$O(\alpha^2)$ corrections to muon lifetime, m_W, and m_Z in the $SU(2)_L \times U(1)$ theory

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We derive formulas for τ_μ, m_W, and m_Z in the $SU(2)_L \times U(1)$ theory which are valid through $O(\alpha^2 \ln m)$, where m is a generic fermion mass; terms of $O(\alpha m_\mu{}^2/m_W{}^2)$ are neglected. The emphasis of the analysis is on the role played by fermion mass singularities. The final results are very simple: the radiative correction of the local $V-A$ theory [through $O(\alpha^2 \ln(m_\mu/m_e))$] factors out and the heavy particles induce a renormalization that can be expressed as an elementary function of the corresponding $O(\alpha)$ contribution slightly modified by fourth-order vacuum-polarization effects.

The existence of the W and Z bosons is one of the basic cornerstones of the present paradigm of electroweak interactions. Precise measurements of their parameters, such as masses and widths, can lead to further tests of these models as fundamental quantum field theories. There exist, of course, other challenging tests, of older vintage, such as the analysis of universality.[1] However, in view of the recent experimental discoveries,[2] the detailed predictions of the basic properties of the intermediate bosons are of particular interest at present.

Using as a starting point the complete $O(\alpha)$ calculation of Ref. 3 (see also Ref. 4), the aim of this paper is to derive formulas for τ_μ, m_W, and m_Z, within the framework of $SU(2)_L \times U(1)$, which incorporate corrections of $O(\alpha^2 \ln m)$ as well as $O(\alpha^n \ln^n m)$ (m stands for a generic fermion mass). The analysis emphasizes the role of mass singularities. From a theoretical standpoint, the results represent one step further than current analyses,[5] which include terms of $O(\alpha^n \ln^n m)$ by means of renormalization-group methods.[6,7] As we will see, the formulas provide a very simple procedure to take into account the higher-order contributions in terms of the $O(\alpha)$ corrections slightly modified by fourth-order vacuum-polarization effects.

In order to discuss these higher-order effects, we consider the perturbative expansion of the muon's total decay rate in the framework of the $SU(2)_L \times U(1)$ theory:

$$\frac{1}{\tau_\mu} = \frac{P}{32}\frac{g^4}{m_W{}^4}(1+e^2 f_1 + e^4 f_2 + \cdots) , \tag{1a}$$

where

$$P = \left[1-8\frac{m_e{}^2}{m_\mu{}^2}\right]\left[1+\frac{3}{5}\frac{m_\mu{}^2}{m_W{}^2}\right] m_\mu{}^5 \Big/ 192\pi^3 . \tag{1b}$$

We recall that in the renormalization framework of Ref. 3 one has the exact relations

$$g = e/\sin\theta_W , \tag{2a}$$

$$m_W = m_Z \cos\theta_W , \tag{2b}$$

where e is the conventionally defined electric charge of the proton, and m_W and m_Z are the physical masses of the intermediate bosons in the $SU(2)_L \times U(1)$ theory. Following that paper, we will neglect terms of $O(\alpha m_\mu{}^2/m_W{}^2)$.

Inspection of Eqs. (38) and (39) of Ref. 3 reveals the existence of fermion mass singularities in f_1. Indeed one encounters large logarithmic corrections of the form $\sum_i Q_i{}^2 \ln(m_Z/m_i)$, where m_i and Q_i are the mass and charge of the ith fermion. At first hand, and from a rather naive point of view, this may seem somewhat surprising because it is easy to see that mass singularities are necessarily absent at the one-loop level in the unrenormalized (but regularized) perturbative expansion.[8] Indeed, if terms of $O(\alpha m_\mu{}^2/m_W{}^2)$ are neglected, such calculation of the $O(\alpha)$ effects splits neatly into two contributions: (i) the $O(\alpha)$ corrections to $1/\tau_\mu$ in the local V-A theory, which will be referred to as QED corrections, and (ii) additional contributions from virtual diagrams involving two massive-boson propagators. In (ii) we can set all fermion masses and external momenta equal to zero, as the presence of the two massive propagators prevents the occurrence of mass singularities. At the level of the differential electron spectrum, the QED corrections contain logarithms of fermion masses but, as is well known, these cancel in the total decay rate.[9,10] The apparent paradox is quickly resolved by inserting $g = e/\sin\theta_W$ in Eq. (1a); the mass singularities emerge in f_1 because the conventional definition of electric charge involves a subtraction at the exceptional invariant momentum $q^2=0$.[11] In other words, their presence in f_1 may be viewed as a consequence of renormalization. Of course, this observation by itself does not explain the logarithms of m_Z accompanying the fermion masses; they occur because m_Z (or m_W) is the natural scale of the electroweak corrections.

In order to study in a systematic way the mass singularities generated in higher orders by renormalization, let us express Eq. (1a) as an expansion in powers of the bare charge e_0 by means of the relations

$$e^2 = \frac{e_0{}^2}{1+e_0{}^2\pi(0)} , \tag{3a}$$

$$\pi(0) = \pi_1(0) + e_0{}^2\pi_2(0) + \cdots , \tag{3b}$$

where $\pi(0)$ is the regularized but unrenormalized vacuum-polarization function evaluated at $q^2=0$. Substitution of Eqs. (2a), (3a), and (3b) in Eq. (1a) leads to

$$\frac{1}{\tau_\mu}=\frac{P}{32}\frac{e_0^4}{\sin^4\theta_W m_W^4}\{1+e_0^2[f_1-2\pi_1(0)]+e_0^4[f_2-3f_1\pi_1(0)+3\pi_1^2(0)-2\pi_2(0)]+\cdots\}\ . \tag{4a}$$

In Eq. (4a) we have not expressed $\sin^2\theta_W$ or m_W^2 in terms of unrenormalized constants. The reason is the following: As seen in Eq. (2b), in the renormalization framework of Ref. 3 these parameters are defined in terms of amplitudes involving nonexceptional external momenta such as $q^2=m_W^2,m_Z^2$. Because of this fact, their expansion in powers of e_0^2 does not generate additional mass singularities.

It is convenient to write

$$f_1=2\pi_1(0)+c_1\ , \tag{4b}$$

$$f_2=3\pi_1(0)[f_1-\pi_1(0)]+2\pi_2(0)+c_2\ , \tag{4c}$$

where c_i $(i=1,2)$ are the coefficients of the $O(e_0^2)$ and $O(e_0^4)$ terms in the unrenormalized perturbative expansion of Eq. (4a). The earlier discussion in this paper informs us that c_1 is free from fermion mass singularities. Furthermore, the cancellation of such singularities in the unrenormalized perturbative expansion of total decay rates[9,10] tells us that c_2 does not contain contributions that diverge as $m_e\to 0$. This statement requires some explanation. In the arguments leading to the cancellation of mass singularities in the unrenormalized perturbative expansion of $1/\tau_\mu$, the muon mass is held fixed while the limit $m_e\to 0$ is considered. Because of this reason, from these arguments alone we do not derive information about logarithms of other masses such as $\ln m_\mu$, $\ln m_\tau$, and $\ln m_q$ $(q=u,d,s,c,\ldots)$. Note in particular that if m_μ is held fixed, it is not obviously clear what physical meaning should be given to the limit $m_\tau\to 0$.[12] Later on in this paper we will discuss the logarithms of fermion masses that appear in c_2 by arguments based on a heuristic examination of the $O(\alpha^2)$ unrenormalized Feynman amplitudes. Equations (4b) and (4c) have a simple meaning. The terms $2\pi_1(0)$ in Eq. (4b) and $3\pi_1(0)[f_1-\pi_1(0)]+2\pi_2(0)$ in Eq. (4c) contain all the mass singularities induced by renormalization in f_1 and f_2, respectively. Any additional term of the form $\ln m$ in c_2, for example, must arise from the unrenormalized amplitudes.

Expressing $\pi_1(0)$ in terms of f_1 [Eq. (4b)] and substituting in Eq. (4c),

$$f_2=\tfrac{3}{4}f_1^2+2\pi_2(0)+\hat{c}_2\ , \tag{5a}$$

where $\hat{c}_2=c_2-\frac{3}{4}c_1^2$. We note that terms proportional to c_1f_1 have canceled in Eq. (5a). There is an important reason for this cancellation explained after Eq. (6). It is convenient to write

$$\pi_2(0)=-\mathrm{Re}\pi_2^r(m_Z^2)+\mathrm{Re}\pi_2(m_Z^2)\ , \tag{5b}$$

where $\pi_2^{(r)}(m_Z^2)\equiv\pi_2(m_Z^2)-\pi_2(0)$ is the fourth-order contribution to the renormalized vacuum-polarization function (with powers of e extracted). As m_Z^2 is a nonexceptional momentum, the second term on the right-hand member (RHM) of Eq. (5b) does not contain fermion mass singularities. The contribution to $\mathrm{Re}\pi_2^{(r)}(m_Z^2)$ involving lepton loops can be obtained from the Jost-Luttinger calculation;[13] a convenient strategy to analyze the contributions involving both hadrons and leptons is briefly discussed after Eq. (12b). Combining Eqs. (5a) and (5b),

$$f_2=\tfrac{3}{4}f_1^2-2\,\mathrm{Re}\pi_2^{(r)}(m_Z^2)+c\ . \tag{6}$$

It is interesting to note that if terms proportional to c_1f_1 had survived in Eq. (5a), and therefore in Eq. (6), the argument of this paper would not be logically consistent. Indeed, as seen in Eq. (4b), c_1 is an ultraviolet-divergent quantity. As f_2, f_1, and $\pi_2^{(r)}$ are finite, such a term could only be cancelled by c. This, however, would imply that c contains terms proportional to $\ln m_e$, in contradiction with the argument. Thus, the apparently mysterious cancellation of c_1f_1 in Eq. (5a) is actually a welcome consistency check for the derivation.

In Eq. (6) we have been able to express the mass-singularity part of f_2 induced by renormalization in terms of $\pi_2^{(r)}$ and f_1. The latter is readily obtained from Ref. 3:

$$e^2f_1=2\,\Delta r+F_{\mathrm{QED}}\ , \tag{7a}$$

where Δr is the electroweak correction of $O(\alpha)$ discussed in Secs. III and IV of that work and

$$F_{\mathrm{QED}}=\frac{\alpha}{2\pi}\left(\tfrac{25}{4}-\pi^2\right) \tag{7b}$$

is the $O(\alpha)$ correction to $1/\tau_\mu$ in the local *V-A* theory.[9]

Let us now discuss the terms involving logarithms of fermion masses not induced by renormalization. From our previous discussion we know that such terms must emerge from the unrenormalized $O(\alpha^2)$ corrections and be contained in c [cf. Eq. (6)]. Furthermore, we know that c cannot contain $\ln m_e$ singularities. It is convenient to separate the $O(\alpha^2)$ corrections into three classes: those involving (1) no photons, (2) one or two photons (virtual or real) and at least two heavy-particle propagators ["heavy" means a particle of mass $\sim O(m_W)$], and (3) two photons (virtual or real) and one heavy propagator.[14] Diagrams of class (1) involve necessarily three heavy propagators and are of $O(1/m_W^4)$ unless the two-loop momenta are large $[k_1^2,k_2^2=O(m_W^2)]$. In these configurations the large loop momenta prevent the vanishing of the denominators associated with the light particles and therefore effectively eliminate the emergence of logarithms of fermion masses. The same is true of many diagrams of class (2) such as Fig. 1(a). There are, however, subsets of diagrams of class (2) in which this is not true. They correspond to vertex renormalizations inserted on the $O(\alpha)$ QED graphs and involving at least one heavy particle. As many of the $O(\alpha)$ photonic contributions contain logarithms of fermion masses, the same is true of these higher-order diagrams. Two sets of such graphs are illustrated in Figs. 1(b)—1(f) and Figs. 2(a)—2(d). In Figs. 1(b)—1(f) the unrenormalized vertex involving the Z^0 does not give rise to $\ln m_\mu$ terms (unless these are suppressed by m_μ^2/m_Z^2 factors) but in the photon-loop integration the photon, the muon, and the electron can be near their mass shells. This gives

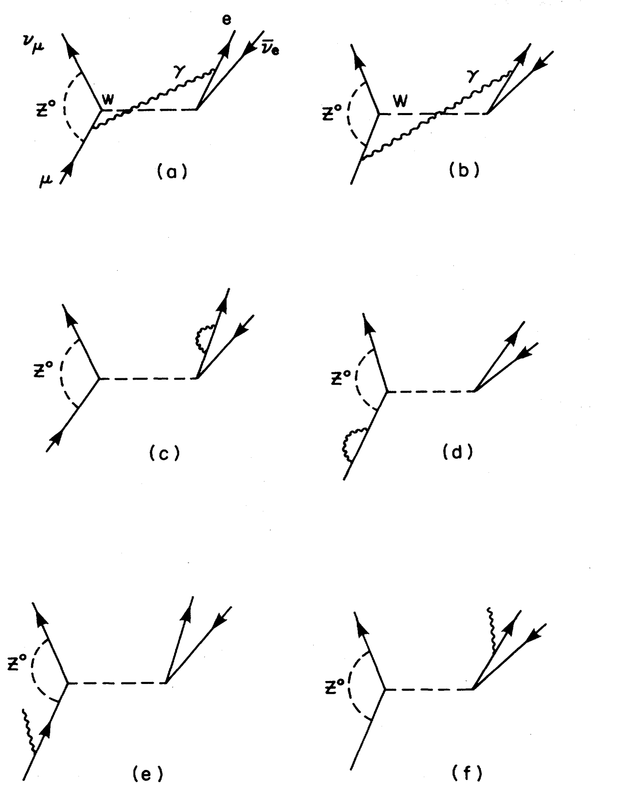

FIG. 1. Examples of two-loop contributions to μ decay. In diagram (a) terms involving $\ln m_\mu$ and $\ln m_e$ are suppressed by a factor of $m_\mu{}^2/m_W{}^2$. The mass singularities of diagrams (b)—(f) cancel to $O(\alpha^2)$ when their combined contribution to $1/\tau_\mu$ is considered (see text).

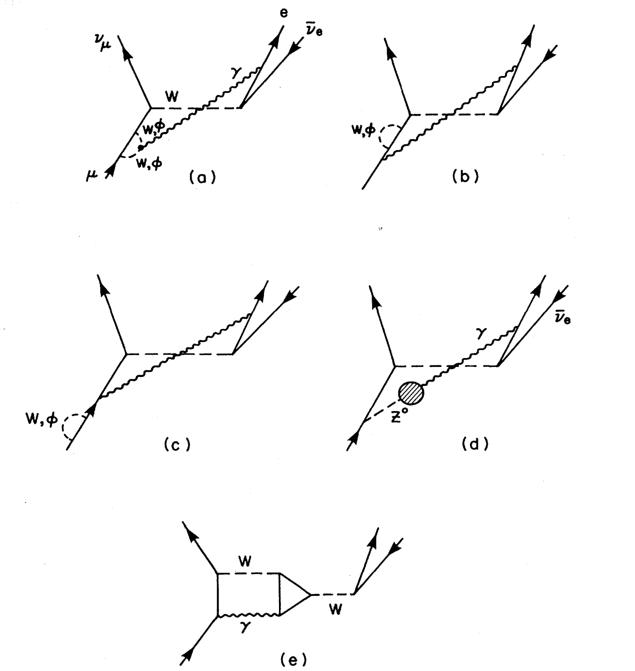

FIG. 2. The sum of the diagrams (a)—(d) is free of mass singularities (see text). Diagram (e) illustrates a typical triangle graph.

rise to terms involving $\ln m_\mu$ and $\ln m_e$. However, it is not difficult to see that such logarithms cancel if one considers the contribution to $1/\tau_\mu$ of all the diagrams of Figs. 1(b)—1(f). Indeed, in analyzing the $\ln m$ terms we may evaluate the Z^0 vertex renormalization with all its external momenta set equal to zero. The Z^0 vertex then becomes a multiplicative constant renormalizing the $O(\alpha)$ QED graphs obtained from Figs. 1(b)—1(f) by shrinking the vertex to a point. But we have seen that the mass singularities associated with such diagrams cancel when their combined contribution to $1/\tau_\mu$ is considered. An analogous argument can be carried out, for example, for the set of diagrams obtained from Fig. 1 by replacing the Z^0 vertex by an insertion involving a second photon attached to one of the charged leptons (μ or e) and the intermediate W. Cutting the photon line we note that the diagrams of Figs. 2(a)—2(d) are part of the electromagnetic-charge form factor of the muon. Their sum vanishes as $k^2 \to 0$ where k is the photon four-momentum.[3] This cancels the k^{-2} behavior of the photon propagator and eliminates the mass singularities. Figure 2(e) illustrates a typical triangle diagram. After the anomalous part is canceled among the various fermion loops, the dependence of the triangle subgraph on the photon four-momentum k is at most logarithmic. Therefore, the photon-loop integral is convergent; as it involves a W propagator, it is suppressed by an additional factor of $O(m_\mu{}^2/m_W{}^2)$. In summary, the logarithms of fermion masses in the diagrams of class (2) either cancel among themselves at the amplitude level [as in Figs. 2(a)—2(d)], or vanish when their contributions to $1/\tau_\mu$ are combined [as in Figs. 1(b)—1(f)], or are suppressed by an additional factor $m_\mu{}^2/m_W{}^2$.

By elementary algebraic splittings of the boson propagators (see the Appendix) it is possible in the 't Hooft—Feynman gauge to identify the diagrams of class (3) with the regularized two-photon graphs appearing in the local $V-A$ theory, plus additional contributions involving two or three massive propagators. The latter can be discussed with arguments analogous to those carried out for classes (1) and (2). In this identification each photon propagator is regularized with a factor $[m_W{}^2/(m_W{}^2-k^2)]$. This corresponds to a "Feynman regulator" with the cutoff Λ set equal to m_W.[15] As μ decay in the local $V-A$ theory is known to be finite (after charge renormalization) to lowest order in G_μ and all orders in α,[16] the difference between setting $\Lambda = m_W$ and $\Lambda = \infty$ is of $O(\alpha^2 m_\mu{}^2/m_W{}^2)$.

The only $O(\alpha^2)$ diagrams of the local $V-A$ theory involving "extraneous fermions" such as $\tau, u, d, s, c, \ldots$ are the vacuum-polarization diagrams illustrated in Fig. 3. (By extraneous we mean here any fermions not participating in μ decay at the tree level.) The unrenormalized-vacuum-polarization insertion gives rise, in each photon propagator, to a correction $1-e^2\pi_1(k^2)$. In the dimensional regularization scheme

$$e^2\pi_1(k^2) = -\frac{2\alpha}{3\pi}\sum_i Q_i^2 \left[\frac{1}{n-4} + C + 3\int_0^1 dx\, x(1-x)\ln\left[\frac{m_i^2 - k^2x(1-x)}{m_Z^2}\right]\right], \tag{8}$$

where Q_i and m_i are the charge and mass of the ith fermion, $C=[\gamma-\ln(4\pi)]/2$ (γ is Euler's constant), and we have set the 't Hooft mass equal to m_Z, which is the natural mass scale accompanying the fermion masses in f_1. Thus, the choice is consistent with our previous strategy of expressing the mass singularities induced by renormalization in terms of f_1 [cf. Eq. (6)]. The $(n-4)^{-1}$ poles in Eq. (8) appear because we are dealing here with the unrenormalized amplitude.[17] Clearly the last term in Eq. (8) can give rise to logarithms of fermion masses. Before the correction factor $1-\pi_1(k^2)$ is included, the sum of the virtual diagrams associated with Figs. 3(a)—3(c) is ultraviolet finite. The same is of course true when the last term of Eq. (8) is inserted. We can then argue that the important range in the integration over the virtual photon momentum is $0<-k^2<m_\mu^2$ because m_μ sets the kinematical mass scale in μ decay. Therefore, for the τ lepton and the heavy quarks we can approximately neglect $k^2x(1-x)$ in comparison with m_i^2 in Eq. (8), so that the contributions of those fermions are essentially the same as in $\pi_1(0)$. The same is of course true in the case of the real-photon diagrams of Figs. 3(d) and 3(e). In these calculations the masses of the light quarks are effective quantities derived from $\sigma(e^+ + e^- \to \text{hadrons})$. In a recent analysis, $m_u = m_d \approx 75$ MeV, $m_s \approx 250$ MeV.[18] These are of the same order as m_μ and clearly we make a small error if for μ, u, d, and s we also neglect $k^2x(1-x)$ in Eq. (8). Indeed the error in this approximation does not involve large logarithms and can be regarded as $O(\alpha^2)$. At first hand the analysis of the contributions of the electron loops and electron-positron pairs in Fig. 3 seems complicated. The diagrams of Figs. 3(d) and 3(e) obviously contain $\ln m_e$ terms and in Eq. (8), we can neglect m_e^2 when the photon is off the mass shell but not when $k^2\approx 0$. Also one must include the diagrams describing the conversion into e^--e^+ pairs [Figs. 3(f) and 3(g)]. Fortunately we know that when all the contributions to $1/\tau_\mu$ are included there can be no electron mass singularity in the unrenormalized $O(\alpha^2)$ terms. A moment's thought reveals therefore that when all the effects are combined, the mass accompanying m_Z in the electron term in Eq. (8) can only be the other parameter of kinematical relevance left in the problem, namely m_μ. The regulator mass m_W will not do because the corrections are finite as $m_W \to \infty$. We reach the conclusion that the contributions of Fig. 3 involving logarithms of fermion masses are of the form

$$e^4c = -e^2\hat{\pi}_1(0)F_{\rm QED} + e^4a\,, \tag{9a}$$

where $\hat{\pi}_1(0)$ is a modified $\pi_1(0)$ in which m_e has been replaced by m_μ, a contains no logarithms of fermion masses (unless suppressed by factors m_μ^2/m_W^2), and $F_{\rm QED}$ is given in Eq. (7b). Writing

$$e^2\hat{\pi}_1 0 = e^2\pi_1(0) - \frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right] = \frac{e^2f_1}{2} - \frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right] - \frac{e^2c_1}{2}\,, \tag{9b}$$

where we employed Eq. (4b), Eq. (9a) becomes

$$e^4c = F_{\rm QED}\left[-\frac{e^2f_1}{2} + \frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right]\right] + e^4\hat{a}\,, \tag{9c}$$

where $\hat{a}$ is free from fermion masses. To complete the argument we must consider the two-photon diagrams of the local $V-A$ theory not included in Fig. 3. These only involve three masses: m_μ, m_e, and the "regulator mass" m_W. The contributions of such diagrams to $1/\tau_\mu$ cannot contain $\ln(m_\mu/m_e)$ because of the cancellation of m_e singularities and cannot contain $\ln(m_W/m_\mu)$ because the corrections are finite as $m_W\to\infty$.[16] Thus, Eq. (9c) is our final answer for c. Combining Eqs. (6) and (9c),

$$e^4f_2 = e^4[\tfrac{3}{4}f_1^2 - 2\,{\rm Re}\pi_2^{(r)}(m_Z^2)] + F_{\rm QED}\left[-\frac{e^2f_1}{2} + \frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right]\right] + e^4\hat{a}\,. \tag{10}$$

Expressing e^2f_1 in terms of $F_{\rm QED}$ and Δr [see Eq. (7a)] and inserting these quantities in Eq. (1a), we find through $O(\alpha^2 \ln m)$

$$\frac{1}{\tau_\mu} = \frac{P}{32}\frac{g^4}{m_W^4}\left\{1 + \frac{\alpha}{2\pi}(\tfrac{25}{4} - \pi^2)\left[1 + \frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right]\right]\right\} \times \left[\frac{1}{1-\Delta r + e^4{\rm Re}\pi_2^{(r)}(m_Z^2)}\right]^2. \tag{11a}$$

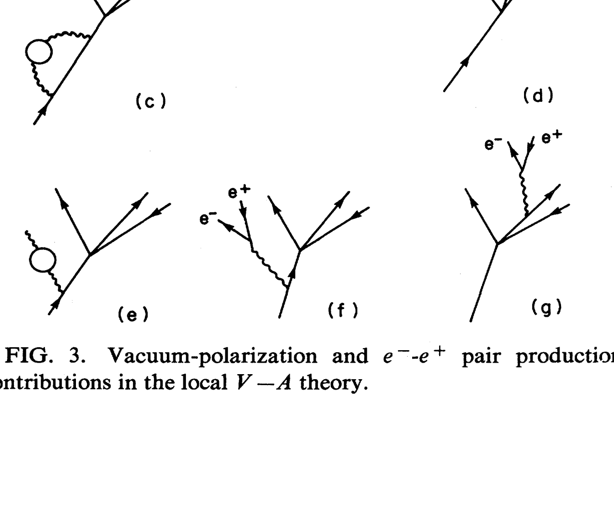

FIG. 3. Vacuum-polarization and e^--e^+ pair production contributions in the local $V-A$ theory.

Equation (11a) has a very simple structure. The first factor is the correction to $(1/\tau_\mu)$ in the local $V-A$ theory through terms of $O(\alpha^2 \ln m_\mu/m_e)$.[19] The second factor represents a renormalization induced by the heavy particles and is given by an elementary function of the complete $O(\alpha)$ electroweak correction Δr slightly modified by the inclusion of the term $-e^4 \mathrm{Re}\pi_2^{(r)}(m_Z^2)$. The factorization of these two effects is clearly related to the neglect of terms of $O(\alpha m_\mu^2/m_W^2)$.

Recalling Eq. (2a) and defining G_μ by means of

$$\frac{1}{\tau_\mu}=G_\mu^2 P\left\{1+\frac{\alpha}{2\pi}\left(\tfrac{25}{4}-\pi^2\right)\left[1+\frac{2\alpha}{3\pi}\ln\left[\frac{m_\mu}{m_e}\right]\right]\right\}, \quad (11b)$$

Eq. (11a) leads to

$$m_W=\left[\frac{\pi\alpha}{\sqrt{2}G_\mu}\right]^{1/2}\frac{1}{\sin\theta_W} \times\left[\frac{1}{1-\Delta r+e^4\mathrm{Re}\pi_2^{(r)}(m_Z^2)}\right]^{1/2}. \quad (12a)$$

Inserting the experimental value of τ_μ,[20] we obtain from Eq. (11b) $G_\mu=(1.166\,34\pm0.000\,02)\times10^{-5}\ \mathrm{GeV}^{-2}$ and Eq. (12a) becomes

$$m_W=\frac{37.281\ \mathrm{GeV}}{\sin\theta_W}\left[\frac{1}{1-\Delta r+e^4\mathrm{Re}\pi_2^{(r)}(m_Z^2)}\right]^{1/2}. \quad (12b)$$

In Eqs. (12a) and (12b) we have kept the terms of $O(\alpha^2\ln^2 m,\alpha^2\ln m)$ but have neglected terms of $O(\alpha^2)$ and higher. On the other hand, Eqs. (12a) and (12b) treat correctly the leading logarithms such as $\alpha^3\ln^3 m$. The term $e^4\mathrm{Re}\pi_2^{(r)}(m_Z^2)$ in Eqs. (11a), (12a), and (12b) plays a minor role in current analyses. For instance, it induces a relative correction of a few parts in 10^5 in the m_W prediction. In order to take into account this small effect, it is useful to recall that an important contribution to the $O(\alpha)$ correction Δr is given by $-e^2\mathrm{Re}\pi_1^{(r)}(m_Z^2)$. Thus, it is natural to combine the two expressions and consider the complete renormalized vacuum-polarization function

$$-e^2\mathrm{Re}[\pi_1^{(r)}(m_Z^2)+e^2\pi_2^{(r)}(m_Z^2)]\ .$$

The effect of the contributions to $\mathrm{Re}\pi_2^{(r)}(m_Z^2)$ involving leptons and containing mass singularities is simply obtained by multiplying the corresponding contributions to $\mathrm{Re}\pi_1^{(r)}(m_Z^2)$ by a factor $1+3\alpha/4\pi$.[13] For the contributions involving hadrons a convenient strategy is to split the dispersion relation for the complete vacuum-polarization function into two parts: (i) a low-energy part which is calculated from experimental data on $\sigma_T(e^+ + e^- \to \text{hadrons})$ and which therefore needs no corrections, and (ii) a high-energy part that is calculated theoretically *including both* QCD *and electromagnetic corrections.*

To illustrate the order of magnitude of the higher-order corrections consider, for example, the explicit calculation of Δr given in Ref. 3. Using $\sin^2\theta_W=0.23$, $m_\phi=m_W$, and $m_{\text{top}}\approx 18$ GeV the answer of that paper was $\Delta r/2=0.0344$ giving rise to an $O(\alpha)$ renormalization factor 1.0344. Inserting Δr into Eq. (12b) we obtain instead 1.0364.[21] Thus, the higher-order terms included in Eq. (12b) represent an additional correction of 2.0×10^{-3} beyond the complete $O(\alpha)$ calculation and translate into a further shift of about $+0.16$ GeV in m_W. It is interesting to compare Eqs. (12a) and (12b) with the renormalization-group analysis. There the leading logarithms of the $O(\alpha)$ calculation are multiplied by the running constant $\alpha(m_W)$ while the remaining terms are evaluated using the standard fine-structure constant. It is easy to see that such a procedure treats the leading logarithms $\alpha^n\ln^n m$ in the same way as Eqs. (12a) and (12b). For this reason the two approaches are numerically close to each other.[22] We emphasize, however, that Eqs. (11a), (12a), and (12b) go one step beyond the present renormalization-group analyses, as they incorporate the $O(\alpha^2\ln m)$ terms and are simpler because they involve elementary functions of the complete $O(\alpha)$ calculation.

Equations (2b) and (12b) can be used to predict m_W and m_Z in terms of the current values of $\sin\theta_W$ derived from deep-inelastic scattering.[5,21] Obviously, it is also possible to eliminate $\sin\theta_W$ between the two equations and relate directly m_W and m_Z[23]

$$m_Z=\frac{m_W}{[1-(A/m_W)^2]^{1/2}}, \quad (12c)$$

where[21]

$$A=37.281\ \mathrm{GeV}\left[\frac{1}{1-\Delta r+e^4\mathrm{Re}\pi_2^{(r)}(m_Z^2)}\right]^{1/2}. \quad (12d)$$

In more general terms, we note that Eqs. (2b) and (12b) relate three parameters which are not accurately known at present: m_W, m_Z, and $\sin\theta_W$. To verify the $SU(2)_L\times U(1)$ theory at the level of the radiative corrections, without involving additional theoretical ideas such as grand unification, at least two of these three parameters must be accurately measured.

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APPENDIX

We discuss briefly how to split the photon propagator in the 't Hooft–Feynman gauge, in order to establish correspondence with diagrams of the local $V-A$ theory in the arguments explained in this paper. A few examples will suffice. We recall that in studying the contributions of Figs. 1(b)–1(f) involving mass singularities, the Z^0 vertex can be regarded as a constant renormalizing the $O(\alpha)$ QED graphs obtained by shrinking the insertion to a point. Let us then assume that the Z^0 vertex has been removed. In Fig. 1(b) the virtual photon and W momenta are the same (we neglect the very small external momenta). The W propagator is proportional to $m_W^{-2}[m_W^2/(m_W^2-k^2)]$. Thus, Fig. 1(b) is the same as we would obtain in the local $V-A$ theory if the W line were also shrunk to a point and the photon propagator endowed with a Feynman regulator with $\Lambda=m_W$. On the other hand, in diagrams 1(c) and 1(d) the photon carries momentum k while the W has essentially zero momentum. The procedure here is identical to that employed in the case of the $O(\alpha)$ corrections;[1,3] the diagram of Fig. 1(b) is left untouched while in Figs. 1(c) and 1(d) one uses the identity

$$k^{-2}=k^{-2}[m_W{}^2/(m_W{}^2-k^2)]+(k^2-m_W{}^2)^{-1}\ .$$

The contributions of the first term of the propagator to Figs. 1(c) and 1(d) plus the unaltered Fig. 1(b) correspond to a regulated form of the $O(\alpha)$ QED corrections in the local $V-A$ theory. In this form each photon propagator is endowed with a Feynman regulator with $\Lambda=m_W$. The contributions to Figs. 1(d) and 1(c) of the $(k^2-m_W{}^2)^{-1}$ part of the photon propagator have no correspondence in the local $V-A$ theory but they involve only massive propagators and do not give rise to terms of the ln$m$ type [in the classification of the text they correspond to class (1) diagrams]. Figure 4 shows four diagrams of class (3). In Fig. 4(a) the k integration is automatically "regulated" by the W propagator and the integrand involves a factor $[\pi_1(k^2)/k^2][m_W{}^2/(m_W{}^2-k^2)]$. In Fig. 4(b) the regulator is absent and we have only a factor $\pi_1(k^2)/k^2$. We then leave Fig. 4(a) unaltered and in Fig. 4(b) were split k^{-2} as before. The contribution of $k^{-2}[m_W{}^2/(m_W{}^2-k^2)]$ to Figs. 4(b) and 4(a) are the regulated versions of Figs. 3(b) and 3(a), respectively. In Fig. 4(c) the k_1 integration is regulated by the W propagator while k_2 is not. We then leave $k_1{}^{-2}$ unaltered and split $k_2{}^{-2}$. The additional contributions arising from the $(k^2-m_W{}^2)^{-1}$ part of the splittings correspond to diagrams with two massive propagators and can be analyzed as class (2) diagrams. Diagram 4(d) is a two-photon box diagram. Here the W propagator carries a momentum k_1+k_2. We write

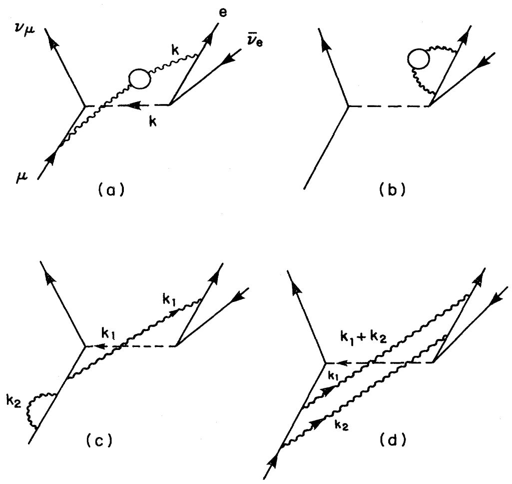

FIG. 4. Some typical two-photon diagrams in μ decay.

$$\frac{1}{m_W{}^2-(k_1+k_2)^2}=\frac{m_W{}^2}{(m_W{}^2-k_1{}^2)}\frac{1}{(m_W{}^2-k_2{}^2)}+\frac{k_1{}^2k_2{}^2+m_W{}^22k_1\cdot k_2}{[m_W{}^2-(k_1+k_2)^2](m_W{}^2-k_1{}^2)(m_W{}^2-k_2{}^2)}\ . \qquad \text{(A1)}$$

The first term regulates the two photons. The term proportional to $k_1{}^2k_2{}^2$ gives a contribution to Fig. 4(d) of class (1). The contribution of the last term to Fig. 4(d) is not strictly of class (1) but it can be analyzed with the same arguments.

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[12]Furthermore, for energy considerations one does not include in the calculation of $1/\tau_\mu$ the emission of τ-$\bar\tau$ pairs. In the limit $m_\tau\to 0$, m_μ fixed, such pairs could be produced and would be degenerate with some of the final states in μ decay. The validity of the theorem on cancellation of mass singularities requires the inclusion of all such degenerate states.

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[22]This is particularly true when the terms not involving $\ln(m_W/m)$ in the $O(\alpha)$ corrections are very small, as it occurs fortuitously for 10 GeV $< m_\phi < m_Z$. For $m_\phi \approx 10 M_Z$ such terms are larger and the more accurate formulas of this paper lead to an additional shift of $\approx +0.065$ GeV for m_W above the prediction of the renormalization-group method explained in the text.

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