Relativistic Eikonal Approximation*

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(Received 23 October 1970)

Our earlier claims in support of the eikonal approximation to generalized ladder graph amplitudes are withdrawn—for the case of scalar-scalar interactions. Justification of the eikonal formula is provided, however, for the more interesting situation where the exchanged objects are vector particles.

A RELATIVISTIC version of the well-known eikonal approximation introduced by Molière[1] for potential scattering has been discussed recently by a number of authors.[2] The relativistic eikonal formula is proposed as the high-energy limit of a model scattering amplitude obtained by summing over all Feynman graphs of the generalized ladder type. For scattering of two scalar "nucleons" which interact via exchange of scalar mesons, the eikonal formula reads

$$M^{\text{eik}}(s,t)=-2is\int d^2b \times e^{-i\mathbf{b}\cdot\Delta}\left[\exp\left(\frac{ig^2}{2s}\int\frac{d^2q}{(2\pi)^2}\frac{e^{-i\mathbf{q}\cdot\mathbf{b}}}{\mu^2+q^2}\right)-1\right], \quad (1)$$

where s is the square of the c.m. energy, $t=\Delta^2$ is the invariant momentum transfer, μ is the meson mass, and g is the coupling constant. For large values of the energy the eikonal amplitude is dominated by the term of order g^2 in its power-series expansion, i.e., by the first Born approximation. So the compact summation over orders does not represent a physically interesting achievement. Nevertheless, we may ask in what sense the eikonal formula is true for generalized ladder graphs with scalar-scalar interactions. In an earlier paper[3] we took as a measure of truth that the eikonal formula, in each order of g^2, should correctly reproduce the leading high-energy behavior for the sum over all ladder graphs in that order. Observe that in order $(g^2)^{n+1}$ the eikonal formula implies $M_{n+1}{}^{\text{eik}}\sim(g^2)^{n+1}/s^n$. Our claim had been that the eikonal formula, for scalar-scalar interactions, could indeed be justified in the above sense. We now assert that this claim is wrong. As will be described below, we had overlooked certain contributions proportional to $1/s^3$ which are present to *all* orders $(g^2)^{n+1}$, $n>3$.

The situation for the eikonal method is nevertheless not so desperate as it may seem. As was said, the scalar-scalar example for generalized ladder graphs is anyhow somewhat academic. Greater physical interest attaches to the case where the exchanged objects are vector particles. Here the eikonal conjecture leads to an expression similar to that of Eq. (1), but with g^2 replaced by an *effective* coupling constant proportional to s. That is, in each order the amplitude now grows linearly with s. So, according to the eikonal approximation for the case of vector-particle exchange, all orders are of comparable importance; and the compact summation over orders represents an important accomplishment. It will be argued here that, for this case of vector-particle exchange, the eikonal approximation *does* seem to reproduce correctly the leading high-energy behavior, order by order, for the sum over generalized ladder graphs. Roughly speaking, this comes about here because of the numerator terms which now appear in the Feynman integrals; these numerators have the effect of enhancing the contributions from those integration regions which the eikonal method presupposes to be the dominant regions.

Let us first see why the eikonal formula fails for the scalar-scalar situation. In determining the asymptotic behavior of a Feynman integral for large s and fixed t, one looks at the function f which multiplies s in the denominator of the integral in its Feynman-parametric form. The function f is a multilinear polynomial in the Feynman parameters $x_1, x_2, \ldots,$ where x_i is the parameter associated with the ith internal line in the graph. For large s, the integral is dominated by those contributions which come from integration regions in the neighborhood of the surfaces $f=0$. As in our earlier paper, we shall *assume* here that only "endpoint" surfaces are important. These are surfaces of the form $x_{i_1}=x_{i_2}=\cdots=x_{i_N}=0$. Graphically, the corresponding set of lines $(i_1, i_2, \ldots, i_N)$ is said to form a "t-path." A t-path is a set of lines of the Feynman graph such that if they are short-circuited, the resulting graph describes an amplitude which depends only on the t variable, i.e., is independent of s. It is clear that the shorter the length of a t-path (by length we mean the number of lines in the path) the stronger is its asymptotic contribution. For the case of scalar-scalar interactions a t-path of length L makes an asymptotic contribution proportional to s^{-L}. If there are M t-paths of equal length L, their joint contribution is asymptotically proportional to $(\ln s)^{M-1}/s^L$. The detailed rules[4] involve certain qualifications, but it will not be necessary to state these here.

* Research sponsored by the U. S. Air Force Office of Scientific Research under Contract No. AF 49(638) 1545.

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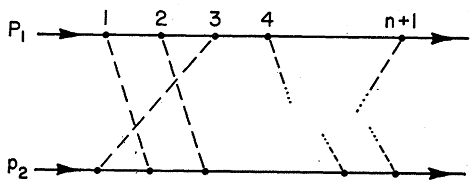

FIG. 1. Generalized ladder graph of order $(g^2)^{n+1}$.

Now in our earlier discussion of the generalized ladder graphs involving purely spinless particles, we focused only on the two t-paths represented by the "nucleon" lines [the upper and lower horizontal lines in the typical order $(g^2)^{n+1}$ graph depicted in Fig. 1]. Since each nucleon t-path has length n, the corresponding asymptotic behavior is $(\ln s)/s^n$, the next leading contribution from these t-paths behaving like s^{-n}. Actually, we found that the $(\ln s)/s^n$ terms cancelled in the sum over all ladder graphs of given order $(g^2)^{n+1}$ and that the s^{-n} terms summed up to give precisely the eikonal result. But for $n>3$, i.e., for orders g^8 and greater, there are other t-paths of comparable or greater importance, and we had overlooked these. For example, consider the particular eighth-order graph which is drawn in two different ways in Fig. 2. In addition to the pair of nucleon t-paths (1234) and (1'2'3'4'), each of length four, there is another pair of paths of comparable length: (12'3'4) and (1'234'). Each pair produces a comparable asymptotic contribution, proportional to $(\ln s)/s^3$. But only the first pair of t-paths is contemplated in the standard eikonal approximation. In general, the eikonal approximation focuses on the twin t-paths which run solely along the nucleon lines. For every generalized graph in order $(g^2)^{n+1}$, a nucleon t-path has length n. However, for $n=3$ we have seen that there are graphs for which other t-paths of comparable length can be found; and for $n>3$ one can always find generalized ladder graphs with t-paths of shorter length. Indeed, however large the order in g^2, there are graphs with t-paths of length $L=3$ [see Fig. 3, with t-path $(abcd)$]. The amplitude corresponding to such a graph behaves asymptotically like s^{-3}, independent of the order of the graph (for $n>3$). We conclude that the eikonal approximation makes no sense for generalized ladder graphs with scalar-scalar interactions, unless a truly miraculous cancellation occurs.[5]

Let us now see how these "non-eikonal" contributions are suppressed in the situation where *vector* particles are exchanged between spin-$\frac{1}{2}$ particles (e.g., electron-electron scattering, with exchange of massive photons). Similar arguments can be given for the scattering of scalar particles, with exchange of vector particles. The basic interaction is taken to have the form $g\bar{\psi}\gamma_\mu\psi A_\mu$; and for simplicity we specialize to the case of forward scattering: $p_1+p_2\to p_1+p_2$, $s=-(p_1+p_2)^2$. Let us simplify further by focusing on the spin-averaged amplitude (it will be clear from the following discussion that the dominant amplitude is helicity-nonflip, helicity-conserving). The Feynman integral associated with any ladder graph of order $(g^2)^{n+1}$ will differ from that for the corresponding graph with all scalar particles only through the appearance of a numerator factor

$$\mathrm{Tr}[(-i\gamma\cdot p_1+m)\gamma_{\mu_1}N_1'\gamma_{\mu_2}N_2'\cdots N_n'\gamma_{\mu_{n+1}}] \times\mathrm{Tr}[(-i\gamma\cdot p_2+m)\gamma_{\nu_1}\bar{N}_1'\cdots\bar{N}_n'\gamma_{\nu_{n+1}}], \quad (2)$$

where the propagator numerators N_i' and $\bar{N}_i'$ are linear in $\gamma\cdot p_1$ and $\gamma\cdot p_2$ and linear in $\gamma\cdot q_1'$, $\gamma\cdot q_2'$, $\ldots$, $\gamma\cdot q_n'$ (q_1', q_2', $\ldots$, q_n' being the integration momenta). In the above expression an appropriate pair-wise contraction on the indices μ_i and ν_j is to be understood. One now conbimes the propagator denominators by means of the Feynman identity and diagonalizes the quadratic form of the denominator by a linear change of integration variables:

$$q_i'=\sum_j R_{ij}q_j+\omega_i p_1+v_i p_2, \quad (2')$$

where the coefficients R_{ij}, ω_i, and v_i are functions of the Feynman parameters x_i, x_2, $\ldots$, x_{3n+1}. Under this change of variables the numerator terms N_i' and $\bar{N}_i'$ are replaced by functions N_i and $\bar{N}_i$ which depend on the Feynman parameters and are linear now in the new integration momenta q_1, q_2, $\ldots$, q_n and linear in p_1 and p_2. The amplitude has the form

$$\int dx_1\cdots dx_{3n+1}\int d^4q_1\cdots d^4q_n\frac{\mathrm{Tr}[(-i\gamma\cdot p_1+m)\gamma_{\mu_1}N_1\cdots]\,\mathrm{Tr}[(-i\gamma\cdot p_2+m)\gamma_{\nu_1}\bar{N}_1\cdots]}{(fs+h+\sum_{i=1}^{n}A_i q_i^2)^{3n+1}}, \quad (3)$$

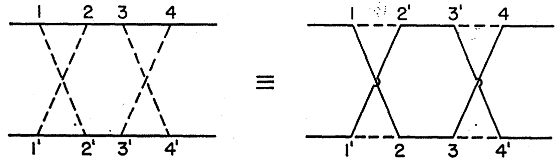

FIG. 2. Eighth-order graph, illustrating equivalence of different t-paths.

where f, h, and A_i all depend on the Feynman parameters x_i. Since the denominator depends only on the squares q_1^2, q_2^2, $\ldots$, the numerator quantities can be

[5] The moral here is that intuition associated with the eikonal approximation in nonrelativistic scattering (where short-wave-length propagation in a smooth instantaneous potential field can be described by rays) may fail in relativistic situations, where retardation effects are an essential feature.

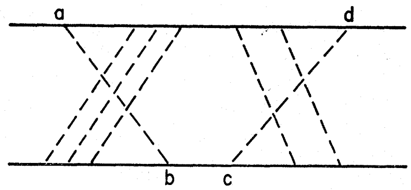

FIG. 3. Graph dominated by a "non-eikonal" t-path.

reduced according to the pattern

$$(p_1\cdot q_1)(p_2\cdot q_2)(q_1\cdot q_2) \to \tfrac{1}{16}(p_1\cdot p_2)q_1^2 q_2^2,$$

etc. Using $p_1\cdot p_2 \sim s$, we may write the numerator as a sum of terms, according to

$$\text{Numerator} = \sum_{k,\alpha_i,\beta_{ij}} s^k C_{k,\{\alpha_i\},\{\beta_{ij}\}} \Big(\prod_i (q_i^2)^{\alpha_i}\Big) \times \Big(\prod_{i<j} (q_i\cdot q_j)^{2\beta_{ij}}\Big), \quad (4)$$

where α_i and β_{ij} are integers and $k+\sum \alpha_i + \sum_{i<j} 2\beta_{ij} = n+1$. In the corresponding problem with spinless particles the numerator is simply $(A_1 A_2 \cdots A_n)^{n-1}$.

Owing to the presence of numerator factors in Eq. (3), it is no longer true, necessarily, that the asymptotic behavior of the amplitude is governed by the length of the shortest t-path. Suppose we scale a parameter λ out of the Feynman variables x_i associated with a particular t-path of length L, so that f is proportional to λ in the limit $\lambda \to 0$. For a particular term in Eq. (4) it may happen that $C_{k,\{\alpha_i\},\{\beta_{ij}\}}$ vanishes like λ^σ. This term will then make a contribution proportional to $s^{k-\sigma-L}$. In order to find the dominant behavior one has therefore to survey all possible t-paths.

We begin by considering the asymptotic contribution coming from the two "eikonal" t-paths, the paths formed exclusively from the two nucleon lines. In the original momentum space integration let q_i' be the momentum carried off by the ith meson in the succession of mesons emitted by nucleon 1. The momentum p_1 is therefore being routed exclusively along the line of nucleon 1, the momentum p_2 along the line of nucleon 2. With this assignment of integration momenta, each N_i' in Eq. (2) contains a $\gamma\cdot p_1$ term, but is independent of p_2; each $\bar{N}_1'$ contains a $\gamma\cdot p_2$ term but is independent of p_1. Let $x_1, x_2, \cdots, x_n$ be the Feynman parameters associated with the lines of nucleon 1. As usual, introduce a scaling parameter λ_1 according to

$$x_i = \lambda_1 x_i', \quad i=1,2,\ldots,n$$

$$\lambda_1 = \sum_{i=1}^{n} x_i .$$

The "λ_1-reduced" graph obtained by short-circuiting the t-path along the line of nucleon 1 does not depend on p_1. Therefore, for small λ_1 we have $f\sim\lambda_1$ and $w_i \sim \lambda_1$. The asymptotic contribution from this t-path is not affected if we set $\lambda_1 = 0$ everywhere in the integrand except in the factor f. Moreover, the numerator in Eq. (3) will be dominated asymptotically by a term proportional to s^{n+1} coming exclusively from the factors $\gamma\cdot p_1$ in all the N_i. In short, the asymptotic contribution from the t-path corresponding to $\lambda_1 \to 0$ can be obtained as follows: In the original Feynman integral set $N_i' \to i\gamma\cdot p_1$ (i.e., omit terms depending on the q_i'); and in fermion propagator denominators, omit terms quadratic in the q_i'. For the rest we can now proceed as in Ref. 3. The approximations described above permit us to sum in compact form over all generalized ladder graphs of the given order $(g^2)^{n+1}$. The resulting integral differs from that for the scalar-scalar case only by the presence of the numerator factor of Eq. (2), of course with $N_i' \to -i\gamma\cdot p_1$. In casting the integral into Feynman parametric form, we find, as in Ref. 3, that its leading behavior now comes from the t-path formed by the lines of nucleon 2. But this means that in the $\bar{N}_i$ we can ignore the part of the origin shift which is proportional to p_2. That is, the dominant term in the numerator of Eq. (2), the term proportional to s^{n+1}, is obtained by setting $\bar{N}_i' \to -i\gamma\cdot p_2$. Moreover, in the propagator denominators for fermion 2 we can drop terms quadratic in the q_i'. To summarize, the contribution to the asymptotic behavior which comes from the twin fermion t-paths is obtained, in the original Feynman integral, by dropping q_i' terms in the numerator and terms quadratic in the q_i' in all fermion propagators. But these are precisely the approximations adopted in the eikonal method and they lead to the eikonal formula of Eq. (1), with $g^2 \to g^2 s$: in order $(g^2)^{n+1}$, $M_{n+1}{}^{\text{eik}} \sim (g^2)^{n+1} s$.

To justify the eikonal formula for vector-particle exchange, we have now to show that all other t-paths make asymptotic contributions which grow less rapidly with s. In fact we will see that these other contributions are weaker by whole powers of s.

A non-eikonal t-path must contain at least one photon line. Consider the graph obtained by short circuiting the t-path to a point T. This reduced graph can now be characterized in the following way. Focus first on the fermion lines of the reduced graph. The four lines attaching to the external fermions all enter the point T. Two of these, in an evident sense, we may speak of as external p_1 lines, and the other two are external p_2 lines. In general there will also be fermion lines which form loops, one end of the line emerging from T, the other end returning to T. Let F_1 be the number of loops referring to fermion 1, F_2 the number referring to fermion 2. Next consider the photon lines of the reduced graph. Let P be the number of photon loops (ends at the point T). Let n_1 be the number of photon lines which run from T to one or another of the external p_1 lines, n_2 the number which run from T to one or another of the external p_2 lines. Denote by m_1 the number of photon lines running from an external p_1 line to a

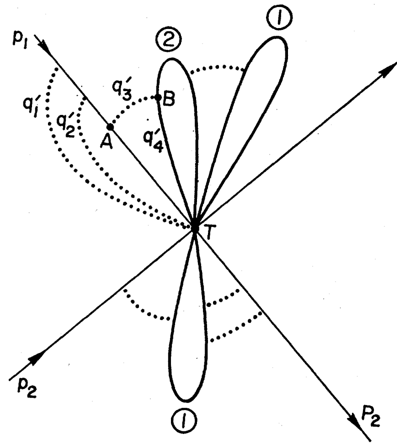

FIG. 4. Example of a reduced graph obtained by shrinking a "non-eikonal" t-path to a point T.

fermion loop (necessarily, a type-2 fermion loop), and denote by m_2 the number of photon lines running from an external p_2 line to a fermion loop. Let M_l be the number of photon lines which run between a fermion loop of type 1 and a loop of type 2, and let m_l be the number of photon lines which run from T to a fermion loop. Finally, let M_i be the number of photon lines which appear on the t-path in question (these lines are all shrunk into the point T in the reduced graph).

For a graph of order $(g^2)^{n+1}$ the length of an eikonal t-path is $L_{\text{eik}}=n$; and evidently

$$L_{\text{eik}}=n=n_1+n_2+m_1+m_2+M_l+m_l+M_i+P-1\,.$$

On the other hand, the number of lines which compose the t-path in question is

$$L_t=n_1+n_2+m_l+2M_i+2P-1\,.$$

Since the numerator factor in Eq. (4) never contributes a term which grows more rapidly than s^{n+1}, and since this limiting growth *is* in fact achieved for the eikonal path, we need never consider situations for which $L_t>L_{\text{eik}}$, i.e., we need focus only on the cases where $m_1+m_2+M_l>M_i+P$. For this purpose let us return to the original Feynman integral. We assign the integration momenta q_i' in such a way that the momenta p_1 and p_2 never appear in the lines which compose the fermion loops of the reduced graph, nor in any photon lines which appear in the reduced graph. Consider the reduced graph shown in Fig. 4. We think of it as being composed of two parts, separated by an imaginary horizontal dividing line drawn through the point T. This dividing line is not crossed by any lines of the reduced graph. The symbol 2 on a fermion loop signifies that it is composed of type-2 fermions; the symbol 1 on a loop has a corresponding meaning.

Now one of the photon lines contributing to the quantity m_1 is the line joining the points A and B. In the numerator expression of Eq. (2) the factors associated with the vertices A and B and with the immediately adjoining propagator factors is displayed in the following expression:

$$\begin{aligned}&\mathrm{Tr}\{(-i\gamma\cdot p_1+m)\cdots[-i\gamma\cdot(p_1-q_1'-q_2')+m]\\&\quad\times\gamma_\mu[-i\gamma\cdot(p_1-q_1'-q_2'-q_3')+m]\cdots\}\\&\quad\times\mathrm{Tr}\{(-i\gamma\cdot p_2+m)\cdots(-i\gamma\cdot q_4'+m)\\&\qquad\qquad\times\gamma_\mu[-i\gamma\cdot(q_3'+q_4')+m]\cdots\}\,.\end{aligned}\tag{5}$$

It is easy to see that for all q_i' appearing in the *upper* half of the reduced graph, the shift parameters v_i of Eq. (2′) will be proportional to the (small) scaling parameter λ associated with the t-path under discussion. Noting that $\lambda p_1\cdot p_2$ counts asymptotically as zeroth order in s, as does $p_1\cdot p_1=-m^2$, we see that the factors displayed in Eq. (5) in the square brackets count altogether as zeroth order in s. Similar remarks hold for factors associated with photon lines contributing to m_2 and M_l. The numerator as a whole, for the t-path in question, grows no faster therefore than $s^{n+1-m_1-m_2-M_l}$. The over-all contribution from the t-path under study is bounded by $s^{n+1-m_1-m_2-M_L-L_t}=s^{1-M_i-P}$. Recall that the eikonal contribution grows linearly with s. Since $M_i>1$ for any non-eikonal path, the contribution from such a path is clearly negligible. This completes our justification of the eikonal approximation for generalized ladder graphs with vector-particle exchange.

It must be recalled that we have been dealing here with the case of fermion-fermion scattering, where for simplicity we specialized to the case of forward scattering. A similar justification of the eikonal formula can be given for the scattering of charged, spinless bosons, interacting via exchange of vector particles. In the fermion case the forward amplitude has in general the c.m. structure

$$A+B\sigma_1\cdot\sigma_2+C\sigma_1\cdot\hat{p}\sigma_2\cdot\hat{p}\,,$$

where $\hat{p}$ is a unit vector along the collision axis. It is the spin-independent amplitude A that we have been discussing. It should be clear, on the basis of trivial variations on our analysis, that the A amplitude dominates over the others for large s. Finally, let us recall an important qualification on the present results: We have focused exclusively on the "endpoint" surfaces $f=0$ [see Eq. (3)] in discussing the asymptotic behavior of our graphs. We are unable to rule out the possibility that there are comparable or more important contributions arising from other surfaces $f=0$.