canonical energy-momentum tensor which only approaches the canonical energy-momentum tensor of the massless free fields instead of $\theta_{\mathrm{free}}{}^{\mu\nu}$ in the limit $f \rightarrow 0$.

To conclude, we have shown that axial-vector current can not have scaling dimension 3 in the presence of the nonlinear realization of chiral symmetry for the conventional models. In other words, if we require the axial-vector current to have scaling dimension 3, then it becomes non-Hermitian. The conflict between scale invariance and chiral symmetry is further reflected by the fact that in no canonical frame can one find a chiral-invariant new energy-momentum tensor. Furthermore, we have also shown that it is impossible to construct a new chiral-invariant tensor for the conventional nonlinear models such that it will approach the new energy-momentum tensor in the free-field limit. The scale invariance is thus essentially broken in the conventional nonlinear Lagrangian models of chiral symmetry for a set of pseudoscalar fields, unless the models are drastically modified.

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Veneziano Model for the Process $V_1+\pi \rightarrow V_2+\pi$*†

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A Veneziano model is presented for the scattering of pions on isoscalar vector mesons of arbitrary mass, including photons. In addition to the leading term of the amplitude, we allow for satellite terms. The first ones are determined by the requirements of the absence of daughters and the correct widths for the first resonances in all channels.

I. INTRODUCTION

In this work we build a Veneziano-type amplitude[1] for the process $\pi+V_1 \rightarrow \pi+V_2$ (V_i is an isosinglet, $C=-1$, vector meson). The main feature distinguishing this work from previous papers[2] is that particular emphasis is given to the structure of the amplitude in the lowest resonance region.

Veneziano-type amplitudes enjoy the benefit of crossing symmetry and correct asymptotic behavior, displayed by the leading term. Their main defect is generally the lack of unitarity.[3] On the other hand, if one wants to describe low-energy scattering, the best-known way to do it is by means of dispersion relations. There we run into the problem of subtraction constants. In order to match these two approaches, the validity of the Veneziano amplitude should be ensured at lower energies. In doing so one has to resort to additional criteria[4] in order to determine the next-to-leading terms. The criteria used by us promise that the Veneziano-type amplitudes will be free of daughter trajectories and we fulfill this requirement, at least in the region of lowest resonances. The amplitude derived here will be matched with a dispersion-relation approach in a theoretical study of the decay $\omega \rightarrow 2\pi+\gamma$.[5]

The plan of this article runs as follows. In Sec. II we define invariant and parity-conserving helicity amplitudes. In Sec. III the asymptotic behavior of the amplitudes is given. In Sec. IV we compute the contributions of the ρ and f_0 exchanges. In Sec. V we build the Veneziano amplitudes, and in Sec. VI the main features of our amplitude are discussed, with particular emphasis on the reasoning that leads us to eliminate daughter trajectories.

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[1] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2] A. Capella, B. Diu, J. M. Kaplan, and D. Schiff, Nuovo Cimento Letters 1, 655 (1969); P. Carruthers and E. Lasley, Phys. Rev. D 1, 1204 (1970); in contrast to these works our model does not contain the $\epsilon(0^+)$ in the s channel. For $\pi=\rho$ scattering see, e.g., E. S. Abers and V. L. Teplitz, Phys. Rev. Letters 22, 909 (1969); Phys. Rev. D 1, 624 (1970).

[3] For attempts of unitarizing Veneziano amplitudes, see C. Lovelace, *Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions* (Argonne National Laboratory, Argonne, Ill., 1969), p. 562.

[4] A Veneziano-type amplitude contains *a priori* an infinite number of satellites whose coefficients are usually determined by additional requirements. In some cases, all satellite coefficients may vanish. This may happen for special values of ϵ, $\epsilon=\alpha(s)+\alpha(t)+\alpha(u)$, such as $\epsilon=2$ in the original work of Veneziano (Ref. 1) or $\epsilon=3$ in our amplitude given in (5.1) and (5.2), while in C. Lovelace's work [Phys. Letters 28B, 264 (1968)], the requirement of Adler's consistency conditions on π-π and π-K amplitudes eliminates the satellite terms.

[5] N. Levy and P. Singer, Phys. Rev. D 3 (to be published).

II. INVARIANT AMPLITUDES AND PARITY-CONSERVING HELICITY AMPLITUDES

The invariant amplitudes for the process $V_1+\pi_i \to V_2+\pi_j$ (see Fig. 1) are defined through the relations

$$S=I-\frac{(2\pi)^4 i\delta^4(p_f-p_i)}{(16E_1E_2E_3E_4)^{1/2}}T, \tag{2.1}$$

$$T_{\lambda_1\lambda_2}=\epsilon_{\lambda_1}{}^{\alpha}(k)M_{\alpha\beta}\epsilon_{\lambda_2}{}^{\beta}(p), \tag{2.2}$$

where k, $\epsilon_{\lambda_1}(k)$ and p, $\epsilon_{\lambda_2}(p)$ are the momenta and polarization vectors for V_1 and V_2, respectively. The general form of $M_{\alpha\beta}$, invariant under C, P, and T transformations, can be expressed in terms of five invariant amplitudes:

$$M_{\alpha\beta}=A(s,t)p_\alpha k_\beta+B(s,t)g_{\alpha\beta}+C(s,t)P_\alpha\Delta_\beta +D(s,t)\Delta_\alpha k_\beta+E(s,t)\Delta_\alpha\Delta_\beta, \tag{2.3}$$

with

$$\Delta_\mu=(q_1-q_2)_\mu, \quad s=(p+k)^2, \quad t=(p+q_2)^2. \tag{2.4}$$

Parity conservation and charge conjugation imply that A,B,E are symmetric, and C,D antisymmetric under the exchange $t\to u$.

There are two particular cases of special interest. (1) $V_1=V_2$, when there is one relation between the five amplitudes:

$$C=-D. \tag{2.5}$$

(2) One of the vectors is an isoscalar photon, $V_1=\gamma$, and there are then two relations which guarantee gauge invariance:

$$E=-C(s-m_2{}^2)/(t-u), \tag{2.6a}$$

$$B=-\tfrac{1}{2}[(s-m_2{}^2)A+(t-u)D]. \tag{2.6b}$$

We now introduce parity-conserving helicity amplitudes $T_{\lambda\mu}{}^{\pm}$ defined by Gell-Mann *et al.*[6] which are especially convenient in analyzing asymptotic behavior. For the process $a+b\to c+d$, one has

$$T_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}{}^{\pm}=(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}T_{\lambda_c,\lambda_d\lambda;a,\lambda_b} \pm(-1)^{\lambda+\lambda_m}\eta_c\eta_d(-1)^{s_c+s_d}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda+\mu|} \times(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda-\mu|}T_{-\lambda_c,-\lambda_d;\lambda_a,\lambda_b}, \tag{2.7}$$

where

$$\lambda=\lambda_a-\lambda_b, \quad \mu=\lambda_c-\lambda_d, \quad \lambda_m=\max(|\lambda|,|\mu|),$$

and η_c,η_d are the intrinsic parities.

In the s channel, for the spinless pions $\lambda_a=\lambda_b=0$, we henceforth denote the amplitudes by the vector-meson helicities only, obtaining

$$T_{1,-1}{}^{s+}=\frac{T_{1,-1}{}^{s}+T_{-1,1}{}^{s}}{\sin^2\theta_s}=4q^2E, \tag{2.8a}$$

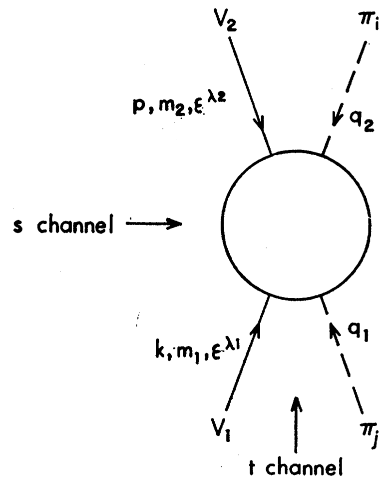

FIG. 1. Definition of kinematics.

$$T_{1,0}{}^{s+}=\frac{T_{1,0}{}^{s}-T_{-1,0}{}^{s}}{\sin\theta_s} =\frac{4q}{\sqrt{2}m_2}[k(\sqrt{s})D-2qp_0\cos\theta_s E], \tag{2.8b}$$

$$T_{0,1}{}^{s+}=\frac{T_{0,1}{}^{s}-T_{0,-1}{}^{s}}{\sin\theta_s} =\frac{4q}{\sqrt{2}m_1}[-k(\sqrt{s})C-2qk_0\cos\theta_s E], \tag{2.8c}$$

$$T_{1,1}{}^{s+}=T_{1,1}{}^{s}+T_{-1,-1}{}^{s}=2B-4q^2\sin^2\theta_s E, \tag{2.8d}$$

$$T_{0,0}{}^{s+}=2T_{0,0}{}^{s}=\frac{2}{m_1m_2}[k^2sA+\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)B +2qk(\sqrt{s})\cos\theta_s(k_0D-p_0C) -k_0p_0(2q)^2\cos^2\theta_s E], \tag{2.8e}$$

$$q=\frac{(s-4\mu^2)^{1/2}}{2},$$

$$k=\frac{\{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{1/2}}{2\sqrt{s}}, \tag{2.9}$$

$$k_0=\frac{s+m_1{}^2-m_2{}^2}{2\sqrt{s}}, \quad p_0=\frac{s+m_2{}^2-m_1{}^2}{2\sqrt{s}},$$

where q and k are the c.m. momenta of pions and vector mesons, respectively. All the $T_{\lambda_1,\lambda_2}{}^{s-}$ amplitudes are zero and $T_{\lambda_1,\lambda_2}{}^{s+}$ induce transitions only between natural-parity states; therefore, these are the only transitions that survive in the s channel.

In the t channel there is one amplitude $T_{1,1}{}^{t-}$ inducing natural-parity transitions and four amplitudes inducing

[6] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

unnatural-parity ones. Here, the pion helicities are $\lambda_b=\lambda_d=0$, and we denote the amplitudes by $T_{\lambda_c;\lambda_a}$.

$$T_{1;1}{}^{t-}=\frac{T_{1;1}{}^{t}}{1+\cos\theta_t}+\frac{T_{1;-1}{}^{t}}{1-\cos\theta_t}=(A-C+D-E)q_1q_2\,, \quad (2.10a)$$

$$T_{1;1}{}^{t+}=\frac{T_{1;1}{}^{t}}{1+\cos\theta_t}-\frac{T_{1;-1}{}^{t}}{1-\cos\theta_t}=-(A-C+D-E)q_1q_2\cos\theta_t-B\,, \quad (2.10b)$$

$$T_{1;0}{}^{t+}=\frac{T_{1;0}{}^{t}-T_{-1;0}{}^{t}}{\sin\theta_t}=\frac{\sqrt{2}}{m_1}[q_1(q_1p_0{}^t-q_2k_0{}^t\cos\theta_t)(A-C+D-E)+2q_1{}^2(\surd t)(E-D)-k_0{}^tB]\,, \quad (2.10c)$$

$$T_{0;1}{}^{t+}=\frac{T_{0;1}{}^{t}-T_{0;-1}{}^{t}}{\sin\theta_t}=\frac{\sqrt{2}}{m_2}[q_2(-k_0{}^tq_2+p_0{}^tq_1\cos\theta_t)(A-C+D-E)-2q_2{}^2(\surd t)(E+C)+p_0{}^tB]\,, \quad (2.10d)$$

$$\begin{aligned}T_{0;0}{}^{t+}=2T_{0;0}{}^{t}=\frac{2}{m_1m_2}[&(p_0{}^tq_1\cos\theta_t-q_2k_0{}^t)\\&\times(q_1p_0{}^t-q_2k_0{}^t\cos\theta_t)(A-C+D-E)\\&+(q_1q_2-k_0{}^tp_0{}^t\cos\theta_t)B+4tq_1q_2E\\&-2q_2(\surd t)(q_1p_0{}^t-q_2k_0{}^t\cos\theta_t)(E+C)\\&+2q_1(\surd t)(p_0{}^tq_1\cos\theta_t-q_2k_0{}^t)(E-D)]\,, \quad (2.10e)\end{aligned}$$

$$q_1=\frac{\{[t-(m_1-\mu)^2][t-(m_1+\mu)^2]\}^{1/2}}{2\surd t}\,,$$

$$q_2=\frac{\{[t-(m_2-\mu)^2][t-(m_2+\mu)^2]\}^{1/2}}{2\surd t}\,, \quad (2.11)$$

$$k_0{}^t=\frac{t+m_1{}^2-\mu^2}{2\surd t}\,,\quad p_0{}^t=\frac{t+m_2{}^2-\mu^2}{2\surd t}\,,$$

where q_1 and q_2 are the c.m. momenta of $V_1-\pi_i$ and $V_2-\pi_j$, respectively.

III. ASYMPTOTIC BEHAVIOR

The asymptotic behavior of the amplitudes is determined by the leading Regge trajectories, which in our case are those of the f_0 in the s channel and the ρ in the t and u channels. As these trajectories are of natural parity, they contribute asymptotically only to natural-parity amplitudes. Reggeization of the parity-conserving helicity amplitudes (2.8) using the procedure of Gell-Mann *et al.*[6] produces the following asymptotic behavior of the invariant amplitudes:

$$\lim_{t\to\infty;\,s=\mathrm{const}}A=A(s)\Gamma(-\alpha_s)\left(\frac{t}{t_0}\right)^{\alpha_s}\frac{1+e^{-i\pi\alpha_s}}{2}\,, \quad (3.1a)$$

$$\lim_{t\to\infty;\,s=\mathrm{const}}B=B(s)\Gamma(-\alpha_s)\left(\frac{t}{t_0}\right)^{\alpha_s}\frac{1+e^{-i\pi\alpha_s}}{2}\,, \quad (3.1b)$$

$$\lim_{t\to\infty;\,s=\mathrm{const}}C=C(s)\Gamma(1-\alpha_s)\left(\frac{t}{t_0}\right)^{\alpha_s-1}\frac{1+e^{-i\pi\alpha_s}}{2}\,, \quad (3.1c)$$

$$\lim_{t\to\infty;\,s=\mathrm{const}}D=D(s)\Gamma(1-\alpha_s)\left(\frac{t}{t_0}\right)^{\alpha_s-1}\frac{1+e^{-i\pi\alpha_s}}{2}\,, \quad (3.1d)$$

$$\lim_{t\to\infty;\,s=\mathrm{const}}E=E(s)\Gamma(2-\alpha_s)\left(\frac{t}{t_0}\right)^{\alpha_s-2}\frac{1+e^{-i\pi\alpha_s}}{2}\,. \quad (3.1e)$$

Similar Reggeization of the amplitudes (2.10) leads to

$$\lim_{s\to\infty;\,t=\mathrm{const}}A=-\frac{f^-}{s_0}[\tfrac{1}{2}(s+m_1{}^2+m_2{}^2)-2(t+\mu^2)]-\frac{f^+}{s_0{}^2}[\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)k_0{}^tp_0{}^t+m_1{}^2m_2{}^2+(2\surd t)(k_0{}^tq_2{}^2+p_0{}^tq_1{}^2)]\,, \quad (3.2a)$$

$$\lim_{s\to\infty;\,t=\mathrm{const}}B=-\frac{f^-}{s_0}[t(s-u)+(m_2{}^2-\mu^2)(m_1{}^2-\mu^2)]-\frac{f^+}{s_0}4tq_1{}^2q_2{}^2\,, \quad (3.2b)$$

$$\lim_{s\to\infty;\,t=\mathrm{const}}C=-\frac{f^-}{s_0}[\tfrac{1}{2}(t-u)+m_1{}^2]-\frac{f^+}{s_0{}^2}[(s-m_1{}^2-m_2{}^2)k_0{}^tp_0{}^t+m_1{}^2m_2{}^2+p_0{}^tq_1{}^22\surd t]\,, \quad (3.2c)$$

$$\lim_{s\to\infty;\,t=\mathrm{const}}D=\frac{f^-}{s_0}[\tfrac{1}{2}(t-u)+m_2{}^2]-\frac{f^+}{s_0{}^2}[(s-m_1{}^2-m_2{}^2)k_0{}^tp_0{}^t+m_1{}^2m_2{}^2+k_0{}^tq_2{}^22\surd t]\,, \quad (3.2d)$$

$$\lim_{s\to\infty;\,t=\mathrm{const}}E=\frac{f^-}{s_0}[\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)]+\frac{f^+}{s_0{}^2}[\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)k_0{}^tp_0{}^t+m_1{}^2m_2{}^2]\,, \quad (3.2e)$$

where f^- and f^+ are defined by

$$f^-\equiv\lim_{s\to\infty;\,t=\mathrm{const}}\frac{T_{1,1}{}^{t-}}{4tq_1q_2}=\frac{f(t)}{s_0}\alpha_t\Gamma(1-\alpha_t)\times\left(\frac{s}{s_0}\right)^{\alpha_t-1}\frac{1-e^{-i\pi\alpha_t}}{2}\,, \quad (3.3a)$$

$$f^+\equiv \lim_{s\to\infty;\,t=\text{const}} \frac{T_{1,1}{}^{t+}}{4tq_1q_2}=\frac{f(t)}{s_0{}^2}\Gamma(2-\alpha_t)\times\left(\frac{s}{s_0}\right)^{\alpha_t-2}\frac{1-e^{-i\pi\alpha_t}}{2}. \quad (3.3b)$$

IV. CONTRIBUTION OF ρ AND f_0 EXCHANGES

In order to construct the amplitude with the correct form in the vicinity of the ρ and f_0 poles, let us compute their residues. The exchange of a vector particle, the ρ, in the t channel is given by

$$T_\rho{}^{(t)\lambda_1\lambda_2}=\frac{g_{V_1\pi\rho}g_{V_2\pi\rho}}{\mu^2}\epsilon^{\alpha\beta\gamma\delta}p_\alpha\epsilon_\beta{}^{\lambda_1}(p)Q^\gamma\frac{g^{\delta\rho}-Q^\delta Q^\rho/Q^2}{t-m_\rho{}^2}\times\epsilon^{\rho\sigma\tau\eta}Q_\sigma\epsilon_\tau{}^{\lambda_2}(k)k_\eta. \quad (4.1)$$

$\epsilon^{\alpha\beta\gamma\delta}$ is the fourth-rank antisymmetric tensor and $Q_\mu=(p+q_2)_\mu$.

From (4.1) we get the contribution of the ρ to the invariant amplitudes:

$$A_\rho{}^{(t)}=-\frac{G}{4(t-m_\rho{}^2)}[\tfrac{1}{2}(s+m_1{}^2+m_2{}^2)-2(t+\mu^2)], \quad (4.2a)$$

$$B_\rho{}^{(t)}=-\frac{G}{4(t-m_\rho{}^2)}[t(s-u)+(m_2{}^2-\mu^2)\times(m_1{}^2-\mu^2)], \quad (4.2b)$$

$$C_\rho{}^{(t)}=-\frac{G}{4(t-m_\rho{}^2)}[\tfrac{1}{2}(t-u)+m_1{}^2], \quad (4.2c)$$

$$D_\rho{}^{(t)}=\frac{G}{4(t-m_\rho{}^2)}[\tfrac{1}{2}(t-u)+m_2{}^2], \quad (4.2d)$$

$$E_\rho{}^{(t)}=\frac{G}{4(t-m_\rho{}^2)}[\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)], \quad (4.2e)$$

$$G\equiv g_{V_1\pi\rho}g_{V_2\pi\rho}/\mu^2. \quad (4.3)$$

The contribution of ρ exchange in the u channel is obtained from (4.2) by crossing symmetry.

The exchange of a spin-2 particle, the f_0, in the s channel is described with the aid of five couplings as follows:

$$\begin{aligned}T_f{}^{(s)\lambda_1\lambda_2}=\tfrac{1}{4}G\Delta_\delta\Delta_\nu[&\tfrac{2}{3}(L^\delta L^\nu L^\rho L^\sigma/L^4)+(1/L^2)(\tfrac{1}{3}g^{\delta\nu}L^\rho L^\sigma\\&+\tfrac{1}{3}g^{\rho\sigma}L^\delta L^\nu-\tfrac{1}{2}g^{\delta\rho}L^\sigma L^\nu-\tfrac{1}{2}g^{\delta\sigma}L^\rho L^\nu-\tfrac{1}{2}g^{\nu\rho}L^\delta L^\sigma\\&-\tfrac{1}{2}g^{\nu\sigma}L^\delta L^\rho)-\tfrac{1}{3}g^{\delta\nu}g^{\rho\sigma}+\tfrac{1}{2}g^{\delta\rho}g^{\nu\sigma}+\tfrac{1}{2}g^{\delta\sigma}g^{\nu\rho}]\\&\times\{\alpha\epsilon_\rho{}^{\lambda_1}(k)\epsilon_\sigma{}^{\lambda_2}(p)+\beta_1[\epsilon^{\lambda_1}(k)\cdot p]\epsilon_\rho{}^{\lambda_2}(p)k_\sigma\\&+\beta_2[\epsilon^{\lambda_2}(p)\cdot k]\epsilon_\rho{}^{\lambda_1}(k)p_\sigma+\gamma[\epsilon^{\lambda_1}(k)\cdot\epsilon^{\lambda_2}(p)]k_\rho p_\sigma\\&+\delta\epsilon_\rho{}^{\alpha\beta\gamma}\epsilon_\alpha{}^{\lambda_2}(p)p_\beta L_\gamma\epsilon_\sigma{}^{\alpha'\beta'\gamma'}\epsilon_{\alpha'}{}^{\lambda_1}(k)k_{\beta'}L_{\gamma'}\}\\&\times(s-m_f{}^2)^{-1}\end{aligned} \quad (4.4)$$

and

$$L_\mu=(q_1+q_2)_\mu.$$

From (4.4) we get the contribution of f_0 to the invariant amplitudes:

$$A_f{}^{(s)}=\left\{-\frac{4}{3}\frac{q^2}{\sqrt{s}}\left(\frac{\alpha}{\sqrt{s}}-\beta_1p_0\beta_2k_0\right)+\delta\left[\frac{8}{3}q^2(k\cdot p)-\frac{(t-u)^2}{4}\right]\right\}\frac{G}{4(s-m_f{}^2)}, \quad (4.5a)$$

$$B_f{}^{(s)}=\{\gamma[\tfrac{4}{3}q^2k^2-\tfrac{1}{4}(t-u)^2]+\alpha(\tfrac{4}{3}q^2)+\delta s[\tfrac{1}{4}(t-u)^2-(8/3)q^2k^2]\}\frac{G}{4(s-m_f{}^2)}, \quad (4.5b)$$

$$C_f{}^{(s)}=\tfrac{1}{2}(t-u)[\beta_1+\tfrac{1}{2}\delta(s+m_1{}^2-m_2{}^2)]\frac{G}{4(s-m_f{}^2)}, \quad (4.5c)$$

$$D_f{}^{(s)}=-\tfrac{1}{2}(t-u)[\beta_2+\tfrac{1}{2}\delta(s+m_2{}^2-m_1{}^2)]\times\frac{G}{4(s-m_f{}^2)}, \quad (4.5d)$$

$$E_f{}^{(s)}=(\alpha-\delta k^2s)\frac{G}{4(s-m_f{}^2)}. \quad (4.5e)$$

V. CONSTRUCTION OF VENEZIANO-TYPE AMPLITUDE

The following conditions are now imposed on the scattering amplitude: (a) Regge asymptotic behavior in all channels. (b) Crossing symmetry. (c) No ancestors or daughters (at least "as far as possible," which in our case is $\alpha_t,\alpha_u>2$, $\alpha_s>3$); this condition translates into the requirement of the appropriate residues at the resonance poles. (d) No parity doubling of the ρ trajectory. (e) The amplitude becomes automatically gauge invariant in the limit $m_1\to 0$. These conditions should hold for arbitrary vector-meson masses and coupling constants, and in order to achieve this, the amplitude is constructed as a product:

$$T_{\lambda\mu}=T_{\lambda\mu}{}'F(s,t). \quad (5.1)$$

F contains the B functions in such a way as to give poles spaced by two units, and $T_{\lambda\mu}{}'$ is made to meet requirements (b)-(e).

Let us write F in the form

$$\begin{aligned}-(\tfrac{1}{8}G\epsilon')^{-1}F=&\{B(1-\alpha_t,2-\alpha_s)+B(1-\alpha_u,2-\alpha_s)\\&+B(1-\alpha_t,1-\alpha_u)\}-\tfrac{1}{2}(\epsilon-3)\{B(2-\alpha_t,3-\alpha_s)\\&+B(2-\alpha_u,3-\alpha_s)+B(2-\alpha_t,2-\alpha_u)\}\\&-(\epsilon-3)\{B(3-\alpha_t,4-\alpha_s)+B(3-\alpha_u,4-\alpha_s)\\&+B(3-\alpha u,3-\alpha_s)\}+\tfrac{1}{12}(\epsilon-3)[(\epsilon-4)(\epsilon-5)-24]\\&\times\{B(4-\alpha_t,5-\alpha_s)+B(4-\alpha_u,5-\alpha_s)\\&+B(4-\alpha_t,4-\alpha_u)\}+\cdots,\end{aligned} \quad (5.2)$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, ϵ' is the slope of the

ρ and f_0 trajectories taken to be exchange degenerate, and

$$\epsilon=\alpha_s+\alpha_t+\alpha_u\,.$$

In (5.2) the first curly bracket is responsible for the asymptotic behavior in all channels. The second one eliminates the poles at $\alpha_t=2$, $\alpha_u=2$, and $\alpha_s=3$. The third one is added to give the residues at $\alpha_t=3$, $\alpha_u=3$, and $\alpha_s=4$ a particular form (which together with $T_{\lambda\mu}'$ will produce the suitable Legendre polynomials). The fourth bracket again eliminates poles at $\alpha_t=4$, $\alpha_u=4$, and $\alpha_s=5$. In principle this procedure can be extended to higher poles by explicit construction step by step, but we have not found the general recipe for it.

It is interesting to mention that the value $\epsilon=3$, for some mysterious reason, plays a special role. For this particular value there is no need to add any satellites since the leading term already possesses all the desired features. This is manifestly displayed by the appearance of the factor $(\epsilon-3)$ in front of the satellite terms.

A few properties of F are

$$\mathrm{Res}(F)_{\alpha_t=1}=\mathrm{Res}(F)_{\alpha_u=1}=\mathrm{Res}(F)_{\alpha_s=2}=-\tfrac{1}{4}G\epsilon'\,, \tag{5.3}$$

$$\mathrm{Res}(F)_{\alpha_t=3}=\tfrac{1}{8}G\epsilon'\alpha_s(\alpha_u+1)\,, \tag{5.4}$$

$$\mathrm{Res}(F)_{\alpha_u=3}=\tfrac{1}{8}G\epsilon'\alpha_s(\alpha_t+1)\,, \tag{5.5}$$

$$\mathrm{Res}(F)_{\alpha_s=4}=\tfrac{1}{8}G\epsilon'(\alpha_t+1)(\alpha_u+1)\,, \tag{5.6}$$

$$\underset{\substack{s\to\infty;\\ \pi>\arg s>0;\\ \cot\alpha_s\to -i}}{F}\sim\Gamma(1-\alpha_t)\tfrac{1}{2}(\epsilon's)^{\alpha_t-1}(1-e^{-i\pi\alpha_t})\,, \tag{5.7}$$

$$\underset{\substack{t\to\infty;\\ \pi>\arg t>0;\\ \cot\alpha_t\to -i}}{F}\sim\Gamma(2-\alpha_s)\tfrac{1}{2}(\epsilon't)^{\alpha_s-2}(1+e^{-i\pi\alpha_s})\,. \tag{5.8}$$

To construct $T_{\lambda\mu}'$, it is more convenient to start with the invariant amplitudes. We begin with A, which is symmetric under $t\to u$, and write it in the form

$$A=\left\{\left[\mathrm{Res}(A_\rho^{(t)})_{t=m_\rho^2}\frac{1-\alpha_u}{2-\alpha_t-\alpha_u}+\mathrm{Res}(A_\rho^{(u)})_{u=m_\rho^2}\frac{1-\alpha_t}{2-\alpha_t-\alpha_u}\right]\frac{1}{G}+\frac{(1-\alpha_t)(1-\alpha_u)}{2-\alpha_t-\alpha_u}\times\left[\frac{(\alpha_s-2)A'}{4-\alpha_t-\alpha_u}+\frac{A''}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right]\right\}F. \tag{5.9a}$$

In this particular construction of A, the first two terms reproduce the correct residues of the ρ in the t and u channels as well as the asymptotic behavior (3.1a). A' is added to ensure correct asymptotic behavior in the limit $s\to\infty$, (3.2a), and A'' in order to fit $\mathrm{Res}(A)_{\alpha_s=2}$ to the value $\mathrm{Res}(A_f^{(s)})_{s=m_f^2}$.

To understand the reasoning leading to this particular form one should take note of the following relations:

$$\frac{1-\alpha_u}{2-\alpha_t-\alpha_u}=1\quad\text{at}\quad\alpha_t=1\,, \tag{5.10a}$$

$$\frac{1-\alpha_t}{2-\alpha_t-\alpha_u}=1\quad\text{at}\quad\alpha_u=1\,, \tag{5.10b}$$

$$\frac{(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}=1\quad\text{at}\quad\alpha_s=2\,. \tag{5.10c}$$

Taking for A′ the value

$$A'=\frac{2(t-u)}{\alpha_t-\alpha_u}\,, \tag{5.11a}$$

A becomes

$$A=\left\{2(t+\mu^2)-\tfrac{1}{2}(s+m_1^2+m_2^2)-\frac{2(t-u)(1-\alpha_t)}{(\alpha_t-\alpha_u)(2-\alpha_t-\alpha_u)}\left[\alpha_t+5-\epsilon+\frac{(\alpha_t-3)(6-\epsilon)}{(4-\alpha_t-\alpha_u)}\right]+\frac{(1-\alpha_t)(1-\alpha_u)A''}{(2-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right\}F, \tag{5.12a}$$

from which one sees that A acquires the prescribed asymptotic behavior provided $A''\to$ const, $s\to\infty$. Since our amplitude obeys crossing symmetry, the correct asymptotic behavior for fixed u is automatically ensured.

The same considerations guide us in constructing all other invariant amplitudes:

$$B=-\left\{t(s-u)+(m_1^2-\mu^2)(m_2^2-\mu^2)-\frac{s(t-u)(1-\alpha_t)}{2-\alpha_t-\alpha_u}+\frac{(1-\alpha_t)(1-\alpha_u)}{2-\alpha_t-\alpha_u}\left[\frac{(\alpha_s-2)B'}{4-\alpha_t-\alpha_u}+\frac{B''}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right]\right\}F. \tag{5.9b}$$

Taking for B' the value

$$B'=(s-m_1^2-m_2^2)\frac{t-u}{\alpha_t-\alpha_u}\,, \tag{5.11b}$$

B becomes

$$B=-\Bigg\{t(s-u)+(m_1^2-\mu^2)(m_2^2-\mu^2)$$
$$-\frac{(1-\alpha_t)(s-m_1^2-m_2^2)}{2-\alpha_t-\alpha_u}\Bigg[\frac{t-u}{\alpha_t-\alpha_u}$$
$$\times\left(\alpha_t+5-\epsilon+\frac{(\alpha_t-3)(6-\epsilon)}{4-\alpha_t-\alpha_u}\right)+m_1^2+m_2^2\Bigg]$$
$$-2(t-\mu^2)(m_1^2+m_2^2)\frac{1-\alpha_t}{2-\alpha_t-\alpha_u}$$
$$+\frac{(1-\alpha_t)(1-\alpha_u)B''}{(2-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\Bigg\}F. \quad (5.12\text{b})$$

As it turns out, the terms giving the ρ residues in C, D, E already have the desired asymptotic behavior; hence $C'=D'=E'=0$. Moreover, in E they are also sufficient in fitting the prescribed form at $\alpha_s=2$ and therefore $E''=0$. Hence

$$C=-\Bigg[\tfrac{1}{2}(t-u)+m_1^2-2m_1^2\frac{1-\alpha_t}{2-\alpha_t-\alpha_u}$$
$$+\frac{(1-\alpha_t)(1-\alpha_u)C''}{(2-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\Bigg]F, \quad (5.12\text{c})$$

$$D=\Bigg[\tfrac{1}{2}(t-u)+m_2^2-2m_2^2\frac{1-\alpha_t}{2-\alpha_t-\alpha_u}$$
$$+\frac{(1-\alpha_t)(1-\alpha_u)D''}{(2-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\Bigg]F, \quad (5.12\text{d})$$

$$E=\tfrac{1}{2}(s-m_1^2-m_2^2)F. \quad (5.12\text{e})$$

We now wish to remark on the following feature of our amplitude. In addition to the poles in F representing the known resonances, singularities of a different nature at $\alpha_t+\alpha_u=2,4$, $\alpha_s+\alpha_u=3$ appear in $T_{\lambda\mu}'$ as a result of the combined requirements of crossing symmetry and correct behavior of resonances. This is not surprising since resonances in one channel lead to cuts in the others when crossing symmetry is imposed. It is interesting to note that for $\epsilon=3$, F has zeros at the very same values so that all these singularities disappear.

We now want to fix the values A'', ..., D''. This is done by comparing (4.5) and (5.12) at the point $\alpha_s=2$. From (4.5) one sees that E depends only on s; C and D have the form $(t-u)f(s)$ and A and B have two parts: one depending only on s and the second one having the form $(t-u)^2g(s)$. This means that on comparing (4.5) and (5.12) we get seven equations for the five parameters: $\alpha,\beta_1,\beta_2,\gamma,\delta$ of Eqs. (4.5). A'',B'', C'',D'' should therefore depend on two additional parameters. Subjected to requirements (e) and (d), the parametrization is taken to be

$$A''=z(t-u)(\alpha_t-\alpha_u)+y8q^2/3(4-\alpha_t-\alpha_u)\,, \quad (5.13\text{a})$$

$$B''=\tfrac{1}{2}zs(t-u)(\alpha_t-\alpha_u)+\tfrac{1}{2}y(s-m_1^2-m_2^2)$$
$$\times 8q^2/3(4-\alpha_t-\alpha_u)\,, \quad (5.13\text{b})$$

$$C''=zm_1^2(\alpha_t-\alpha_u)\,, \quad (5.13\text{c})$$

$$D''=zm_2^2(\alpha_t-\alpha_u)\,, \quad (5.13\text{d})$$

where $z=x/(4-\alpha_t-\alpha_u)(6-\alpha_t-\alpha_u)$ and x,y are free parameters.

Inserting (5.13) into (5.12) and comparing with (4.5), we finally obtain for the seven parameters

$$\alpha=k\cdot p+\delta k^2s\,, \quad (5.14\text{a})$$

$$-\beta_1=1+2m_1^2\frac{\epsilon'(1+z)}{2-\alpha_t-\alpha_u}+\delta k_0\sqrt{s}\,, \quad (5.14\text{b})$$

$$-\beta_2=1+2m_2^2\frac{\epsilon'(1+z)}{2-\alpha_t-\alpha_u}+\delta p_0\sqrt{s}\,, \quad (5.14\text{c})$$

$$\gamma-\delta s=1+\frac{2\epsilon's(1+z)}{2-\alpha_t-\alpha_u}\,, \quad (5.14\text{d})$$

$$-\delta=\frac{4\epsilon'(1+z)}{2-\alpha_t-\alpha_u}\,, \quad (5.14\text{e})$$

$$1+z=\frac{2-\alpha_t-\alpha_u}{s}\,\frac{s+2\mu^2}{s-4\mu^2}\,, \quad (5.15\text{a})$$

$$y=(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)\,. \quad (5.15\text{b})$$

It should be remembered that the right-hand sides of (5.14) and (5.15) are evaluated at the point $\alpha_s=2$.

Our resulting amplitudes are then

$$A=\Bigg\{-(4q^2+k\cdot p)-\frac{2}{\epsilon'}\frac{(1-\alpha_t)(1-\alpha_u)(2-\alpha_s)}{(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)}$$
$$+y\frac{8}{3}\frac{q^2(1-\alpha_t)(1-\alpha_u)}{(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}$$
$$+(t-u)^2\Bigg[1+\frac{z(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\Bigg]$$
$$\times\frac{\epsilon'}{2-\alpha_t-\alpha_u}\Bigg\}F, \quad (5.16\text{a})$$

$$B=-\left\{-(k^2s+4q^2k\cdot p)-\frac{2k\cdot p}{\epsilon'}\frac{(1-\alpha_t)(1-\alpha_u)(2-\alpha_s)}{(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)}\right.$$
$$+\frac{8}{3}\frac{yq^2k\cdot p(1-\alpha_t)(1-\alpha_u)}{(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}$$
$$+\frac{(t-u)^2}{4}\left[1+\frac{\epsilon' s}{(2-\alpha_t-\alpha_u)}\right.$$
$$\left.\left.\times\left(1+\frac{z(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right)\right]\right\}F, \quad (5.16b)$$

$$C=-\frac{(t-u)}{2}\left\{1+\frac{2m_1{}^2\epsilon'}{2-\alpha_t-\alpha_u}\right.$$
$$\left.\times\left[1+\frac{z(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right]\right\}F, \quad (5.16c)$$

$$D=\frac{(t-u)}{2}\left\{1+\frac{2m_2{}^2\epsilon'}{2-\alpha_t-\alpha_u}\right.$$
$$\left.\times\left[1+\frac{z(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right]\right\}F, \quad (5.16d)$$

$$E=\tfrac{1}{2}(s-m_1{}^2-m_2{}^2)F. \quad (5.16e)$$

VI. DISCUSSION

In the preceding sections we presented a procedure for building a Veneziano amplitude subjected to certain requirements [(a)–(e) of Sec. V], and fulfilling them independently of the external masses. In particular, the resulting amplitude factorizes as a consequence of the process of fitting it at each resonance separately.

The requirement of the absence of daughters deserves special comment and justification because it is quite uncommon in the contemporary literature on Veneziano amplitudes.[7] First, there is as yet no experimental evidence for the existence of particles lying on daughter trajectories (with the possible exception of the ϵ). As to the theoretical argumentation concerning daughters, they were introduced by Freedman and Wang[8] for the unequal-mass case. Assuming Regge behavior in the s channel, one finds in this case that the amplitude becomes singular at $t=0$. The daughters are then introduced in order to eliminate these undesired singularities. However, one should remember that these singularities enter as a result of expanding the Legendre functions in the limit $\cos\theta_t\to\infty$. Expressing $\cos\theta_t$ and $\cos\theta_s$ in terms of s and t:

$$\cos\theta_t=\frac{2st+t^2-t\sum_i m_i{}^2+(m_1{}^2-m_3{}^2)(m_2{}^2-m_4{}^2)}{\{[t-(m_1-m_3)^2][t-(m_1+m_3)^2][t-(m_2-m_4)^2][t-(m_2+m_4)^2]\}^{1/2}}, \quad (6.1)$$

$$\cos\theta_s=\frac{2st+s^2-s\sum_i m_i{}^2+(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)}{\{[s-(m_1-m_2)^2][s-(m_1+m_2)^2][s-(m_3-m_4)^2][s-(m_3+m_4)^2]\}^{1/2}}. \quad (6.2)$$

This shows that for elastic scattering, i.e., $m_1=m_3$, $m_2=m_4$, one has

$$\cos\theta_t|_{t=0}=\frac{s-m_1{}^2-m_2{}^2}{2m_1m_2}\xrightarrow[\cos\theta_s=1]{s\to\infty;}\infty,$$

whereas for $m_1\neq m_3$, $m_2\neq m_4$, the two points $t=0$ and $\cos\theta_s=1$ do not coincide while at both points $|\cos\theta_t|=1$ independently of s. Thus, the asymptotic expansion becomes meaningless here, which is indeed reflected in the need to take into account all other terms in the expression, i.e., the "daughters." Consequently, models which exhibit Regge behavior which is not derived by improper expansions are not compelled to contain daughters. In this spirit our amplitude does not contain the $\epsilon(0^+)$.[2]

Our procedure for removing the daughters and ensuring the correct polynomial behavior of the resonance residues has been carried through only for the first few points. It becomes more tedious as one goes to higher resonances. However, the contributions of these additional terms at low and high energies is negligible in actual physical processes.[5] Therefore, until a satisfactory way to unitarize the Veneziano-type amplitude is found, this amplitude is incomplete at intermediate energies; hence it does not seem worthwhile to include the higher corrections at this stage.

Finally, let us remark on the fulfillment of our requirement (d) that there is no parity doubling of the ρ; i.e., it does not contribute to wrong-parity amplitudes. This is ensured by the insertion of the factors (5.10), as can be easily checked by inserting (5.16) into (2.10).

[7] E. Predazzi, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Gordon and Breach, New York, to be published), Vol. XII.

[8] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).