This simply means that a factor τ_i occurs whenever the corresponding ν_i in the argument of $\Delta_{\{M_i\}}^{\{X_i,\tilde{C}_i\}}$ in (A4) is -1. The internal helicity-flip factors $e^{i\pi M_i}$ appear whenever neighboring parameters ν_i, ν_{i+1} have opposite signs in $\Delta_{\{M_i\}}^{\{X_i,\tilde{C}_i\}}$.

The full signatured amplitude $f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\})$ is defined as having $f_{\{M_i\}}^{\{j_i,\tau_i\}}(\{t_i\})$ as its $O(2,1)$ partial waves. It has only right-hand cuts in all subenergies s_i, and is expressed in terms of the $f_{\{M_i\}}^{\{X_i\}}(\{s_i,t_i\})$ amplitudes by

$$f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\}) = \sum_{\{X_i=R,L\}} \mu_1\cdots\mu_{n-1}f^{\{X_i\}}(\{\nu_i s_i,\, t_i,\, \omega_i-\nu_{i,i+1}\pi\}),$$

where

$$\mu_i=\begin{cases}1, & \nu_i=1\\ \tau_i, & \nu_i=-1\end{cases} \tag{A5}$$

and ν_i and $\nu_{i,i+1}$ are defined above. The $(\omega_i-\nu_{i,i+1}\pi)$ dependence is due to the extra $e^{i\pi M_i\nu_{i,i+1}}$ factors in the Fourier M_i sums for $f^{\{X_i\}}$.

The full amplitude is expressed in terms of the signatured full amplitudes by

$$f(\{s_i,t_i,\omega_i\}) = \frac{1}{2^{n-1}}\sum_{\{\tau_i\}}\sum_{\{\nu_i\}}\mu_1\cdots \times\mu_{n-1}f^{\{\tau_i\}}(\{\nu_i s_i,\, t_i,\, \omega_i-\nu_{i,i+1}\pi\}). \tag{A6}$$

A multi-Regge pole occurs as a set of j_i-plane poles in the signatured multi-partial-wave amplitude $f_{\{M_i\}}^{\{j_i,\tau_i\}}(\{t_i\})$, since that is the amplitude which supposedly can be analytically continued in the j_i variables. The multi-Regge pole also occurs with a factorized residue (see the Introduction). The resulting asymptotic behavior of the full signatured amplitude is therefore factorized as well,

$$f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\})\sim 2^{n-1}g_{a1}(t_1)g_{bn}(t_{n-1})\prod_{i=1}^{n-1}(-s_i)^{\alpha_i(t_i)} \times\prod_{j=1}^{n-2}\beta_j(t_j,\eta_j,t_{j+1}). \tag{A7}$$

Use of Eq. (A6) then yields the asymptotic form for the full amplitude, as described in the text. For a given multi-Regge pole, the $\{\tau_i\}$ sum is absent.

Note added in proof. Professor I. Halliday has emphasized to us that cuts in the subenergies s_i arising from unitarity in dependent variables and Gram-determinant conditions render the analytic continual tion of some of the terms in Eq. (A6) to the physical region to be more complicated than the asymptotic s_i continuations used here. There is the possibility that mixed $i\epsilon$ prescriptions for the η_i in these anomolous terms may recover factorization for the full amplitude in the ordinary sense. For a perturbation theory calculation which factorizes, see D. K. Campbell, Phys. Rev. **188**, 2471 (1969).

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Dimension of Operators in Broken Scale Invariance

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It is shown that if the Hamiltonian can be split into a part which is scale invariant and a part which breaks scale invariance by means of a dilaton, then, if the latter part has a unique dimension, this dimension must be 1 if the vacuum does not realize the invariance under scaling. This implies that there must exist a term which breaks scale invariance in addition to that which breaks chiral $SU(3)\times SU(3)$ symmetry in order to avoid a contradiction with Gell-Mann's argument.

I. INTRODUCTION

THE study of the relation of scaling transformations to the dynamics of strong interactions and to deep inelastic electroproduction has been the subject of many recent investigations.[1] In the study of the dynamical consequences of broken scale invariance, two main approaches have in general been pursued. These may be classified as to whether it is assumed that the vacuum is or is not invariant under scale transformations. In the former case, renormalized field theories have been the focus of attention and many new and important results concerning the relationship between

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[1] Good reviews on the subject to the present are given by the following: (a) G. Mack and A. Salam, Ann. Phys. (N. Y.) **53**, 174 (1969); (b) M. Gell-Mann, Symmetry Violation in Hadron Physics, Summer School of Theoretical Physics, University of Hawaii, 1969 (unpublished); and (c) P. Carruthers, Phys. Rept. (to be published).

scaling and the renormalization group, in soluble models or in perturbation theory for simple models, have been obtained.[2] It is not clear how the conclusions of these studies will be affected if it is assumed that the vacuum is not invariant under scaling. Such spontaneous breaking has recently been shown to be physically interesting.[3,4] In Refs. 3 and 4 and in the lectures by Gell-Mann,[1] there have been attempts made to ascertain whether (a) the part of the Lagrangian which breaks scale invariance can be ascribed to the same part which breaks chiral $SU(3)\times SU(3)$ invariance or (b) a chiral $SU(3)\times SU(3)$ scalar is needed to break scale invariance when the $SU(3)\times SU(3)$ breaking is turned off. It might be thought that the second case must hold, particularly in view of the recent analysis by Kim and von Hippel[5] of meson-baryon scattering lengths. This analysis seems to show that the nucleon matrix element $\langle N|u_0+cu_8|N\rangle$ is small [where u_0 and u_8 are scalar components of a $(3,\bar{3})+(\bar{3},3)$ representation]. However, there are possibly large systematic errors arising in this evaluation from the complicated extrapolation and approximation procedure used, so that this result [albeit the most compelling evidence at the present time for the existence of an $SU(3)\times SU(3)$ singlet] is not sufficient[3,4] to decide between alternatives (a) and (b).

Another interesting problem which has been considered in the spontaneous-breakdown approach is to try to determine if the non-scale-invariant operators have a unique dimension and, if so, what that dimension d is. The phenomenological tests proposed so far[1(b),3,4] have not been able to settle the first of these two points. The data, under certain assumptions at least, are compatible with the unique-dimension hypothesis. On the value of the dimension there does, however, appear to be some contradiction between Ref. 3 (which, on considering baryon matrix elements and making certain assumptions about the smoothness of form factors in the scale-invariant limit, finds that $d=1$ in lowest order) and Gell-Mann[1] (who considers the shift in energy of pseudoscalars to show that d should be 2 if only a chiral-symmetry-breaking term is present).

A resolution of this contradiction was shown in Ref. 3 to follow if there exists an $SU(3)\times SU(3)$ scalar which breaks scale invariance. The present paper is devoted to analyzing further the problem of dimension. In particular, we shall strengthen the proof given in Ref. 3 by removing some of the assumptions contained therein. We shall isolate the crucial step in this proof, which involves consideration of the behavior at infinity of massless Goldstone particles. In Sec. II we develop the notation and discuss the problems arising from the introduction of spontaneous breaking of scale invariance. Section III gives, in detail, the proof of the statement that if the Lagrangian which breaks scale invariance has a unique dimension, then to lowest order this dimension is unity. Section IV concludes and summarizes the work.

II. SPONTANEOUS BREAKDOWN OF SCALE INVARIANCE

From the work of Callan *et al.*,[6] we know that we can define the energy-momentum tensor $\theta_{\mu\nu}$ in such a way that the generator of dilations D may be written

$$D=\int d^3x\, x^\mu\theta_{\mu 0}(x)\,.$$

Such a stress-energy tensor has the property that in renormalizable field theories its matrix elements are finite. Breaking of scale invariance is then simply related to $\theta_\mu{}^\mu$ since

$$\dot{D}=\int d^3x\, \theta_\mu{}^\mu(x)\,.$$

Now suppose that we may write the energy density θ_{00} as a sum of two parts, one $(\bar{\theta}_{00})$ which is scale invariant and thus has the dimension $d=4$, and a remainder (world scalar) $-\epsilon L'$ which breaks scale invariance. That is,

$$\begin{aligned}\theta_{00}&=\lim_{\epsilon\to 0}\theta_{00}-\epsilon L'\\&=\bar{\theta}_{00}-\epsilon L'\,,\end{aligned}$$

where we have defined $\bar{\theta}_{00}=\lim_{\epsilon\to 0}\theta_{00}$. If L' consists of pieces w_n of dimension d_n, then we have the "virial theorem"[1(b),3]

$$\theta_\mu{}^\mu=\sum_n (4-d_n)w_n\,.$$

Since, as mentioned in the Introduction, the evidence for the existence of different d_n is scanty, we shall assume that all of the parts making up L' have the same dimension d, in which case

$$\theta_\mu{}^\mu=\epsilon(d-4)L'\,.$$

We note here, in passing, that this equation may not be as simple as it looks when scale invariance is broken spontaneously in the Goldstone way. A simple field-theory example illustrates the difficulty.[4] Consider the Lagrangian density of a simple scalar-meson model:

$$L=\tfrac{1}{2}(\partial_\mu\phi)^2+f\phi^4+\epsilon\phi\,,$$

with

$$\theta_\mu{}^\mu=-3\epsilon\phi\,.$$

[2] See, e.g., the recent papers by K. Wilson, Phys. Rev. D **2**, 1473 (1970); **2**, 1478 (1970); SLAC Report No. 737 (unpublished). Also see the following: C. G. Callan Jr., Caltech. Report Nos. CALT-68-257 and 259 (unpublished); R. Jackiw, MIT report (unpublished). Further references and discussion, are given in Ref. 1(c).

[3] S. P. de Alwis and P. J. O'Donnell, Phys. Rev. D **2**, 1023 (1970).

[4] P. Carruthers, Phys. Rev. D **2**, 2265 (1970).

[5] J. K. Kim and F. von Hippel Phys. Rev. Letters **20**, 740 (1969); F. von Hippel and J. K. Kim, Phys. Rev. D **1**, 151 (1970).

[6] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) **59**, 42 (1970).

Now redefine ϕ by

$$\phi \to \phi' = \phi - \langle\phi\rangle \equiv \phi - \phi_0,$$

where $\langle\phi'\rangle = 0$. In this case the Lagrangian density becomes

$$L' = \tfrac{1}{2}(\partial_\mu\phi')^2 - \tfrac{1}{2}m^2\phi'^2 + g\phi'^3 + f\phi'^4,$$

with

$$\theta'_\mu{}^\mu = m^2\phi'^2 - g\phi'^3.$$

Here we have defined $m^2 = -12f\phi_0^2$ and the external (no linear term present) condition is

$$\epsilon = -4f\phi_0^3.$$

Now either Lagrangian L or L' forms a suitable one for scalar-meson theory, but in this latter case there are mixed dimensions in the terms which break scale invariance. Thus in model calculations, in lowest-order perturbation theory, at least, one needs some additional criterion to determine a starting point for the theory when the field can have a nonzero vacuum expectation value. This example also shows where much of the trouble arises in trying to trace backwards from the real world to what we may consider as the symmetric world which underlies this real world. For when the field ϕ is massless and has nonzero vacuum expectation value, the difference between $\theta_{\mu\nu}$ and $\theta'_{\mu\nu}$ will not necessarily go to zero fast enough at large distances.[7] In Sec. III we shall discuss this further since the basic step in our proof of the value of d is concerned with just such a problem.

III. BROKEN SCALE INVARIANCE AND DIMENSIONS

In this section we shall assume that the part of the Lagrangian density which breaks scale invariance has a unique dimension d. For definiteness we shall only consider nucleon matrix elements. This has an advantage over considering pseudoscalar matrix elements in that we shall not be led into comparing terms involving the pseudoscalar masses with at least comparable terms coming from symmetry breaking.[3]

The matrix element of $\theta_{\mu\nu}$ between nucleon states is

$$\langle \mathbf{p}'|\theta_{\mu\nu}(0)|\mathbf{p}\rangle = \bar{u}(\mathbf{p}')\Big\{\tfrac{1}{4}(\gamma_\mu P_\nu + \gamma_\nu P_\mu)F_1(k^2) + \tfrac{1}{2}P_\mu P_\nu F_2(k^2) + (g_{\mu\nu}k^2 - k_\mu k_\nu)\Big[\frac{f_\sigma g_{\sigma NN}}{m_\sigma^2 - k^2} + F_3(k^2)\Big]\Big\}u(\mathbf{p}), \quad (3.1)$$

where $P_\mu = (p'+p)_\mu$, $k_\mu = (p'-p)_\mu$ and where we have explicitly displayed the pole term arising from the "Goldstone" particle with the quantum numbers of the vacuum, the dilaton denoted by σ.[8] Since $\theta_{0\mu}(x)$ is the energy-momentum density, we have[9] (where M is the nucleon mass)

$$F_1(0) + 2MF_2(0) = 1. \quad (3.2)$$

Taking the trace in Eq. (3.1) and evaluating at $k^2 = 0$ gives[3,4,10]

$$3f_\sigma g_{\sigma NN} = M. \quad (3.3)$$

The proof of the dimension of L' now proceeds as follows. First, note that

$$\lim_{\mathbf{k}\to 0}\langle \mathbf{p}'|\theta_{00}|\mathbf{p}\rangle = \frac{E^2}{M}[F_1(0) + 2MF_2(0)] = E^2/M. \quad (3.4)$$

Hence

$$\lim_{\epsilon\to 0}\lim_{\mathbf{k}\to 0}\langle \mathbf{p}'|\theta_{00}|\mathbf{p}\rangle = \frac{E^2}{M}.$$

The limit $\mathbf{k}\to 0$ ensures the absence of the σ-pole term; the limit $\epsilon \to 0$ is taken to emphasize that we are working to zeroth order in ϵ. As mentioned in Sec. II, we may write

$$\theta_{00} = \lim_{\epsilon\to 0}\theta_{00} - \epsilon L'.$$

Hence

$$\langle \mathbf{p}'|\theta_{00}|\mathbf{p}\rangle = \langle \mathbf{p}'|(\lim_{\epsilon\to 0}\theta_{00} - \epsilon L')|\mathbf{p}\rangle = \langle \mathbf{p}'|\lim_{\epsilon\to 0}\theta_{00}|\mathbf{p}\rangle - \langle \mathbf{p}'|\theta_\mu{}^\mu|\mathbf{p}\rangle/(d-4), \quad (3.5)$$

using the broken-scale-invariance condition[3]

$$\theta_\mu{}^\mu = (d-4)\epsilon L'.$$

Without a complete theory of strong interactions we cannot calculate the matrix element $\langle \mathbf{p}'|\lim_{\epsilon\to 0}\theta_{00}|\mathbf{p}\rangle$. Since such a matrix element involves a massless (Goldstone) particle, we cannot assume that the limit may be taken outside the matrix element when the states $|\mathbf{p}\rangle$ are the usual (covariant) momentum states. This is related to the point mentioned in Sec. II with regard to the example of scalar-meson theories.

We can proceed, however, if we first normalize the nucleon states in a box of volume V (less than ϵ^{-3}) and then let $V\to\infty$ to restore relativistic invariance. With this procedure understood, we may write

$$\langle \mathbf{p}'|\lim_{\epsilon\to 0}\theta_{00}|\mathbf{p}\rangle = \lim_{\epsilon\to 0}\langle \mathbf{p}'|\theta_{00}|\mathbf{p}\rangle. \quad (3.6)$$

We also have[3]

$$\lim_{\mathbf{k}\to 0}\langle \mathbf{p}'|\theta_\mu{}^\mu|\mathbf{p}\rangle = 3f_\sigma g_{\sigma NN},$$

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[7] C. G. Callan and P. Carruthers (unpublished).

[8] The normalization of the fermion states is $\langle p'|p\rangle = (2\pi)^3(p_0/M) \times \delta^3(\mathbf{p}'-\mathbf{p})$ and $\bar{u}u = 1$. The constant f_σ is defined by $\langle 0|\theta_{\mu\nu}|\sigma(k)\rangle = f_\sigma(g_{\mu\nu}k^2 - k_\mu k_\nu)$.

[9] H. Pagels, Phys. Rev. **144**, 1250 (1966); D. Gross and J. Wess, Phys. Rev. D **2**, 753 (1970). The latter authors also state that $F_2(0) = 0$ follows from considering $x_\mu\theta_{0\nu}(x) - x_\nu\theta_{0\mu}(x)$ as the angular momentum density. We do not need to use this result explicitly in our proof.

[10] We shall always be considering the limit $\epsilon\to 0$ so that all results are to order ϵ^0. Note that $m_\sigma^2 = O(\epsilon)$.

so that Eq. (3.5) gives [using Eq. (3.6)]

$$\lim_{\epsilon\to 0}\lim_{\mathbf{k}\to\mathbf{0}}\langle\mathbf{p}'|\theta_{00}|\mathbf{p}\rangle=\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\epsilon\to 0}\langle p'|\theta_{00}|\mathbf{p}\rangle-\frac{3f_\sigma g_{\sigma NN}}{(d-4)},\quad(3.7)$$

showing that for arbitrary (but finite $\mathbf{p}$), the interchange of limits of the matrix element of θ_{00} is nonuniform, as already noted in Ref. 3.

In addition, we see from the presence of the pole term on the right-hand side of Eq. (3.7) that, irrespective of the validity of Eq. (3.6), we must have a pole term in the matrix element $\langle p'|\lim_{\epsilon\to 0}\theta_{00}|p\rangle$ if the splitting up of θ_{00} into a scale-invariant and scale-noninvariant part is valid. In Ref. 3 the right-hand side of Eq. (3.7) was then calculated *assuming* that the form factor $F_1(0)$ behaves smoothly as a function of ϵ as $\epsilon\to 0$. We shall now prove that the form factors $F_1(0)$ and $F_2(0)$ are regular functions of ϵ in this limit of $\epsilon\to 0$. To do this, we first of all calculate the right-hand side of Eq. (3.7) in an arbitrary frame using Eq. (3.1). This gives

$$\frac{E^2}{M}[\bar{F}_1(0)+2M\bar{F}_2(0)]-\lim_{\mathbf{k}\to 0}(\mathbf{k}^2/k^2)f_\sigma g_{\sigma NN}-3f_\sigma g_{\sigma NN}/(d-4),\quad(3.8)$$

where $\bar{F}_i(0)$ denotes the form factor $F_i(0)$ calculated with ϵ set to zero. This is permissible since we see that if we now let $|\mathbf{p}|\to\infty$, then Eqs. (3.4) and (3.8) give

$$\lim_{\mathbf{k}\to\mathbf{0}}\lim_{\epsilon\to 0}\langle\mathbf{p}'|\theta_{00}|\mathbf{p}\rangle=\lim_{\epsilon\to 0}\lim_{\mathbf{k}\to 0}\langle\mathbf{p}'|\theta_{00}|\mathbf{p}\rangle;\quad(3.9)$$

i.e., in the infinite-momentum frame, the matrix element of θ_{00} becomes uniform under the interchange of the two limits, the pole terms arising from the zero-mass Goldstone particles vanishing in this frame. Equation (3.9) also tells us that $\bar{F}_1(0)+2M\bar{F}_2(0)=1$, the result assumed in Ref. 3. If we use this and now evaluate Eq. (3.7) in the rest frame $\mathbf{p}=\mathbf{0}$, we obtain $d=1$.

IV. CONCLUSIONS

In Sec. III we have strengthened the proof first given Ref. 3 of the dimension of the broken scale-invariant part of the Lagrangian. The crucial steps in this proof consist of using the infinite-momentum frame to determine the behavior of the form factors as functions of $\epsilon\sim 0$, and also being able to extract the limit $\epsilon\to 0$ from inside the matrix element. To do this, we proceeded by normalizing in a box (or alternatively using a narrow Gaussian distribution), in which case the limit as $\epsilon\to 0$ may be taken outside the matrix element. This procedure is necessary since in theories with massless particles the integrands may not go to zero fast enough at infinity to make the integrals finite. This same behavior will also affect the field-theory calculation of Ref. 2 for, e.g., Ward identities may now have additional surface terms present. In view of the fact that most of the physical content of broken scale invariance has come from considering a degenerate vacuum (Goldstone) approach,[1(b),3,4] it would seem necessary to allow for such a case in the model field-theory approach.[2]

If we now assume that d remains one when higher-order terms in ϵ are introduced, then, as already noted in Ref. 3, the contradiction between this value and those considered in Ref. 1(b) is removed by the presence of an $SU(3)\times SU(3)$ scalar term δ in L' in addition to the usual breaking term u_0+cu_8.

With two terms δ and u now breaking scale invariance, there is, of course, no reason for both to have the same dimension, and indeed there are various choices of dimensions d_u and d_δ which give reasonable solutions.[1(b),4] In the present case we have also shown that there exists the possibility that these are equal, in which case $d_u=d_\delta=d=1$. There is a further implication that can be drawn from having $d=1$. Carruthers[4] and Gell-Mann[1] have shown that a solution of the virial theorem is

$$m(\nu,\lambda)=c\nu^{1/2x}\lambda^{1/2y}f(z),$$

where $z=\lambda^{1/2y}/\nu^{1/2x}$, provided the dimensions are independent of ν and λ. Here we assume that the part of θ_{00} which breaks scale invariance is written as $\gamma\delta+\lambda u$, with δ an $SU(3)\times SU(3)$ scalar. Also $x=4-d_\delta$, $y=4-d_u$, where d_δ and d_u are the dimensions of δ and u. If we suppose that these dimensions are the same and are both unity, $x=y=3$. In order to obtain $m^2\propto\lambda$ for pseudoscalars and $M\propto\lambda$ for baryons, we must have $f_M(z)\propto z^2$ and $f_B(z)\propto z^5$, respectively. Such behaviors for $f(z)$ will be singular in ν as $\nu\to 0$ unless we write

$$\nu\delta+\lambda u=\nu(\delta+\eta u),$$

i.e., unless we switch off both terms in going to the scale-invariant limit.

It should be emphasized that in our work we have assumed the absence of interactions which break scale invariance and which arise from sources other than spontaneous breaking. If these interactions have a zero matrix element between dilations and the vacuum, they will not affect our arguments since our results about the dimension of the terms which break scale invariance are derived in the limit of turning off the breaking part of the Lagrangian. Within such limits, it would seem that this work, together with that of Gell-Mann,[1] requires that scale invariance be broken by at least two terms δ and u. It is still an open question whether these terms have the same or different dimensions.

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