}\right)-2\,\mathrm{Li}_2(1)-\ln^2\left|\frac{N_3(t_2)}{N_3-N_1}\right|,$$

$$L_{34}-L_{43}=\left[2\ln|N_3-N_4|\ln|N_3|-\ln|N_3|\ln|N_4|\right]_{t_1}^{t_2}+2\,\mathrm{Li}_2\left(\frac{N_4(t_2)}{N_4(t_2)-N_3(t_2)}\right)+2\,\mathrm{Li}_2\left(\frac{N_3(t_1)}{N_3(t_1)-N_4(t_1)}\right)-2\,\mathrm{Li}_2(1)+2\ln\left|\frac{N_3(t_2)}{N_3(t_2)-N_4(t_2)}\right|\ln\left|\frac{N_4(t_2)}{N_4(t_1)-N_3(t_2)}\right|.$$

Hence we can write

$$L=L_1+L_2+L_3,$$

where

$$L_1=\Bigg[-\tfrac{1}{2}\ln^2\left|\frac{N_1}{N_4}\right|+\tfrac{1}{2}\ln^2\left|\frac{N_2}{N_3}\right|+2\ln|N_1-N_2|\ln|N_2|-\ln|N_1|\ln|N_2|-2\ln|N_1-N_3|\ln|N_3|+\ln|N_1|\ln|N_3|$$
$$+2\ln|N_2-N_4|\ln|N_4|-\ln|N_2|\ln|N_4|+2\ln|N_3-N_4|\ln|N_3|-\ln|N_3|\ln|N_4|\Bigg]_{t_1}^{t_2}$$
$$-2\ln\left|\frac{N_2(t_1)}{N_2(t_1)-N_1(t_1)}\right|\ln\left|\frac{N_1(t_1)}{N_1(t_1)-N_2(t_1)}\right|-\ln^2\left|\frac{N_3(t_2)}{N_3(t_2)-N_1(t_2)}\right|-\ln^2\left|\frac{N_4(t_1)}{N_4(t_1)-N_2(t_1)}\right|$$
$$+2\ln\left|\frac{N_3(t_2)}{N_3(t_2)-N_4(t_2)}\right|\ln\left|\frac{N_4(t_2)}{N_4(t_2)-N_3(t_2)}\right|,$$

$$L_2=-4\,\mathrm{Li}_2(1)-2\left[\mathrm{Li}_2\left(\frac{N_1(t_1)}{N_1(t_1)-N_2(t_1)}\right)+\mathrm{Li}_2\left(\frac{N_2(t_2)}{N_2(t_2)-N_1(t_2)}\right)+\mathrm{Li}_2\left(\frac{N_3(t_2)-N_1(t_2)}{N_3(t_2)}\right)+\mathrm{Li}_2\left(\frac{N_4(t_1)-N_2(t_1)}{N_4(t_1)}\right)\right],$$

$$L_3=-2\,\mathrm{Li}_2\left(\frac{N_3(t_1)}{N_3(t_1)-N_1(t_1)}\right)-2\,\mathrm{Li}_2\left(\frac{N_4(t_2)}{N_4(t_2)-N_2(t_2)}\right)+2\,\mathrm{Li}_2\left(\frac{N_3(t_1)}{N_3(t_1)-N_4(t_1)}\right)+2\,\mathrm{Li}_2\left(\frac{N_4(t_2)}{N_4(t_2)-N_3(t_2)}\right).$$

We also find

$$M=\left[\ln|N_4|-\ln|N_3|+\ln|N_2|-\ln|N_1|\right]_{t_1}^{t_2}.$$

Using

$$\frac{\sigma-1}{\sigma+1}=\frac{N_3-N_4}{N_1-N_2},$$

we get after a long calculation

$$M\ln\frac{\sigma-1}{\sigma+1}+L_1=-\tfrac{1}{2}\ln^2\left|\frac{N_1N_2}{N_4(N_1-N_2)}\right|-\tfrac{1}{2}\ln^2\left|\frac{N_3}{N_1-N_2}\right|+\ln\left|1-\frac{N_4}{N_3}\right|\ln\left|\frac{(N_3-N_4)N_2}{N_1N_4}\right|+\ln\left|1-\frac{N_4}{N_2}\right|\ln\left|\frac{N_4^2}{N_2(N_2-N_4)}\right|+(y\to -y),\quad\text{(A27)}$$

where all the N's have argument $t=t_2$. Now for $r\to 0$, we find (for $t=t_2$)

$$N_1=2(\sigma+y),\quad N_2=-2(1-y),$$
$$N_3=2(\sigma-1),\quad N_4=\frac{r^2}{2x}\frac{(\sigma-1)(1-y)}{\sigma+y},$$

so that we finally get in this approximation

$$M\ln\frac{\sigma-1}{\sigma+1}+L_1=-\tfrac{1}{2}\ln^2\left[\frac{4x}{r^2}\frac{(\sigma+y)^2}{(\sigma^2-1)}\right]-\tfrac{1}{2}\ln^2\left[\frac{\sigma-1}{\sigma+1}\right]-\tfrac{1}{2}\ln^2\left[\frac{4x}{r^2}\frac{(\sigma-y)^2}{(\sigma^2-1)}\right]-\tfrac{1}{2}\ln^2\left(\frac{\sigma-1}{\sigma+1}\right).$$

Also in this same approximation, we get

$$L_2=-4\,\mathrm{Li}_2(1)-2\left[\mathrm{Li}_2\left(\frac{1+y}{1+\sigma}\right)+\mathrm{Li}_2\left(\frac{1-y}{\sigma+1}\right)+\mathrm{Li}_2\left(\frac{1+y}{1-\sigma}\right)+\mathrm{Li}_2\left(\frac{1-y}{1-\sigma}\right)\right],$$

$L_3=0$.

Hence

$$J(q_1,q_2)=\frac{1}{q^2}\bigg\{-\tfrac{1}{4}\ln^2\bigg[\frac{4x}{r^2}\frac{(\sigma+y)^2}{(\sigma^2-1)}\bigg]-\tfrac{1}{4}\ln^2\bigg[\frac{4x}{r^2}\frac{(\sigma-y)^2}{(\sigma^2-1)}\bigg]-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma-1}{\sigma+1}\bigg)-\frac{\pi^2}{3}-\mathrm{Li}_2\bigg(\frac{1+y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1+y}{1-\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1-\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1+\sigma}\bigg)\bigg\},$$

which exposes the symmetries very clearly. Remark also that the integral is dominated by negative terms. All terms except the last two are negative. This expression is valid for $x\gg r^2$, in which case we also have $\sigma^2-1\gg r^2$. We now find the functions $G_1(x)$ and $G_2(x,y)$:

$$G_1(x)=\frac{1+\beta^2}{2\beta}\ln\frac{1+\beta}{1-\beta}-2,\tag{A28}$$

$$G_2(x,y)=-\tfrac{1}{4}\ln^2\bigg[\frac{4x}{r^2}\frac{(\sigma+y)^2}{(\sigma^2-1)}\bigg]-\tfrac{1}{4}\ln^2\bigg[\frac{4x}{r^2}\frac{(\sigma-y)^2}{(\sigma^2-1)}\bigg]-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma-1}{\sigma+1}\bigg)-\frac{\pi^2}{3}-\mathrm{Li}_2\bigg(\frac{1+y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1+y}{1-\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1-\sigma}\bigg)+\ln\bigg[\frac{4x}{r^2}\frac{(\sigma^2-y^2)}{(\sigma^2-1)}\bigg].$$

This expression can be reduced into an even more compact form by combining the logarithmic terms:

$$G_2(x,y)=\tfrac{1}{2}-\tfrac{1}{4}\bigg\{1-\ln\bigg[\frac{4x}{r^2}\frac{(\sigma+y)^2}{(\sigma^2-1)}\bigg]\bigg\}^2-\tfrac{1}{4}\bigg\{1-\ln\bigg[\frac{4x}{r^2}\frac{(\sigma-y)^2}{(\sigma^2-1)}\bigg]\bigg\}^2-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma-1}{\sigma+1}\bigg)-\frac{\pi^2}{3}-\mathrm{Li}_2\bigg(\frac{1+y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1+y}{1-\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1-\sigma}\bigg).$$

Introducing the variable $R=4x/r^2=q^2/m_e^2$, we obtain

$$G_2(x,y)=-\tfrac{1}{2}\ln^2R-\ln R\bigg(-1+\ln\frac{\sigma^2-y^2}{\sigma^2-1}\bigg)-\tfrac{1}{4}\ln^2\bigg[\frac{(\sigma+y)^2}{\sigma^2-1}\bigg]-\tfrac{1}{4}\ln^2\bigg[\frac{(\sigma-y)^2}{\sigma^2-1}\bigg]-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma-1}{\sigma+1}\bigg)-\frac{\pi^2}{3}+\ln\frac{\sigma^2-y^2}{\sigma^2-1}-\mathrm{Li}_2\bigg(\frac{1+y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1+\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1+y}{1-\sigma}\bigg)-\mathrm{Li}_2\bigg(\frac{1-y}{1-\sigma}\bigg).$$

By means of Abel's relation,[23] we get

$$\mathrm{Li}_2\bigg(\frac{1+y}{1+\sigma}\bigg)+\mathrm{Li}_2\bigg(\frac{1+y}{1-\sigma}\bigg)+\mathrm{Li}_2\bigg(\frac{1-y}{1+\sigma}\bigg)+\mathrm{Li}_2\bigg(\frac{1-y}{1-\sigma}\bigg)=-\mathrm{Li}_2\bigg(\frac{1-y^2}{\sigma^2-y^2}\bigg)-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma+y}{\sigma-y}\bigg)-\ln\frac{\sigma+y}{\sigma-1}\ln\frac{\sigma-y}{\sigma-1}-\ln\frac{\sigma+y}{\sigma+1}\ln\frac{\sigma-y}{\sigma+1},$$

so that we can finally reduce $G_2(x,y)$ to

$$G_2(x,y)=-\tfrac{1}{2}\ln^2R-(\ln R-1)\bigg[-1+\ln\bigg(\frac{\sigma^2-y^2}{\sigma^2-1}\bigg)\bigg]+1-\frac{\pi^2}{3}+\mathrm{Li}_2\bigg(\frac{1-y^2}{\sigma^2-y^2}\bigg)-\tfrac{1}{2}\ln^2\bigg(\frac{\sigma+y}{\sigma-y}\bigg).\tag{A29}$$

[23] See Ref. 20, p. 245.