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Multiperipheral Nonfactorization, Signature, and Toller-Angle-Variable Cuts*

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It is shown that the assumption of a multi-Froissart-Gribov definition of signature for multiparticle amplitudes coupled with the existence of discontinuities of internal Regge-residue coupling functions in Toller-angle variables in general gives rise to a nonfactorizable expression for the asymptotic behavior of the full amplitude, even when the signatured amplitudes factorize. Factorizability occurs only in certain exceptional cases, including strict exchange degeneracy. The formalism can be cast into a two-vector form, however, which does factorize in a matrix sense. Chew-Goldberger-Low–type equations thus become 2×2 matrix equations in general. Multi-Froissart-Gribov formalism problems involving analyticity and unitarity of multiparticle amplitudes are ignored.

I. INTRODUCTION

ONE of the most common ideas connected with the multiperipheral model has been the notion of factorizability. That is, the $2\rightarrow n$ amplitude at asymptotic subenergies should be proportional to the $2\rightarrow n-1$ amplitude with a factor depending on the added vertex. On the other hand, the factorization of the Regge residues in the $2\rightarrow 2$ amplitude is proved using unitarity in the momentum transfer for the signatured partial-wave amplitudes. One would therefore suspect that the factorization of residues for the multiparticle amplitude should in principle be formulated for the signatured multi-$O(2,1)$ partial-wave amplitude. In the $2\rightarrow 2$ and $2\rightarrow 3$ amplitudes this assumption leads to factorization of the asymptotic behavior of the full amplitude. Even in the case of three final particles, however, a surprising result emerges which is connected with the discontinuity of the internal Regge residue in a variable η related to the Toller angle ω. The likelihood of these cuts has been discussed by Drummond, Landshoff, and Zakrzewski,[1] where the crucial η dependence is of the form $(-\eta)^{-\alpha_i}$. The discontinuity gives rise to an additional phase beyond that of the product of the two signature factors.[2] For the $2\rightarrow n$ case with $n>3$, more than one Toller angle ω_i enters, and the Regge coupling functions will generally have discontinuities in the corresponding variables η_i. We show that these η_i discontinuities not only lead to additional phases beyond the product of the signature factors, but also result in a nonfactorizable asymptotic behavior for the full amplitude. Factorization occurs only if these discontinuities vanish or cancel. This occurs in the cases of strict exchange degeneracy and when Regge trajectories assume physical right-signature j values. If the strengths of the η_i discontinuities are small, due to approximate exchange degeneracy, near right-signature values for trajectories, or extreme peripherality of interactions, the asymptotic full amplitude should factorize approximately. In any case, we show that factorization in a two-vector formalism is always possible. Unitarity equations of the Chew-Goldberger-Low (CGL) type[3] then become 2×2 matrix equations.

The preceding results all rely on a multi-Froissart-Gribov (MFG) definition of signature. This definition is defective. Actually, an acceptable concept of signature for a multiparticle process using analyticity has not been formulated.[4] The main difficulties have been threefold. First, one must write dispersion relations for multiparticle amplitudes taking into account Gram-determinant conditions for singularities in dependent variables. It is required that right (left) half-plane branch points in each independent subenergy reflected from the dependent-variable singularities have branch cuts in the right (left) half subenergy planes. That is, subenergy branch cuts must not cross the imaginary axes.[5] The positions of these reflected subenergy singularities are in general mutually dependent.

The second difficulty is that the signatured amplitudes must satisfy unitarity including multiparticle intermediate states. Drummond[6] has shown that the ordinary definition of signature is incompatible with three-particle intermediate states, even for the $2\rightarrow 2$ amplitude. The third difficulty involves divergent infinite-helicity sums[7] and will be discussed separately.[8]

We assume here that it is possible to write all appro-

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[1] I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Nucl. Phys. **B11**, 383 (1969).
[2] I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Phys. Letters **28B**, 676 (1969).
[3] G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters **22**, 208 (1969); I. G. Halliday, Nuovo Cimento **60A**, 177 (1969).
[4] It is not clear that the group-theoretic definition of signature for multiparticle amplitudes avoids the difficulties mentioned here, or that it does not also lead to nonfactorizable asymptotic amplitudes if it is made consistent with crossing. See M. Toller, Nuovo Cimento **53A**, 671 (1968); **54A**, 295 (1968); and Rivista, Nuovo Cimento **1**, 403 (1969). I wish to thank M. Ciafaloni for discussions on this point.
[5] For fixed timelike momentum transfers, such unwanted cuts do appear [S. Mandelstam (private communication)]. We are concerned here only with asymptotic properties of amplitudes at spacelike momentum transfers which, as far as we know, may or may not have these cuts.
[6] I. T. Drummond, Phys. Rev. **140**, 1368 (1965); **153**, 1565 (1967).
[7] R. L. Omnès and V. A. Alessandrini, Phys. Rev. **136**, 1137 (1964).
[8] J. W. Dash, preceding paper, Phys. Rev. D **3**, 1012 (1971).

priate dispersion relations. The results we achieve are admittedly based on a nonunitary theory. It would seem, however, that since the ultimate definition of signature (if it exists) will be more complicated than the one used here, only miraculous cancellations in the unitary theory will recover factorization.

In Sec. II we treat the $2 \to 4$ amplitude which is the simplest case where nonfactorization occurs. We also discuss exceptional cases in which factorization is retained, and we formulate the 2×2 matrix CGL equation. In Sec. III we treat the general $2 \to n$ case. The Appendix outlines the MFG formalism.

II. $2 \to 4$ AMPLITUDE AND MATRIX CGL EQUATION

We choose the independent variables for the $2\to 4$ amplitude with spinless particles as indicated in Fig. 1, namely, the three neighboring subenergies s_i, the three neighboring momentum transfers t_i, and the two Toller angles ω_i. In terms of the full signatured amplitudes $f_{2\to 4}{}^{\tau_1\tau_2\tau_3}(s_1s_2s_3;\, t_1t_2t_3;\, \omega_1\omega_2)$, where $\tau_i = \pm 1$, the full amplitude is [see Appendix, Eq. (A6)]

$$8f_{2\to 4}(s_1s_2s_3;\, t_1t_2t_3;\, \omega_1\omega_2) = \sum_{\tau_1\tau_2\tau_3}\sum_{\nu_1\nu_2\nu_3} \mu_1\mu_2\mu_3 \times f_{2\to 4}{}^{\tau_1\tau_2\tau_3}(\nu_1s_1, \nu_2s_2, \nu_3s_3;\, t_1t_2t_3;\, \omega_1 - \nu_{12}\pi, \omega_2 - \nu_{23}\pi)\,, \tag{2.1}$$

where

$$\nu_i = \pm 1$$

and

$$\mu_i = \begin{cases} 1, & \nu_i = 1 \\ \tau_i, & \nu_i = -1, \end{cases}$$

$$\nu_{i,i+1} = \begin{cases} 1, & \nu_i\nu_{i+1} = -1 \\ 0, & \nu_i\nu_{i+1} = 1. \end{cases}$$

For a multi-Regge pole in the signatured multipartial-wave amplitude, we have the factorized asymptotic expression for the full signatured amplitude [see Fig. 2 and Eq. (A7)]

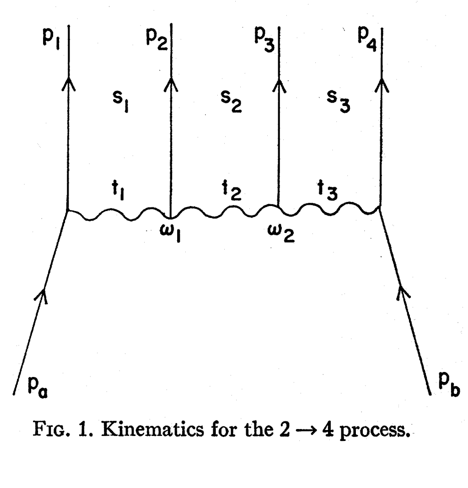

FIG. 1. Kinematics for the $2 \to 4$ process.

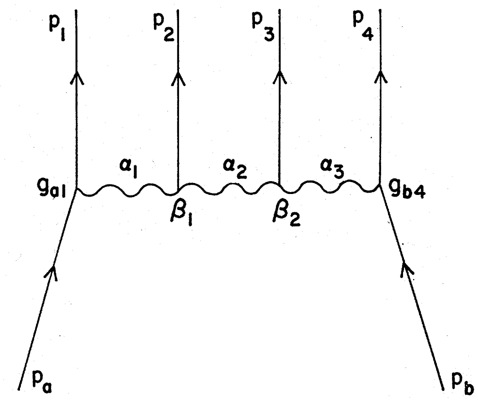

FIG. 2. Regge residue and trajectory notation for the $2 \to 4$ process.

$$f_{2\to 4}{}^{\tau_1\tau_2\tau_3}(s_1s_2s_3;\, t_1t_2t_3;\, \omega_1\omega_2) \underset{s_i\to\infty}{\sim} 8g_{a1}(t_1)g_{b4}(t_3) \times(-s_1)^{\alpha_1(t_1)}(-s_2)^{\alpha_2(t_2)}(-s_3)^{\alpha_3(t_3)} \times\beta_1(t_1\eta_1t_2)\beta_2(t_2\eta_2t_3)\,, \tag{2.2}$$

where

$$\eta_i^{-1} \sim [m_{i+1}{}^2 - t_i - t_{i+1} + 2(t_it_{i+1})^{1/2} \times(|s_i|\,|s_{i+1}|/s_is_{i+1})\cos\omega_i]\lambda^{-1}(t_it_{i+1}m_{i+1}{}^2) \sim s_{i,i+1}/s_is_{i+1}\,, \tag{2.3}$$

where $s_{i,i+1}$ is the three-particle subenergy $(p_i+p_{i+1}+p_{i+2})^2$.

The $i\epsilon$ prescriptions for the η_i are obtained as in Ref. 2. The result is that η_i has a $-i\epsilon$ prescription if the two neighboring subenergies s_i and s_{i+1} both have $-i\epsilon$ prescriptions (ν_i and $\nu_{i+1} = -1$). Otherwise η_i has a $+i\epsilon$ prescription. Since the full signatured amplitudes $f_{2\to 4}{}^{\tau_1\tau_2\tau_3}$ have only right-hand s_i cuts, we may analytically continue from $s_i < 0$, where $f_{2\to 4}{}^{\tau_i}(s_i,t_i,\omega_i)$ is real, to $s_i > 0$ with $a + i\epsilon$ prescription asymptotically using $s_i = (-s_i)e^{-i\pi}$ ($s_i > 0$). We obtain from Eqs. (2.1) and (2.2)

$$f_{2\to 4}(s_1s_2s_3;\, t_1t_2t_3;\, \omega_1\omega_2) \underset{s_i\to\infty}{\sim} \xi_1(t_1)\xi_2(t_2)\xi_3(t_3) \times s_1{}^{\alpha_1(t_1)}s_2{}^{\alpha_2(t_2)}s_3{}^{\alpha_3(t_3)}g_{a1}(t_1)g_{b4}(t_3)R_{2\to 4}(t_1t_2t_3\eta_1\eta_2), \tag{2.4}$$

where

$$R_{2\to 4} = \beta_2\left(\beta_1 - \frac{\tau_1\tau_2}{\xi_1\xi_2}\Delta\beta_1\right) - \frac{\Delta\beta_2\tau_2\tau_3}{\xi_2\xi_3}\left(\beta_1 - \frac{\tau_1}{\xi_1}\Delta\beta_1\right), \tag{2.5}$$

$$\xi_i = e^{-i\pi\alpha_i} + \tau_i\,,$$

$$\Delta\beta_i = \beta_i(t_i,\, \eta_i + i\epsilon,\, t_{i+1}) - \beta_i(t_i,\, \eta_i - i\epsilon,\, t_{i+1})\,. \tag{2.6}$$

We see that an extra phase besides that of the product of signature factors $\xi_1\xi_2\xi_3$ is present in $R_{2\to 4}$, similar to the result obtained for the $2 \to 3$ amplitude.[2] However, it is easy to see that $R_{2\to 4}$ does not factorize into any expression of the form

$$(a\beta_1+b\Delta\beta_1)(c\beta_2+d\Delta\beta_2)\,.$$

For calculating (bd) and $(ad)(bc)/(ac)$, we get both $\tau_1\tau_2\tau_3/\xi_1\xi_2\xi_3$ and $\tau_1\tau_2{}^2\tau_3/\xi_1\xi_2{}^2\xi_3$, which are not equal.

Now let us assume that $\Delta\beta_2=0$. This happens, for example, when α_3 becomes physical. We must also assume its value is of right signature so that $\xi_3\neq0$. We then get a factorized expression for $R_{2\to4}$:

$$R_{2\to4}|_{\Delta\beta_2=0}=\beta_2(\beta_1-(\tau_1\tau_2/\xi_1\xi_2)\Delta\beta_1)\equiv\beta_2R_{2\to3}\,. \quad (2.7)$$

We recognize the second factor as the expression obtained in Ref. 2 for the Regge residue of the $2\to3$ amplitude, namely $f_{2\to3}\sim g_{a1}g_{b3}\xi_1\xi_2s_1{}^{\alpha_1}s_2{}^{\alpha_2}R_{2\to3}$.

Next, assume that there are two exactly exchange-degenerate trajectories $\alpha_3{}^+$ and $\alpha_3{}^-$ with equal residues and

$$\xi_3{}^\pm=e^{-i\pi\alpha_3}+\tau_3{}^\pm \quad (\tau_3{}^\pm=\pm1)\,,$$

respectively. The τ_3 sum in Eq. (2.1) produces a cancellation of the $\Delta\beta_2$ terms in the full amplitude, again yielding factorization. We obtain

$$f_{2\to4}\sim\xi_1\xi_2s_1{}^{\alpha_1}s_2{}^{\alpha_2}s_3{}^{\alpha_3}g_{a1}g_{b4}R_{2\to4}{}^{\mathrm{EX}}\,, \quad (2.8)$$

where

$$\begin{aligned}R_{2\to4}{}^{\mathrm{EX}}&=\beta_2(\xi_3{}^++\xi_3{}^-)[\beta_1-(\tau_1\tau_2/\xi_1\xi_2)\Delta\beta_1]\\&\quad-\Delta\beta_2(\tau_3{}^++\tau_3{}^-)(\tau_2/\xi_2)[\beta_1-(\tau_1/\xi_1)\Delta\beta_1]\\&=2e^{-i\pi\alpha_3}\beta_2[\beta_1-(\tau_1\tau_2/\xi_1\xi_2)\Delta\beta_1]\,. \end{aligned}\quad (2.9)$$

We have seen that factorization of the $2\to4$ amplitude has been spoiled by the presence of the $\Delta\beta_i$ discontinuity terms. If $\Delta\beta_i$ is small for some dynamical reason, we will obtain approximate factorization. Two obvious cases are approximate exchange degeneracy and trajectories assuming approximate right-signature physical values. Another possibility arises from a peripherality argument. If $\beta_i(t_i\eta_it_{i+1})$ is highly peaked in t_i, t_{i+1} so that only $t_i\approx t_{i+1}\equiv\langle t\rangle\ll m_{i+1}{}^2$ contributes, we see from Eq. (2.3) that η_i becomes approximately independent of the $\cos\omega_i$ term. We get

$$\eta_i{}^{-1}\sim\frac{1}{m_{i+1}{}^2}\left[1+\frac{2\langle t\rangle}{m_{i+1}{}^2}\left(1+\frac{|s_i|\,|s_{i+1}|}{s_is_{i+1}}\cos\omega_i\right)\right]$$

so that $\Delta\beta_i\approx0$.

We next exhibit the two-vector factorization. We set

$$S_{2\to3}=(\tau_2/\xi_2)[\beta_1-(\tau_1/\xi_1)\Delta\beta_1]$$

and

$$V_{2\to3}=\begin{pmatrix}R_{2\to3}\\S_{2\to3}\end{pmatrix}. \quad (2.10)$$

We obtain from Eqs. (2.5), (2.7), and (2.10)

$$R_{2\to4}=\beta_2R_{2\to3}-(\tau_3/\xi_3)\Delta\beta_2S_{2\to3} \quad (2.11)$$

In Sec. III we will obtain a definition for $S_{2\to4}$ in such a way that

$$\begin{aligned}V_{2\to4}&=\begin{pmatrix}R_{2\to4}\\S_{2\to4}\end{pmatrix}\\&=\begin{pmatrix}\beta_2 & -(\tau_3/\xi_3)\Delta\beta_2\\(\tau_3/\xi_3)\beta_2 & -(\tau_3/\xi_3)\Delta\beta_2\end{pmatrix}\begin{pmatrix}R_{2\to3}\\S_{2\to3}\end{pmatrix},\end{aligned} \quad (2.12)$$

or

$$V_{2\to4}=K(\text{vertex }2)V_{2\to3}\,. \quad (2.13)$$

We will also show that this matrix factorization is a general property of the $2\to n$ amplitude. Thus we will obtain

$$V_{2\to n}=K(\text{vertex }n-2)V_{2\to n-1}\,. \quad (2.14)$$

With this accomplished, we may imagine writing equations of the CGL type.[3] From unitarity, we have (suppressing the variable dependence which duplicates that of the CGL equation)

$$\mathrm{Im}f_{2\to2}=\sum_n\int f_{2\to n}f_{2\to n}{}^*\rho_n\,. \quad (2.15)$$

We consider the following matrix equation, where the 11 element is the unitarity equation:

$$\begin{aligned}\begin{pmatrix}\mathrm{Im}f_{2\to2} & \cdots\\ \cdots & \cdots\end{pmatrix}&=\sum_n\int\begin{pmatrix}f_{2\to n}f_{2\to n}{}^* & f_{2\to n}\tilde f_{2\to n}{}^*\\ \tilde f_{2\to n}f_{2\to n}{}^* & \tilde f_{2\to n}\tilde f_{2\to n}{}^*\end{pmatrix}\rho_n\\&=\sum_n\int F_{2\to n}F_{2\to n}{}^\dagger\rho_n\,,\end{aligned} \quad (2.16)$$

where $\tilde f_{2\to n}$ and $F_{2\to n}$ are defined by

$$F_{2\to n}=\begin{pmatrix}f_{2\to n}\\ \tilde f_{2\to n}\end{pmatrix}\sim g_{a1}g_{bn}\Big(\prod_{i=1}^{n-1}\xi_is_i{}^{\alpha_i}\Big)V_{2\to n}\,. \quad (2.17)$$

But from Eq. (2.14)

$$V_{2\to n}V_{2\to n}{}^\dagger=K(V_{2\to n-1}V_{2\to n-1}{}^\dagger)K^\dagger\,. \quad (2.18)$$

Suppressing irrelevant factors and "backing up a rung,"[3] we obtain

$$\begin{aligned}B&\equiv\sum_n\int\cdots V_{2\to n}V_{2\to n}{}^\dagger\\&=\sum_n\int\cdots K(V_{2\to n-1}V_{2\to n-1}{}^\dagger)K^\dagger\\&=B_0+\int KBK^\dagger\,.\end{aligned} \quad (2.19)$$

This is the generalized CGL equation, a 2×2 matrix equation. Once the matrix B has been calculated, we add back the last rung and obtain $\mathrm{Im}f_{2\to2}$ from the 11 matrix element of the result in Eq. (2.16).

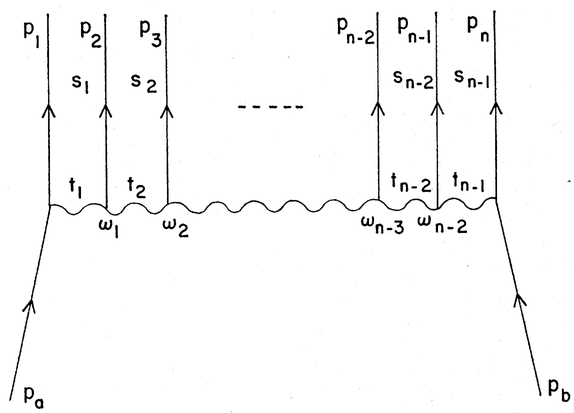

FIG. 3. Kinematics for the $2\to n$ process.

III. $2\to n$ AMPLITUDE FOR GENERAL $n\geqslant 4$

In this section we generalize the results of Sec. I. In particular, we prove that the full asymptotic $2\to n$ amplitude factorizes in the matrix sense described in Sec. II, and that strict exchange degeneracy recovers factorization in the multiplicative sense.

We consider the $2\to n$ amplitude for spinless particles $f=f_{2\to n}$ with kinematics defined in Fig. 3 as the $n-1$ neighboring subenergies s_i, the $n-1$ neighboring momentum transfers t_i, and the $n-2$ Toller angles w_i. From Eq. (A6) of the Appendix, we see that the full amplitude is the sum of the signatured full amplitudes, each of which factorizes for a given multi-Regge pole. We define $R_{2\to n}$ by

$$f\sim g_{a1}g_{bn}\Big(\prod_{i=1}^{n-1}\xi_i s_i^{\alpha_i}\Big)R_{2\to n}\,, \tag{3.1}$$

where, as before, $\xi_i=e^{-i\pi\alpha_i}+\tau_i$. Pictorially, $R_{2\to n}$ results from the sum of signatured amplitudes with both signs for all subenergies s_i (see Fig. 4). We also define $S_{2\to n}$ using the sum of signatured amplitudes with both signs for all subenergies except for s_{n-1}, which is negative. Explicitly, $S_{2\to n}$ is defined by [see Eq.

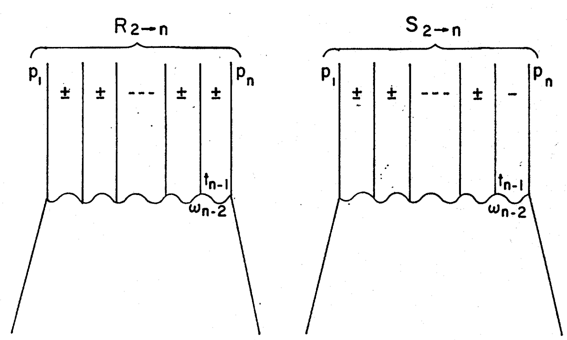

FIG. 4. Diagrammatic definition of $R_{2\to n}$ and $S_{2\to n}$. See Eqs. (3.1) and (3.2).

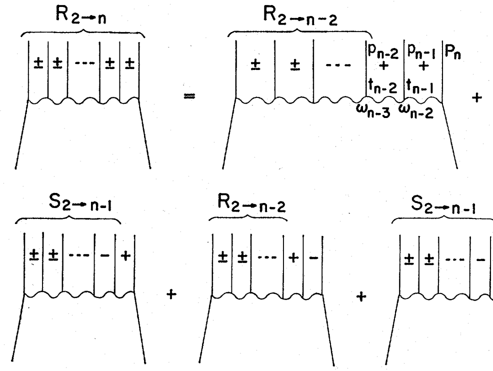

FIG. 5. Diagrammatic equation for $R_{2\to n}$ leading to Eq. (3.5).

(A6)]

$$\tau_{n-1}\sum_{\nu_1\cdots\nu_{n-2}}\mu_1\cdots\mu_{n-2}f^{\{\tau_i\}}(\{\nu_i s_i,t_i,\omega_i-\nu_{i,i+1}\pi\})\big|_{\nu_{n-1}=-1}$$
$$\sim 2^{n-1}g_{a1}g_{bn}\Big(\prod_{i=1}^{n-2}\xi_i s_i^{\alpha_i}\Big)s_{n-1}^{\alpha_{n-1}}(\xi_{n-1}S_{2\to n})\,. \tag{3.2}$$

We also have

$$\sum_{\nu_1\cdots\nu_{n-2}}\mu_1\cdots\mu_{n-2}f^{\{\tau_i\}}(\{\nu_i s_i,t_i,\omega_i-\nu_{i,i+1}\pi\})\big|_{\nu_{n-1}=1}$$
$$\sim 2^{n-1}g_{a1}g_{bn}\Big(\prod_{i=1}^{n-2}\xi_i s_i^{\alpha_i}\Big)s_{n-1}^{\alpha_{n-1}}(\beta_{n-2}R_{2\to n-1})\,. \tag{3.3}$$

Using Eqs. (3.1) and (A6), we obtain

$$\xi_{n-1}R_{2\to n}=\beta_{n-2}R_{2\to n-1}+\xi_{n-1}S_{2\to n}\,. \tag{3.4}$$

Using similar reasoning (see Fig. 5), we get

$$\begin{aligned}\xi_{n-2}\xi_{n-1}R_{2\to n}=&\,\beta_{n-3}\beta_{n-2}R_{2\to n-2}+\xi_{n-2}\beta_{n-2}S_{2\to n-1}\\&+\tau_{n-1}\beta_{n-3}\beta_{n-2}R_{2\to n-2}\\&+(\beta_{n-2}-\Delta\beta_{n-2})\xi_{n-2}\tau_{n-1}S_{2\to n-1}\\=&\,\xi_{n-1}\beta_{n-2}(\beta_{n-3}R_{2\to n-2}+\xi_{n-2}S_{2\to n-1})\\&-\Delta\beta_{n-2}\xi_{n-2}\tau_{n-1}S_{2\to n-1}\,.\end{aligned} \tag{3.5}$$

Using Eq. (3.4), we get

$$R_{2\to n}=\beta_{n-2}R_{2\to n-1}-\Delta\beta_{n-2}(\tau_{n-1}/\xi_{n-1})S_{2\to n-1}\,. \tag{3.6}$$

From Eqs. (3.4) and (3.6), we obtain

$$\begin{aligned}\xi_{n-1}S_{2\to n}=&\,\xi_{n-1}[\beta_{n-2}R_{2\to n-1}\\&-\Delta\beta_{n-2}(\tau_{n-1}/\xi_{n-1})S_{2\to n-1}]-\beta_{n-2}R_{2\to n-1}\\=&\,\tau_{n-1}(\beta_{n-2}R_{2\to n-1}-\Delta\beta_{n-2}S_{2\to n-1})\end{aligned} \tag{3.7a}$$

or, using Eqs. (3.1) and (3.2) and Fig. 6, we have

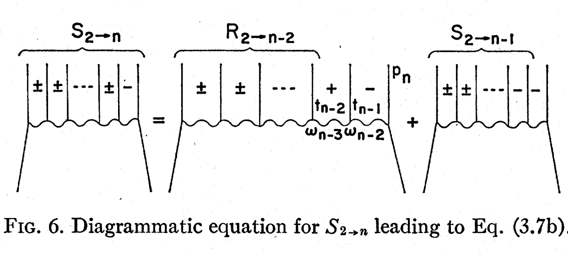

FIG. 6. Diagrammatic equation for $S_{2\to n}$ leading to Eq. (3.7b).

directly

$$\xi_{n-2}\xi_{n-1}S_{2\to n}=\beta_{n-3}\beta_{n-2}\tau_{n-1}R_{2\to n-2}+\xi_{n-2}(\beta_{n-2}-\Delta\beta_{n-2})\tau_{n-1}S_{2\to n-1}\,, \quad (3.7b)$$

and using Eq. (3.4), we again obtain Eq. (3.7a).

Finally we obtain

$$\begin{pmatrix}R_{2\to n}\\ S_{2\to n}\end{pmatrix}=\begin{pmatrix}\beta_{n-2} & -(\tau_{n-1}/\xi_{n-1})\Delta\beta_{n-2}\\ (\tau_{n-1}/\xi_{n-1})\beta_{n-2} & -(\tau_{n-1}/\xi_{n-1})\Delta\beta_{n-2}\end{pmatrix}\times\begin{pmatrix}R_{2\to n-1}\\ S_{2\to n-1}\end{pmatrix}, \quad (3.8)$$

which is of the form Eq. (2.14). If $\Delta\beta_{n-2}/\xi_{n-1}=0$, we obtain multiplicative factorization $R_{2\to n}=\beta_{n-2}R_{2\to n-1}$. If some $\Delta\beta_i/\xi_{i+1}=0$ or if exchange degeneracy occurs in the $i+1$ rung, the amplitude factors multiplicatively into two parts, depending on variables to the right and left of that rung.

We explicitly consider the effect of exchange degeneracy in the $n-1$ rung. We have from Eq. (3.6)

$$f_{2\to n}\sim g_{a1}g_{bn}\left(\prod_{i=1}^{n-2}\xi_i s_i^{\alpha_i}\right)\times s_{n-1}^{\alpha_{n-1}}\left[(\xi_{n-1}^{+}+\xi_{n-1}^{-})\beta_{n-2}R_{2\to n-1}-\Delta\beta_{n-2}(\tau_{n-1}^{+}+\tau_{n-1}^{-})S_{2\to n-1}\right].$$

But $\xi_{n-1}^{+}+\xi_{n-1}^{-}=2e^{-i\pi\alpha_{n-1}}$ and $\tau_{n-1}^{+}+\tau_{n-1}^{-}=0$, proving multiplicative factorization.

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix we briefly outline the multi-Froissart-Gribov formalism for the $2\to n$ amplitude, since most of it has already been presented.[7,9,10] The procedure parallels the $2\to 2$ case. The kinematics are illustrated in Fig. 3. We use the neighboring subenergies s_i interchangeably with the more conventional z_i, and take the case with all particles spinless.

The full amplitude $f(\{s_i,t_i,\omega_i\})$ with Fourier components $f_{\{M_i\}}(\{s_i,t_i\})$ is expanded in a multi-$O(2,1)$ expansion[11] in terms of multi-partial-wave amplitudes $f_{\{M_i\}}{}^{\{j_i\}}(\{t_i\})$. We separate out the right- and left-hand subenergy-plane cuts of $f_{\{M_i\}}(\{s_i,t_i\})$ (but see the Introduction):

$$f_{\{M_i\}}(\{s_i,t_i\})=\sum_{\{X_i=R,L\}}f_{\{M_i\}}{}^{\{X_i\}}(\{s_i,t_i\})\,, \quad (A1)$$

where an R (L) ith place superscript in $f_{\{M_i\}}{}^{X_1\cdots X_{n-1}}$ indicates the presence of only right (left) half s_i-plane cuts.

Writing a dispersion relation for each $f_{\{M_i\}}{}^{\{X_i\}}(\{s_i,t_i\})$ function in the s_i at fixed $t_i\leq 0$, we have (neglecting kinematic singularities)

$$f_{\{M_i\}}{}^{\{X_i\}}(\{s_i,t_i\})=\frac{1}{\pi^{n-1}}\int_{C_1}\frac{ds_1'}{s_1'-s_1}\times\int_{C_2(s_1')}\frac{ds_2'}{s_2'-s_2}\cdots\int_{C_{n-1}(s_1'\cdots s_{n-2}')}\frac{ds_{n-1}'}{s_{n-1}'-s_{n-1}}\times\Delta_{\{M_i\}}{}^{\{X_i,C_i\}}(\{s_i',t_i\})\,, \quad (A2)$$

where the discontinuities are taken over subenergy contours C_i that are in general mutually interdependent due to Gram-determinant conditions, and which lie in the R (L) half s_i plane accordingly as $X_i=R$ (L). Since many singularities are present, C_i generally represents unions of contours. If we now insert Eqs. (A2) and (A1) into the multi-$O(2,1)$ expansion inverse formula for $f_{\{M_i\}}{}^{\{j_i\}}(\{t_i\})$,[11] and use the integral identity

$$e_{\lambda\mu}{}^{j}(z)=e^{\pm i\pi(j-\lambda)}\left[-\frac{\sin\pi(j-\lambda)}{\pi}\int_1^\infty\frac{e_{\lambda\mu}{}^{j}(z')dz'}{z'-z}-\frac{1}{2}\int_{-1}^{1}\frac{d_{\lambda-\mu}{}^{j}(-z')dz'}{z'-z}\right], \quad (A3)$$

we obtain a formula for $f_{\{M_i\}}{}^{\{j_i\}}(\{t_i\})$ in terms of integrals over the $\Delta_{\{M_i\}}{}^{\{X_i,C_i\}}$. [In doing this, one must add in the finite integrals $\int_{-1}^{1}d^{j_i}\cdots$ in appropriate places in order to use Eq. (A3). Doing so makes no contribution to the principal-series integrals over $\mathrm{Re}\,j_i=-\frac{1}{2}$ because of symmetry under $j_i\to -j_i-1$.[9]] For a left half s_i-plane contour C_i, the s_i variable in $\Delta_{\{M_i\}}{}^{\{X_i,C_i\}}(\{s_i,t_i\})$ will get a minus sign, and a factor $e^{-i\pi j_i}$ will occur. The signatured multi-partial-wave amplitudes $f_{\{M_i\}}{}^{\{j_i,\tau_i\}}(\{t_i\})$ are defined by replacing all such $e^{-i\pi j_i}$ factors by τ_i ($\tau_i=\pm 1$). The result is the multi-Froissart-Gribov formula

$$f_{\{M_i\}}{}^{\{j_i,\tau_i\}}(\{t_i\})=\sum_{\{X_i=R,L\}}\prod_{i=1}^{n-1}\frac{1}{\pi}\int_{\tilde{C}_i}dz_i'e_{M_{i-1}',M_i'}{}^{j_i}(z_i')\times W\Delta_{\{M_i\}}{}^{\{X_i,\tilde{C}_i\}}(\{\nu_i s_i',t_i\})\,, \quad (A4)$$

where the $\tilde{C}_i$ are the C_i, but flipped over into the right-half s_i plane if C_i lies in the left-half s_i plane. Also,

$$\nu_i=\begin{cases}1, & X_i=R\\ -1, & X_i=L\end{cases},\quad W=\prod_{i=1}^{n-1}\tau_i^{(1-\nu_i)/2}\prod_{i=1}^{n-2}e^{i\pi M_i\nu_{i,i+1}},$$

$$\nu_{i,i+1}=\begin{cases}1, & \nu_i\nu_{i+1}=-1\\ 0, & \nu_i\nu_{i+1}=1,\end{cases}\qquad M_1'=\begin{cases}-M_1, & \nu_1=\nu_2=-1\\ +M_1, & \text{otherwise}\end{cases}$$

and $M_{i+1}'=\nu_{i,i+1}M_i'$.

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[9] J. F. Boyce, J. Math. Phys. **8**, 675 (1967).

[10] J. B. Hartle and C. E. Jones, Phys. Rev. **184**, 1564 (1969); T. K. Gaisser and C. E. Jones, *ibid.* **184**, 1602 (1969).

[11] See, M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964); M. Toller, Nuovo Cimento **37**, 631 (1965); and N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

This simply means that a factor τ_i occurs whenever the corresponding ν_i in the argument of $\Delta_{\{M_i\}}{}^{\{X_i,\tilde{C}_i\}}$ in (A4) is -1. The internal helicity-flip factors $e^{i\pi M_i}$ appear whenever neighboring parameters ν_i, ν_{i+1} have opposite signs in $\Delta_{\{M_i\}}{}^{\{X_i,\tilde{C}_i\}}$.

The full signatured amplitude $f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\})$ is defined as having $f_{\{M_i\}}{}^{\{j_i,\tau_i\}}(\{t_i\})$ as its $O(2,1)$ partial waves. It has only right-hand cuts in all subenergies s_i, and is expressed in terms of the $f_{\{M_i\}}{}^{\{X_i\}}(\{s_i,t_i\})$ amplitudes by

$$f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\}) = \sum_{\{X_i=R,L\}} \mu_1\cdots\mu_{n-1} f^{\{X_i\}}(\{\nu_i s_i,\, t_i,\, \omega_i-\nu_{i,i+1}\pi\}),$$

where

$$\mu_i = \begin{cases} 1, & \nu_i=1 \\ \tau_i, & \nu_i=-1 \end{cases} \tag{A5}$$

and ν_i and $\nu_{i,i+1}$ are defined above. The $(\omega_i-\nu_{i,i+1}\pi)$ dependence is due to the extra $e^{i\pi M_i \nu_{i,i+1}}$ factors in the Fourier M_i sums for $f^{\{X_i\}}$.

The full amplitude is expressed in terms of the signatured full amplitudes by

$$f(\{s_i,t_i,\omega_i\}) = \frac{1}{2^{n-1}} \sum_{\{\tau_i\}} \sum_{\{\nu_i\}} \mu_1\cdots \times\mu_{n-1} f^{\{\tau_i\}}(\{\nu_i s_i,\, t_i,\, \omega_i-\nu_{i,i+1}\pi\}). \tag{A6}$$

A multi-Regge pole occurs as a set of j_i-plane poles in the signatured multi-partial-wave amplitude $f_{\{M_i\}}{}^{\{j_i,\tau_i\}}(\{t_i\})$, since that is the amplitude which supposedly can be analytically continued in the j_i variables. The multi-Regge pole also occurs with a factorized residue (see the Introduction). The resulting asymptotic behavior of the full signatured amplitude is therefore factorized as well,

$$f^{\{\tau_i\}}(\{s_i,t_i,\omega_i\}) \sim 2^{n-1} g_{a1}(t_1) g_{bn}(t_{n-1}) \prod_{i=1}^{n-1} (-s_i)^{\alpha_i(t_i)} \times \prod_{j=1}^{n-2} \beta_j(t_j,\eta_j,t_{j+1}). \tag{A7}$$

Use of Eq. (A6) then yields the asymptotic form for the full amplitude, as described in the text. For a given multi-Regge pole, the $\{\tau_i\}$ sum is absent.

Note added in proof. Professor I. Halliday has emphasized to us that cuts in the subenergies s_i arising from unitarity in dependent variables and Gram-determinant conditions render the analytic continual tion of some of the terms in Eq. (A6) to the physical region to be more complicated than the asymptotic s_i continuations used here. There is the possibility that mixed $i\epsilon$ prescriptions for the η_i in these anomolous terms may recover factorization for the full amplitude in the ordinary sense. For a perturbation theory calculation which factorizes, see D. K. Campbell, Phys. Rev. **188**, 2471 (1969).

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Dimension of Operators in Broken Scale Invariance

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It is shown that if the Hamiltonian can be split into a part which is scale invariant and a part which breaks scale invariance by means of a dilaton, then, if the latter part has a unique dimension, this dimension must be 1 if the vacuum does not realize the invariance under scaling. This implies that there must exist a term which breaks scale invariance in addition to that which breaks chiral $SU(3)\times SU(3)$ symmetry in order to avoid a contradiction with Gell-Mann's argument.

I. INTRODUCTION

THE study of the relation of scaling transformations to the dynamics of strong interactions and to deep inelastic electroproduction has been the subject of many recent investigations.[1] In the study of the dynamical consequences of broken scale invariance, two main approaches have in general been pursued. These may be classified as to whether it is assumed that the vacuum is or is not invariant under scale transformations. In the former case, renormalized field theories have been the focus of attention and many new and important results concerning the relationship between

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[1] Good reviews on the subject to the present are given by the following: (a) G. Mack and A. Salam, Ann. Phys. (N. Y.) **53**, 174 (1969); (b) M. Gell-Mann, Symmetry Violation in Hadron Physics, Summer School of Theoretical Physics, University of Hawaii, 1969 (unpublished); and (c) P. Carruthers, Phys. Rept. (to be published).