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Torsion-Free World Lines in Curved Space-Time*

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A torsion-free world line is shown to correspond to a particle whose three-acceleration vector points in a fixed direction relative to its Fermi-transported local inertial rest frame. In flat space-time this is equivalent to the three-acceleration vector pointing in a constant direction.

SOME time ago one of us proposed a definition of what might be meant by the motion with uniform acceleration of a test particle in curved space-time.[1] That definition, which was arrived at by generalizing the geometric properties of such motion in flat space-time ("hyperbolic" motion), consisted in requiring the world line of the particle to be torsion-free and of constant curvature. More recently, Gautreau[2] has shown—in essence—that this definition is equivalent to requiring the particle to have a constant three-acceleration vector relative to its Fermi-transported local Minkowski rest frame. In physical terms, these frames are a sequence of local inertial frames positioned along the particle's world line, such that the particle finds itself momentarily at rest always in one of them, and their base vectors are related by Fermi transport, i.e., without rotation.

We now ask ourselves what is the necessary and sufficient kinematic condition for a particle to have a torsion-free world line *without* the extra requirement of constant curvature. The condition turns out to be the following: (A) The three-acceleration vector has a constant direction relative to the Fermi-transported local inertial rest frame. In flat space-time this is equivalent to the following condition: (B) The three-acceleration vector has a constant direction relative to any inertial frame. (It is perhaps in itself a reasonably interesting result that the constancy of the direction of the three-acceleration is a Lorentz-invariant property.)

The condition for a world line to be torsion-free is[3]

$$\frac{D}{d\tau}\left(\frac{A^\mu}{\alpha}\right)=\alpha U^\mu, \tag{1}$$

where τ=proper time, $D/d\tau$ stands for absolute differentiation, $A^\mu=DU^\mu/d\tau$=four-acceleration, $U^\mu=dx^\mu/d\tau$=four-velocity, and $\alpha=(-g_{\mu\nu}A^\mu A^\nu)^{1/2}$=proper acceleration; our metric has signature $(---+)$ and our units are chosen so that c=speed of light=1. (These conventions agree with those of Ref. 1.) It was shown in Ref. 1 that only two of the four equations (1) are independent. Equation (1) is trivially satisfied in the limit by $A^\mu\equiv 0$ (geodetic motion), as can be seen by performing the differentiation and multiplying by α^2. For later reference we note[4] that in any local Minkowski frame

$$U^\mu=\gamma(\mathbf{u},1)\,,\quad \gamma^{-2}=1-u^2\,, \tag{2}$$

and consequently

$$A^\mu=\gamma(\dot\gamma\mathbf{u}+\gamma\mathbf{a},\,\dot\gamma)\,, \tag{3}$$

where $\dot\gamma=d\gamma/dt$, $\mathbf{u}$=three-velocity, and $\mathbf{a}$=three-acceleration.

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[1] W. Rindler, Phys. Rev. **119**, 2082 (1960). [A minor correction should be made to that paper: Of the two conditions there labeled (16), the second is in fact a consequence of the first and should be omitted. For (14) implies (16) (i); conversely, if (16) (i) is multiplied by U_μ one finds, using the parenthetical remark following (15), that $\alpha^2=-A_\mu A^\mu$; differentiating this expression absolutely with respect to τ, substituting from (16) (i), and using $A_\mu U^\mu=0$, one deduces α^2=const.]

[2] R. Gautreau, Phys. Rev. **185**, 1662 (1969).

[3] Reference 1, Eq. (14) (i).

[4] See, for example, W. Rindler, *Special Relativity* (Interscience, New York, 1966), Eqs. (4.14) and (4.15).

To interpret Eq. (1), we now write $A^\mu=\alpha V^\mu$ (assuming $\alpha\neq 0$), where V^μ is a unit spacelike vector, which must be orthogonal to U^μ since A^μ is so. Equation (1) implies that

$$\frac{DV^\mu}{d\tau}=\alpha U^\mu=-\alpha^{-1}A_\nu A^\nu U^\mu=-V_\nu A^\nu U^\mu. \qquad (4)$$

But this is precisely the condition for the Fermi transport of V^μ along the particle's world line.[5] Relative to an orthonormal Fermi-transported tetrad of which V^μ forms the first base vector and U^μ the fourth, A^μ thus has only a first component, α. Relative to the corresponding local inertial frame, $\mathbf{a}$ always points in the x direction, for, in virtue of Eq. (3), A^μ reduces to $(\mathbf{a},0)$ in any local Minkowski *rest* frame. The necessity of our condition (A) is therefore established. Its sufficiency also follows from Eqs. (4), read in reverse order; for if $\mathbf{a}$ has constant direction in the Fermi-transported local Minkowski rest frame, the unit direction V^μ of A^μ is Fermi transported, whence

$$\frac{DV^\mu}{d\tau}=-V_\nu A^\nu U^\mu=-\alpha^{-1}A_\nu A^\nu U^\mu=\alpha U^\mu,$$

and Eq. (1) is satisfied.

It is evident that, when augmented by the requirement $\alpha=$constant, condition (A) amounts to the *constancy* of the three-acceleration relative to the Fermi-transported local rest frame. This recovers Gautreau's above-mentioned result.

Simple examples of torsion-free motion in curved space-times—apart from the trivial case of geodetic motion ($A^\mu\equiv 0$)—are provided by arbitrary radial motions in all spherically symmetric metrics of the form

$$ds^2=A\,dt^2-B\,dr^2-Cr^2(d\theta^2+\sin^2\theta d\phi^2),$$

where A, B, and C are functions of t and r only. Schwarzschild space and the Friedmann cosmological models are cases in point.

The special case of flat space-time is most easily discussed *ab initio*. For this purpose we shall use the fact that Eq. (1) possesses a general solution of the form

$$U^\mu=L^\mu\cosh\theta+M^\mu\sinh\theta, \qquad (5)$$

where

$$\theta=\int\alpha d\tau \qquad (6)$$

and L^μ and M^μ are parallelly propagated unit vectors, timelike and spacelike, respectively, and orthogonal to each other. This is easily seen by introducing the variable $\theta=\int\alpha d\tau$ into Eq. (1), whereupon the latter reduces to $D^2U^\mu/d\theta^2=U^\mu$; and this evidently has a solution of the form (5). The metric requirements on L^μ and M^μ become evident on setting $\theta=0$ in Eq. (5) and in the equation derived from it by the operation $D/d\theta$. We also note that the functional form (6) of θ is implicit in Eq. (5): It can be obtained by absolutely differentiating Eq. (5) with respect to τ in order to get A^μ, and then calculating $\alpha^2=-A_\mu A^\mu$.

[5] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), p. 15.

Let us now specialize to flat Minkowski space-time (x,y,z,t). Then L^μ and M^μ will simply be constant. From Eqs. (2) and (3), we have

$$A^\mu-\dot\gamma U^\mu=\gamma^2(\mathbf{a},0). \qquad (7)$$

Thus *if* the motion is torsion-free, substitution from (5) into (7) yields

$$(\mathbf{a},0)=\phi L^\mu+\psi M^\mu \qquad (8)$$

for *some* functions ϕ, ψ of τ. Consequently, $\phi L^4+\psi M^4=0$; $L^4\neq 0$ since L^μ is timelike, and thus $\phi=-\psi(M^4/L^4)$. Substituting this back into Eq. (8), we find

$$a^i=\psi\left(M^i-\frac{M^4}{L^4}L^i\right),\quad i=1,2,3$$

i.e., $\mathbf{a}$ is a multiple of a constant three-vector, as asserted in (B).

To prove the converse, let us assume

$$\mathbf{a}=\psi\mathbf{b},\quad [\mathbf{b}=\text{constant and unit},\ \psi=\psi(t)].$$

Integration then yields

$$\mathbf{u}=\int\psi dt\,\mathbf{b}+\mathbf{d},\quad (\mathbf{d}=\text{constant})$$

$$=\chi\mathbf{b}+\mathbf{p},\qquad (\mathbf{p}\cdot\mathbf{b}=0), \qquad (9)$$

where $\chi=\int\psi dt+\mathbf{d}\cdot\mathbf{b}$ and $\mathbf{p}=\mathbf{d}-(\mathbf{d}\cdot\mathbf{b})\mathbf{b}$. Consequently,

$$\gamma^{-2}=1-u^2=1-\chi^2-p^2. \qquad (10)$$

We now define

$$L^\mu=(1-p^2)^{-1/2}(\mathbf{p},1),\quad M^\mu=(\mathbf{b},0),$$

so that $L_\mu L^\mu=1$, $M_\mu M^\mu=-1$, and $L_\mu M^\mu=0$. Then, using Eq. (9) in the second step, we have

$$U^\mu=\gamma(\mathbf{u},1)=\gamma(\chi\mathbf{b}+\mathbf{p},\ 1)=\gamma(1-p^2)^{1/2}L^\mu+\gamma\chi M^\mu.$$

Because of Eq. (10) we may put $\gamma\chi=\sinh\theta$, $\gamma(1-p^2)^{1/2}=\cosh\theta$. The motion thus satisfies Eq. (5), and consequently also Eq. (1); in other words, it is torsion-free, as we wished to prove.

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