Divergent Regge Helicity Sums, Distributions, and Toller Angles*

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Divergent infinite-helicity sums appearing in complex J continuations of multiparticle unitarity equations including the Faddeev equations, and in the multi-Froissart-Gribov formalism are made convergent by regarding the amplitude as a distribution in the Toller azimuthal angles conjugate to the helicities. Some aspects of dynamical calculations using this formalism are discussed.

I. INTRODUCTION

THE interpretation of divergent infinite-helicity sums which arise in considering complex angular momenta in multiparticle amplitudes has been a serious problem. Such sums arise in the extension of partial-wave Faddeev equations and partial-wave unitarity equations for general processes to complex J, as well as in the straightforward (though nonunitary) multi-Froissart-Gribov (MFG) formulation of signature for production processes. We present a formalism which treats divergent helicity sums in the sense of a certain type of distribution, with elliptic theta functions used as test smearing functions. Such sums are treated as boundary values of functions analytic in upper half-planes of certain auxiliary parameters τ_i. It is argued that physical-J unitarity equations as well as Regge residue helicity sums, arising in the MFG formalism, do not require such an interpretation. Dynamical calculations utilizing unitarity at unphysical J can be done formally for $\mathrm{Im}\tau>0$ with the limit $\mathrm{Im}\tau\to 0$ taken at the end.

Our philosophy is that multiparticle amplitudes are indeed square integrable in the multiple $O(2,1)$ group parameters. Although the MFG representation apparently leads to divergent-helicity sums, consistency implies that these divergences cancel out when integrated over the lines $\mathrm{Re}J_i=-\frac{1}{2}$. Regularizing these sums on displaced background integrals would seem to be necessary to obtain multi-Regge behavior for $\mathrm{Re}\alpha_i<-\frac{1}{2}$. Similarly, with explicitly convergent helicity sums in Faddeev and general unitarity equations at complex J, Regge parameters may at least in principle be calculated directly from unitarity.

Actually, the occurrence of distributions is quite common when representations of noncompact groups are considered. Traces of representations and Plancherel-like formulas for noncompact groups are mathematically handled by introducing a set of smearing functions, called the group ring. Well-defined quantities are obtained by smearing the original distribution expressions with group-ring elements.[1]

As a simple example, the trace of an $O(3)$ representation function is (after rotating coordinates to a frame where the axis of rotation points along the z direction)

$$\mathrm{Tr}D^J(g)=\sum_{M=-J}^{J}e^{iM\phi}=\frac{\sin[(J+\frac{1}{2})\phi]}{\sin\frac{1}{2}\phi},\quad g\in O(3).$$

However, the trace of a discrete-series representation of $O(2,1)$ involves infinite-helicity sums. For suitable $g\in O(2,1)$ we obtain (after a boost to the appropriate rest frame[1a])

$$\mathrm{Tr}D^k(g)=\sum_{M=k}^{\infty}e^{iM\phi}=\pi\delta(\phi)+\text{finite}\qquad(0\leq\phi<2\pi,\ k\geqslant 0\ \text{integer}).$$

We see that the infinite-helicity sum has led to a distribution in an azimuthal angle.

In Sec. II we outline the MFG formalism and examine the resulting divergent-helicity sums. In Sec. III we consider the formalism of distributions defined on a class of elliptic theta functions. Section IV is concerned with applying the formalism to multiparticle amplitudes.

II. MFG FORMALISM

We consider the expansion of the amplitude for the multiparticle process $2\to n$ of $n+2$ spinless particles, assumed to be square integrable over the group $O(2,1)$ corresponding to an intermediate link in the multiparticle chain. We have[2]

$$f(g,v)=\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\tilde{\mu}\,(J)\sum_{M,M'=-\infty}^{\infty}D_{M,M'}{}^J(g)f_{M,M'}{}^J(v)+\sum_{J=0}^{\infty}\sum_{|M|,|M'|>J}\hat{D}_{M,M'}{}^J(g)\hat{f}_{M,M'}{}^J(v),\quad(2.1)$$

where M and M' are interpretable in the multiperipheral model as neighboring Regge helicities, $g\in O(2,1)$, and v represents the remaining variables. The form of $d\tilde{\mu}(J)$ will not concern us. The discrete series is composed solely of nonsense terms ($J<|M|,|M'|$). The in-

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[1] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964), Chaps. 13, 14.

[1a] W. Rühl, Rockefeller University report (unpublished); G. Fuchs and P. Renovard, J. Math. Phys. 11, 2617 (1970).

version formula for the partial-wave amplitude $f_{M,M'}{}^{J}(v)$ is[2]

$$f_{M,M'}{}^{J}(v)=\int_{O(2,1)} D_{M,M'}{}^{J*}(g)f(g,v)d\mu(g)\,. \quad (2.2)$$

Certain conditions must obviously exist for the M, $M'\to\infty$ behavior of $f_{M,M'}{}^{J}(v)$ so that the helicity sums in the expansion formula [Eq. (2.1)] converge. It is thus probable that some arbitrary assumption for the form of $f_{M,M'}{}^{J}(v)$ will lead to divergent-helicity sums. That is in fact what happens here. We remove the most obvious ζ-kinematic singularities, defining amplitudes $\tilde{f}_{M,M'}(\zeta,v)$ by

$$K_{M,M'}(\zeta)\tilde{f}_{M,M'}(\zeta,v)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{iM\phi}e^{iM'\phi'}f(\phi\zeta\phi',v)d\phi d\phi', \quad (2.3)$$

where $g=(\phi,\zeta,\phi')$ and

$$K_{M,M'}(\zeta)=\left(\frac{1+\zeta}{2}\right)^{(M+M')/2}\left(\frac{1-\zeta}{2}\right)^{|M-M'|/2}.$$

We write a dispersion relation for $\tilde{f}_{M,M'}(\zeta,v)$ for fixed v (assuming no subtractions)

$$\tilde{f}_{M,M'}(\zeta,v)=\sum_i\int_{C_i}\Delta_{M,M'}{}^{i}(\zeta',v)\frac{d\zeta'}{(\zeta'-\zeta)}, \quad (2.4)$$

where the C_i are contours which generally lie in the complex ζ' plane. Hence, we obtain

$$f(\phi\zeta\phi',v)=\sum_i\int_{C_i}\sum_{MM'}K_{M,M'}(\zeta)\Delta_{M,M'}{}^{i}(\zeta',v)\times e^{-iM\phi}e^{-iM'\phi'}\frac{d\zeta'}{\zeta'-\zeta} \quad (2.5)$$

and, using integral relations for $e_{M,M'}{}^{J}(\zeta)$ functions along with the symmetry of the principal-series integral under $J\to -J-1$, we obtain[2–4]

$$f_{M,M'}{}^{J\tau}(v)=\sum_i\int_{\tilde{C}_i}K_{M,M'}(\pm\zeta')\Delta_{M,M'}{}^{i}(\pm\zeta',v)\times e_{M,\pm M'}{}^{J}(\zeta')\tau_{M,M'}{}^{i}d\zeta', \quad (2.6)$$

where the contours $\tilde{C}_i$ have $\mathrm{Re}\zeta'>0$. All $e^{-i\pi J}$ factors have been replaced by $\tau_i=\pm1$, and $\tau_{M,M'}{}^{i}\propto\tau_i$ may also have factors $e^{i\pi M}$ or $e^{i\pi M'}$ (see Ref. 4 for a more complete discussion of the FG formalism).

Equation (2.6) is the Froissart-Gribov formula. If a multiple expansion is made in angular momenta along the multiperipheral chain, we call the result a MFG expansion.[5]

Resumming the partial waves, we obtain

$$f(\phi\zeta\phi',v)=\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\tilde{\mu}(J)\sum_i\int_{\tilde{C}_i}g^{iJ\tau_i}(\phi\phi'\zeta\zeta',v)+\text{discrete}, \quad (2.7)$$

where

$$g^{iJ\tau_i}(\phi\phi'\zeta\zeta',v)=\sum_{M,M'=-\infty}^{\infty}e^{-iM\phi}e^{-iM'\phi'}d_{M,M'}{}^{J}(\zeta)\times e_{M,\pm M'}{}^{J}(\zeta')\tau_{M,M'}{}^{i}K_{M,M'}(\pm\zeta')\Delta_{M,M'}{}^{i}(\pm\zeta',v)\,. \quad (2.8)$$

As Omnès and Alessandrini noticed,[3] the exponentially increasing behavior of $d_{M,M'}{}^{J}(\zeta)e_{M,M'}{}^{J}(\zeta')$ for arbitrary ζ, ζ' as M, $M'\to\infty$ leads to divergence of the sum in Eq. (2.8). The product $K_{M,M'}(\pm\zeta')\Delta_{M,M'}{}^{i}\times(\pm\zeta',v)$ is well behaved as $M,M'\to\infty$ since the sums in Eq. (2.5) are assumed to converge. At any rate, its $M,M'\to\infty$ behavior is dynamically determined, and cannot *a priori* be assumed to give rise to the cancellations needed for convergence.

II. DISTRIBUTION FORMALISM

We have been led to consider sums of the form

$$g(\phi)=\sum_{M=-\infty}^{\infty}a_Me^{-iM\phi}, \quad (3.1)$$

where $|a_M|\sim O(e^{\lambda M})$ as $M\to\infty$. We shall treat $g(\phi)$ as a distribution on the class of elliptic theta functions $\vartheta_3(\frac{1}{2}(\phi'-\phi),q)$ where $q=e^{i\pi\tau}$ is a complex parameter with $|q|<1$ ($\mathrm{Im}\tau>0$). These functions are defined by[6]

$$\vartheta_3(z,q)=\sum_{M=-\infty}^{\infty}e^{2iMz}q^{M^2}. \quad (3.2)$$

The convergence factor q^{M^2} damps all divergences of exponential order $e^{\lambda M}$ in M, and the theta function is an entire function of z and is analytic in τ for $\mathrm{Im}\tau>0$. If we set $z=\frac{1}{2}(\phi'-\phi)$ where $\phi,\phi'\in(0,2\pi)$, we obtain

$$\vartheta_3(\tfrac{1}{2}(\phi'-\phi),q)=\sum_{M=-\infty}^{\infty}e^{iM(\phi'-\phi)}q^{M^2}. \quad (3.3)$$

As $q\to1$, we have

$$\vartheta_3(\tfrac{1}{2}(\phi'-\phi),q)\to 2\pi\delta(\phi'-\phi)\,.$$

[2] M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964); M. Toller, Nuovo Cimento **37**, 631 (1965). See also M. Toller, *ibid.* **54A**, 295 (1968). Here, Toller considers the amplitude as a linear functional in order to extend group expansions to non-L^2 functions.

[3] R. L. Omnès and V. A. Alessandrini, Phys. Rev. **136**, B1137 (1964).

[4] J. B. Hartle and C. E. Jones, Phys. Rev. **184**, 1564 (1969); T. K. Gaisser and C. E. Jones, *ibid* **184** 1602 (1969); J. W. Dash, following paper, Phys. Rev. D **3**, 1016 (1971).

[5] We ignore the formidable difficulties involved with Gram-determinant conditions for singularities in dependent variables and involved in defining unitary signatured amplitudes. See I. T. Drummond, Phys. Rev. **140**, B1368, (1965); **153**, 1565 (1967). We are concerned here only with the convergence properties of the helicity sums.

We define smeared functions $g^q(\phi)$ by

$$g^q(\phi)=\frac{1}{2\pi}\int_0^{2\pi}\vartheta_3(\tfrac{1}{2}(\phi'-\phi),q)g(\phi')d\phi'. \qquad (3.4)$$

We have

$$g^q(\phi)=\sum_{M=-\infty}^{\infty}a_M q^{M^2}e^{-iM\phi}. \qquad (3.5)$$

This sum now converges for all ϕ, and is entire in ϕ. Thus all singularities in ϕ of $g(\phi)$ have been removed (i.e., pushed off to ∞).

IV. APPLICATION TO MULTIPARTICLE SCATTERING

We choose the independent variables for the $2\to n$ amplitude for spinless particles to be the $n-1$ neighboring subenergies $s_i=(p_i+p_{i+1})^2$, the $n-1$ neighboring momentum transfers t_i, and the $n-2$ Toller angles[7] ϕ_i (see Fig. 1). The Toller angles are the conjugate variables to the "Regge" helicities M_i which are responsible for the divergent-helicity sums (we set $M_0=M_{n-1}=0$). Smearing the amplitude in each Toller angle ϕ_i with $\vartheta_3(\frac{1}{2}(\phi_i'-\phi_i),q_i)$ leads to convergent-helicity sums because of factors $q_i^{M_i^2}$. Since all helicity sums are now convergent in complex-J unitarity equations (see below) or in the MFG formalism, we may imagine formally carrying out dynamical calculations at complex J, obtaining Regge trajectories and residues, and letting all $q_i\to 1$ at the end. Of course, it must be demonstrated that the results are independent of the manner in which the $q_i\to 1$. As a partial solution to this problem of limits, we first notice that since physical t_i, integral J_i crossed-channel partial-wave amplitudes have only finitely many possible helicities, these helicity sums do not require convergence factors at all. [To see this explicitly, we imagine an inverse Sommerfeld-Watson transform being performed on the ith $O(2,1)$ expansion. Boyce[8] has explicitly demonstrated the cancellation of the nonsense terms from the principal-series integral with those of the discrete series, so that the crossed-channel $O(3)$ expansion possesses only sense terms, as it should.] Hence the $q_i\to 1$ limit in this important case is trivial.

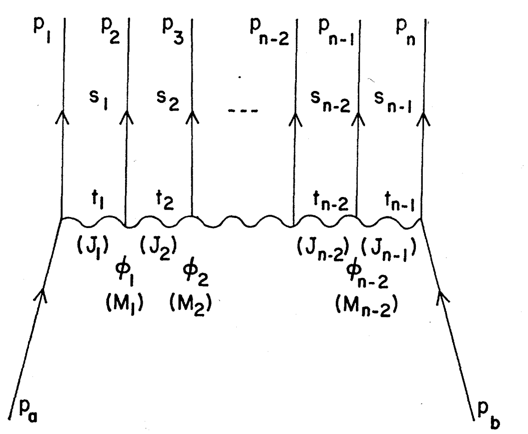

FIG. 1. Kinematics for the multiparticle $2\to n$ process.

Next we show heuristically that the $q_i\to 1$ limit for the asymptotic behavior $(s_i\to\infty)$ of the amplitude is well behaved even though the $q_i^{M_i^2}$ factors are required in the MFG formalism for the background integral. The point is that the helicity dependence of the internal Regge-residue coupling $\beta_{M_i}(t_i,t_{i+1})$ as $M_i\to\infty$ is completely different from the $M_i\to\infty$ behavior of the $e_{M_{i-1},M_i}{}^{J_i}(\zeta_i')$ function appearing in the Froissart-Gribov formula [Eq. (2.6)]. Neglecting dependence on irrelevant variables, we have

$$\beta_{M_{i-1}}(t_{i-1},t_i)\beta_{M_i}(t_i,t_{i+1}) =\lim_{J_i\to\alpha_i(t_i)}[J_i-\alpha_i(t_i)]f_{M_{i-1},M_i}{}^{J_i\tau_i}. \qquad (4.1)$$

If we substitute Eq. (2.6) into Eq. (4.1), we must first look at the $\zeta_i'\to\infty$ behavior of $e_{M_{i-1},M_i}{}^{\alpha_i}(\zeta')$ since the divergence of the FG integral produces the Regge pole. We have[3]

$$|\lim_{\zeta_i'\to\infty} e_{M_{i-1},M_i}{}^{\alpha_i}(\zeta_i')| \propto[\Gamma(\alpha_i-M_i+1)\Gamma(\alpha_i+M_i+1)]^{1/2}\zeta_i'^{-\alpha_i-1}. \qquad (4.2)$$

The $M_i\to\infty$ behavior of Eq. (4.2) is power bounded in M_i of order $M_i^{\alpha_i+1/2}$ for nonintegral α_i, and is thus much better behaved than the exponential $e^{\lambda(\zeta')M_i}$ behavior of $e_{M_{i-1},M_i}{}^{\alpha_i}(\zeta')$ for fixed ζ' which led to the original divergence problems. We have already seen that the $M_i\to\infty$ behavior of $K_{M_{i-1},M_i}\Delta_{M_{i-1},M_i}$ must be reasonable in order for the series [Eq. (2.5)] to converge away from singularities in ϕ_{i-1}, ϕ_i. In fact, it must decrease,[9] so without much loss of generality we assume it to be of order $O(M_i^{-\gamma_i})$, $\gamma_i>0$. Then we obtain $\beta_{M_i}\sim O(M_i^{\alpha_i+\frac{1}{2}-\gamma_i})$ which decreases if $\mathrm{Re}\,\alpha_i+\frac{1}{2}<\gamma_i$. The contribution of the Regge pole to the total amplitude is proportional to

$$f\sim\cdots\sum_{M_i}\tau_{M_i}e^{-iM_i\phi_i}[\Gamma(\alpha_i+M_i+1)\Gamma(\alpha_i-M_i+1)]^{-1/2} \times\beta_{M_i}(t_i,t_{i+1})\zeta_i^{\alpha_i}\quad(|\tau_{M_i}|=1), \qquad (4.3)$$

where we have used the asymptotic form for $d_{M_{i-1},M_i}(\zeta_i)$ as $\zeta_i\to\infty$. The M_i sum converges without a convergence factor $q_i^{M_i^2}$.

Thus we see that calculation of some important quantities of physical interest do not require a distribution interpretation, even though the original helicity sums from which these limits were taken did diverge.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U.P., Cambridge, England, 1963), Chap. 21.

[7] Actually each Toller angle is the sum of two azimuthal angles. For details, see N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

[8] J. F. Boyce, J. Math. Phys. **8**, 675 (1967).

[9] E. C. Titchmarsh, *The Theory of Functions* (Oxford U. P., London, 1964), p. 404.

Finally we show that the bilinear terms appearing in crossed-channel unitarity equations possess convergence factors in each helicity sum, thus providing a formalism for analytically continuing the unitarity equations to complex J.[10] The crossed-channel $O(3)$ expansion in J_i partial waves is accomplished by analytically continuing the $2\rightarrow n$ amplitude into the $n-i+1\rightarrow 1+i$ amplitude (see Fig. 2). We define $\cos(i\theta_i)=\zeta_i$ and $z_i=\cos\theta_i$. Smearing in ϕ_{i-1} and ϕ_i, we obtain the $O(3)$ partial-wave expansion for the smeared functions

$$f^{q_{i-1}q_i}(\phi_{i-1}z_i\phi_i,v) = \sum_{J_i=0}^{\infty} \sum_{|M_{i-1}|,|M_i|\leqslant J_i} D_{M_{i-1},M_i}{}^{J_i}(\phi_{i-1}z_i\phi_i) \times q_{i-1}{}^{M_{i-1}^2} q_i{}^{M_i^2} f_{M_{i-1},M_i}{}^{J_i q_{i-1} q_i}(v)\,. \quad (4.4)$$

The unitarity equation for a process $I\rightarrow F$, where the initial (final) state I (F) is characterized in body-fixed frames by variables v_I (v_F), is written as

$$(1/2i)[f_{FI}(r_{FI}v_F{}^+t^+v_I{}^+)-f_{FI}(r_{FI}v_F{}^-t^-v_I{}^-)] = \sum_{K=2}^{N(t)} \int dv_K dr_K \rho_K(t,v_K) f_{FK}(r_{FK}v_F{}^-t^-v_K{}^-) \times f_{KI}(r_{KI}v_K{}^+t^+v_I{}^+)\,. \quad (4.5)$$

Here r_{FI} is the rotation between final and initial frames with parameters $(\phi_{i-1}z_i\phi_i)$ and equals $r_I{}^{-1}r_F$, where r_I (r_F) rotates a standard frame into the initial (final) body-fixed frames. $t^{\pm}=t\pm i\epsilon$ specifies the $i\epsilon$ prescription of the total c.m. energy, where t is defined by

$$t=t_i=(p_a-\sum_{j=1}^{i} p_j)^2=(\sum_{j=i+1}^{n} p_j-p_b)^2 \quad (4.6)$$

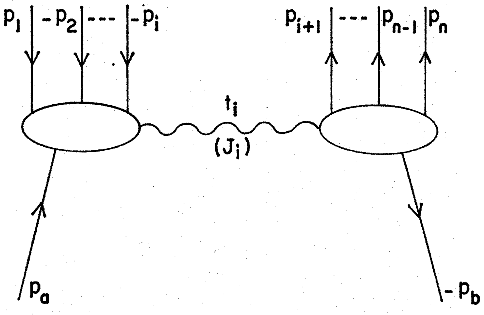

FIG. 2. Kinematics for the $2\rightarrow n$ amplitude continued to the timelike t_i region.

and the amplitude has been continued to the region where t_i is timelike. Also, ρ_K is the phase-space factor, and the $\pm$ on the variables v_F, ..., specify $i\epsilon$ prescriptions for these variables.

We take the smeared functions $f_{FI}{}^{q_Fq_I}$ to satisfy unitarity among themselves. This ensures that the smearing procedure is consistent with unitarity regardless of the q_i parameters. For simplicity we take the q_i to be real. Using the expansion Eq. (4.4) in the unitarity equation (4.5) in which the original functions f_{FI} are replaced by $f_{FI}{}^{q_Fq_I}$, we obtain

$$(1/2i)[f_{M_FM_I}{}^{J_iq_Fq_I}(v_F{}^+t^+v_I{}^+)-f_{M_FM_I}{}^{J_iq_Fq_I}(v_F{}^-t^-v_I{}^-)] = \sum_{K=2}^{N(t)} \sum_{M_K=-J_i}^{J_i} \int dv_K \rho_K(t,v_K) f_{M_FM_K}{}^{J_iq_Fq_K}(v_F{}^-t^-v_K{}^-) \times f_{M_KM_I}{}^{J_iq_K'q_I}(v_K{}^+t^+v_I{}^+) q_K{}^{M_K^2} q_K'{}^{M_K^2} \quad (4.7)$$

(for $k=2$, the M_k sum is absent).

Since convergence factors are now provided in each helicity sum, the unitarity equation (4.7) can now be analytically continued in J_i, and the resulting infinite-helicity sums will continue to converge. The three-body Faddeev equations are treated in the special case $n=4$, $i=2$, where all but $k=3$ body intermediate states are excluded. The complex-J extensions of these equations then have bounded kernels.

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[10] This crossed-channel unitarity approach is complementary to equations of the Chew-Goldberger-Low or Halliday-Saunders type. There, Regge parameters at spacelike arguments t_i are examined using asymptotic forms for the amplitudes in s-channel unitarity equations rather than using J-plane analytic continuations of the crossed t_i-channel unitarity at timelike t_i. See G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters **22** 208 (1969); I. G. Halliday, Nuovo Cimento **60A**, 177 (1969); I. G. Halliday and L. M. Saunders, *ibid.* **60A**, 494 (1969). For discussions of unitarity calculations with infinite helicity sums, see I. T. Drummond and G. A. Winbow, Phys. Rev. **161**, 1401 (1967); G. A. Winbow, *ibid.* **171**, 1517 (1968).