x-space analysis for the photon structure functions in QCD

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QCD predictions for the photon structure functions are analyzed with emphasis on the need of retaining hadronlike contributions. The Altarelli-Parisi formalism for the photon structure function is generalized to include next-to-leading-order corrections. Numerical results of the Altarelli-Parisi equations are presented.

I. INTRODUCTION

In recent years the structure functions of the photon have been studied by several authors using perturbative QCD:[1-7] next-to-leading order predictions have been obtained both for the Bjorken limit (where $P^2=-p^2\approx 0$, $Q^2=-q^2$ large, $0<x=Q^2/2p\cdot q<1$) and for the kinematical configuration $\Lambda^2 \ll P^2 \ll Q^2$; here p and q are the momenta (spacelike) of the target and probe photon, respectively, and Λ is the QCD scale parameter.

Two types of terms contribute to the photon structure functions in the Bjorken limit. Hadronlike terms exhibit the same dependence on Q^2 as hadronic deep-inelastic scattering structure functions, whereas pointlike terms are described by a series in integer powers of the effective strong coupling constant $\alpha(Q^2)$: The x dependence of pointlike terms can be calculated in perturbative QCD; hadronlike terms, on the other hand, are not completely calculable by present methods: perturbative QCD predicts only their Q^2 evolution. Since in the limit $Q^2\to\infty$ pointlike terms appear to dominate hadronlike parts, the claim has often been made that the photon structure functions in the Bjorken limit have the unique status of being calculable in perturbative QCD. However, the results for the pointlike parts are affected by spurious power singularities at $x=0$, which must be canceled by corresponding singularities in the hadronlike terms: these singularities become more severe at each order in perturbation theory, thereby compromising the convergence of the perturbation expansion for the calculable part at larger and larger values of x. Unphysical results are avoided only if hadronlike terms are taken into account. This can be done performing an analysis similar to the one used for deep-inelastic scattering off hadrons: Using data taken at a certain $Q^2=Q_0{}^2$ as boundary conditions for the Q^2 evolution, one can obtain the structure functions at different values of Q^2. Vector-meson-dominance estimates can only account for part of the hadronlike terms, since they fail to reproduce the singularity structure of the perturbation expansion.

As first pointed out by the authors of Ref. 5, in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$ the hadronlike contributions can be computed and the additional scale P^2 can be used to fix the renormalization point so that the calculability of the x behavior is not subject to the limitations discussed above. It should be noticed that several of the features characteristic of the Bjorken limit are preserved in this region: in particular, structure functions other than those present in the Bjorken limit are of order P^2/Q^2 and can be neglected; moreover, the various contributions to the structure functions can be consistently separated as leading, next to leading, etc., according to their order with respect to the large logarithm $\ln Q^2/\Lambda^2\approx 1/\alpha(Q^2)$.

The next-to-leading-order predictions have all been derived[2,5] using Wilson's operator-product-expansion (OPE) techniques. The results obtained in this way are, in fact, the Mellin transform of the structure functions and numerical inversion of these results back to x space is necessary to obtain the structure functions themselves. In this paper we show that the next-to-leading-order results can be obtained using an appropriate generalization of the Altarelli-Parisi formalism.[8] In leading order this method has the advantage over the OPE of a more transparent physical interpretation for the quantities involved: in particular, the Altarelli-Parisi equations follow directly from a simple diagrammatic analysis.[3,4] The immediacy of this approach is somewhat reduced beyond leading order by the absence of a unique prescription to define parton distributions. Nevertheless, one still has the advantage of dealing with equations already written in x space whose solutions give the structure functions directly without the need for additional inversion of the moments. Indeed, the x-space formalism is intrinsically more accurate and it is better suited for comparison with experimental results.

This paper is organized as follows. In Sec. II, a review of the results of the OPE approach is presented. In Sec. III, the Altarelli-Parisi formalism is discussed and generalized to include next-to-leading-order effects. Section IV is devoted to the x-space solution of the equations and to a discussion of our numerical results. Explicit formulas for the quantities entering the next-to-leading-order equations are collected in the Appendix.

II. SUMMARY OF OPE RESULTS

The standard OPE approach consists in writing the moments of the structure functions in terms of products of c-number coefficients with reduced matrix elements in target photon states of appropriate operators; for example, the leading-twist contributions to the structure function $F_2^\gamma(n,Q^2)$ in the Bjorken limit[9] are given by

$$F_2^\gamma(n,Q^2)=\int_0^1 dx\, x^{n-2}F_2^\gamma(x,Q^2)$$
$$=\sum_i C_n^i\left[\frac{Q^2}{\mu^2},g^2,\alpha_{\rm EM}\right]\langle\gamma\,|\,O_n^i(\mu)\,|\,\gamma\rangle$$
$$+C_n^\gamma\left[\frac{Q^2}{\mu^2},g^2,\alpha_{\rm EM}\right]\langle\gamma\,|\,O_n^\gamma(\mu)\,|\,\gamma\rangle\ ,$$
$$i=\psi,G,{\rm NS}\ . \quad (2.1)$$

Here g^2 is the renormalized strong coupling constant, μ is the subtraction scale at which the theory is renormalized; the sum contains reduced matrix elements between target photon states of the nonsinglet-fermion $O_n^{\rm NS}$, singlet-fermion O_n^ψ, and gluon O_n^G spin-n, twist-two operators: $C_n^{\rm NS}$, C_n^ψ and C_n^G are the coefficient functions relative to these operators. As first observed by Witten, the distinguishing feature of photon-photon scattering is the presence in (2.1) of terms containing matrix elements of the photon operators O_n^γ: These are absent in the corresponding analysis of deep-inelastic scattering off hadrons. In leading order in $\alpha_{\rm EM}$, the matrix elements of O_n^γ between photon states is simply equal to 1; therefore, despite the fact that the coefficients C_n^γ are $O(\alpha_{\rm EM})$, the last term in the right-hand side (RHS) of (2.1) is of the same order in $\alpha_{\rm EM}$ as the part containing hadronic operators [in these terms the matrix elements are $O(\alpha_{\rm EM})$ and the coefficients are $O(1)$]. In formula (2.1) only leading-twist contributions are included. This should be a good approximation as long as terms of order $O(P^2/Q^2)$ and quark-mass effects can be neglected; such effects will be neglected throughout.

In the Bjorken limit the matrix elements of hadronic operators cannot be calculated perturbatively. They are μ^2 dependent and Q^2 independent so that the Q^2 evolution is completely described by the coefficient functions. Uematsu and Walsh[5] pointed out that in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$, these matrix elements are calculable: Intuitively this can be interpreted by noting that real photons behave like a hadron target, while virtual photons with $P^2 \gg \Lambda^2$ exhibit hard pointlike behavior which can be dealt with perturbatively. Explicit expressions for the matrix elements can be found in the literature.[10,11] At lowest order they are $O(1)$ in the strong coupling constant; they are renormalization-convention and infrared-cutoff dependent: the expressions obtained setting the quark masses equal to zero and using P^2 as infrared cutoff are appropriate for the kinematics under consideration. One can use the additional scale P^2 to fix the subtraction point: letting $\mu^2=P^2$, one can rewrite (2.1) as

$$F_2^\gamma(n,Q^2,P^2)=\frac{\alpha_{\rm EM}}{4\pi}\sum_i C_n^i\left[\frac{Q^2}{P^2},\bar g^2(P^2),\alpha_{\rm EM}\right]A_n^{(2)i}$$
$$+C_n^\gamma\left[\frac{Q^2}{P^2},\bar g^2(P^2),\alpha_{\rm EM}\right],\ i=\psi,G,{\rm NS}\ . \quad (2.2)$$

Here $\bar g^2(P^2)$ is the strong coupling constant evaluated at P^2, and

$$(A_n^{(2)\psi},A_n^{(2)G},A_n^{(2)\rm NS})=6f(\langle e^2\rangle,0,\langle e^4\rangle-\langle e^2\rangle^2)\tilde A_{nG}^{(2)\psi}\ , \quad (2.3)$$

where $f\equiv$ number of flavors and

$$f\langle e^{2,4}\rangle=\sum_{i=1}^f e_i^{2,4}$$

with $e_i\equiv$ the charge of the quark of flavor i in units of e. One has to make sure that the various quantities entering (2.2) are calculated in the same renormalization scheme: we shall work in the modified minimal-subtraction ($\overline{\rm MS}$) scheme and use for $\tilde A_{nG}^{(2)\psi}$ the expression found from Ref. 10 and reported in the Appendix.

The Q^2 evolution can be obtained by solving the renormalization-group equations for the coefficient functions: one solves such equations using as boundary conditions the perturbatively calculable expressions for the coefficients $C_n^i(1,\bar g^2)$ at $\mu^2=Q^2$. Consider, for example, the nonsinglet part of $F_2^\gamma(n,Q^2)$ in the Bjorken limit: one gets

$$F_2^{\rm NS}(n,Q^2)=\left[\frac{\alpha_{\rm EM}}{4\pi}A_{\rm NS}^n(Q_0{}^2)M_{\rm NS}^n(\bar g^2,\bar g_0{}^2)+X_{\rm NS}^n(\bar g^2,\bar g_0{}^2,\alpha_{\rm EM})\right]C_n^{\rm NS}(1,\bar g^2)\ . \quad (2.4)$$

Here $\alpha_{\rm EM}A_{\rm NS}^n(Q_0{}^2)/4\pi$ is the reduced matrix element of the fermion nonsinglet operator taken between real photon states, $\bar g^2$ and $\bar g_0{}^2$ are the renormalized strong coupling constants at Q^2 and at the subtraction scale $Q_0{}^2=\mu^2$, respectively [$\alpha(Q^2)=\bar g^2/4\pi$], and

$$M_{\rm NS}^n(\bar g^2,\bar g_0{}^2)=\exp\int_{\bar g}^{\bar g_0}dg\frac{\gamma_{\rm NS}^n(g)}{\beta(g)}\ , \quad (2.5a)$$

$$X_{\rm NS}^n(\bar g^2,\bar g_0{}^2,\alpha_{\rm EM})=\int_{\bar g}^{\bar g_0}dg\frac{K_{\rm NS}^n(g,\alpha_{\rm EM})}{\beta(g)}\times M_{\rm NS}^n(\bar g^2,g^2)\ . \quad (2.5b)$$

Using (2.4) and (2.5), one can write $F_2^{\rm NS}(n,Q^2)$ explicitly in terms of the coefficients of the perturbative expansions for the anomalous dimensions $\gamma_{\rm NS}^n(g)$ and $K_{\rm NS}^n(g,\alpha)$, for $C_n^{\rm NS}(1,g^2)$, and for the β function $\beta(g)$:

$$\gamma_{\rm NS}^n(g)=\frac{g^2}{16\pi^2}\gamma_{\rm NS}^{0,n}+\frac{g^4}{16\pi^2}\gamma_{\rm NS}^{1,n}+\cdots\ , \quad (2.6a)$$

$$K_{\rm NS}^n(g,\alpha_{\rm EM})=\frac{\alpha_{\rm EM}}{4\pi}\left[-K_{\rm NS}^{0,n}-\frac{g^2}{16\pi^2}K_{\rm NS}^{1,n}\cdots\right]\ , \quad (2.6b)$$

$$C_n^{\rm NS}(1,g^2)=1+\frac{g^2}{16\pi^2}B_{\rm NS}^n+\cdots\ , \quad (2.6c)$$

$$\beta(g)=-\beta_0\frac{g^3}{16\pi^2}-\beta_1\frac{g^5}{(16\pi^2)}-\cdots\ . \quad (2.6d)$$

The explicit form of the coefficients in the expansions above is available in the literature up to next-to-leading order and can be obtained, for example, from Refs. 2 and

12. Using the expansions (2.6) in the integrands in the RHS of (2.5), one gets

$$\frac{\gamma_{\rm NS}^n(g)}{\beta(g)}=-\frac{1}{g}\sum_{p=0}^{\infty}\Gamma_p(n)\left[\frac{g^2}{16\pi^2}\right]^l$$

$[\Gamma_0(n)=\gamma_{\rm NS}^{0,n}/\beta_0=2d_{\rm NS}^n\ ,\ \Gamma_1(n)=(\gamma_{\rm NS}^{1,n}-2\beta_1 d_{\rm NS}^n)/\beta_0]$,

so that

$$M_{\rm NS}^n(\bar g^2,\bar g_0{}^2)=\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^{d_{\rm NS}^n}\frac{f(n,\bar g)}{f(n,\bar g_0)} \tag{2.7}$$

with

$$f(n,g)=\exp\sum_{l=1}^{\infty}\frac{\Gamma_l(n)}{2l}\left[\frac{g^2}{16\pi^2}\right]^l =1+\frac{\Gamma_1(n)}{2}\frac{g^2}{16\pi^2}+\cdots$$

and

$$\frac{K_{\rm NS}^n(g,\alpha_{\rm EM})}{\beta(g)f(n,g)}=\frac{4\pi\alpha_{\rm EM}}{g^3}\sum_{m=0}^{\infty}\Omega_m(n)\left[\frac{g^2}{16\pi^2}\right]^m$$

$[\Omega_0(n)=K_{\rm NS}^{0,n}/\beta_0,\ldots]$, so that

$$X_{\rm NS}^n(\bar g^2,\bar g_0{}^2,\alpha_{\rm EM})=\frac{\alpha_{\rm EM}}{4\pi}f(n,\bar g)\sum_{m=0}^{\infty}\pi_m(n)\left[\left[\frac{\bar g^2}{16\pi^2}\right]^{m-1}-\left[\frac{\bar g_0{}^2}{16\pi^2}\right]^{m-1}\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^{d_{\rm NS}^n}\right] \tag{2.8}$$

with

$$\pi_m(n)=\frac{1}{2}\frac{\Omega_m(n)}{d_{\rm NS}^n+1-m}\ . \tag{2.9}$$

Inserting (2.7) and (2.8) in (2.4), one gets

$$F_2^{\rm NS}(n,Q^2)=\frac{\alpha_{\rm EM}}{4\pi}C_{\rm NS}^n(1,\bar g^2)f(n,\bar g)\left\{\sum_{m=0}^{\infty}\pi_m(n)\left[\frac{\bar g^2}{16\pi^2}\right]^{m-1}+\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^{d_{\rm NS}^n}\left[\frac{A_{\rm NS}^n(Q_0{}^2)}{f(n,\bar g_0)}-\sum_{m=0}^{\infty}\pi_m(n)\left[\frac{\bar g_0{}^2}{16\pi^2}\right]^{m-1}\right]\right\}. \tag{2.10}$$

Note that $F_2^{\rm NS}(n,Q^2)$ contains a calculable (pointlike) part which can be written as an expansion in integer powers of the strong coupling constant, and a part which includes terms noncalculable in perturbative QCD and which exhibits the same Q^2 behavior as hadronic deep-inelastic-scattering structure functions. In fact, the treatment above can be repeated for the singlet sector and the complete result can be summarized by

$$F_2^\gamma(n,Q^2)=\frac{\alpha_{\rm EM}}{4\pi}\left[\frac{4\pi}{\beta_0\alpha}a(n)+b(n)+\sum_{l=1}^{\infty}r_l(n)\alpha^l+\sum_i\sum_{l=0}^{\infty}h_{i,l}(n)\alpha^{d_i^n+l}\right],$$

$$i=+,-,{\rm NS}\ . \tag{2.11}$$

The quantities $a(n)$, $b(n)$, and $r_l(n)$ appearing here are calculable in perturbative QCD, while the quantities $h_{i,l}(n)$ contain noncalculable terms; the exponents $d_\pm^n$ are related to the eigenvalues $\lambda_\pm^n$ of the singlet one-loop anomalous-dimension matrix by $d_\pm^n=\lambda_\pm^n/2\beta_0$.

Since the exponents d_i^n are positive for $n>2$ (see Table I) and since in the Bjorken limit Q^2 is assumed to be large, there has been a tendency in the literature to formally treat the hadronlike parts as higher-order corrections whose effect could be neglected at sufficiently large Q^2. On this basis the claim has often been made that both the x and Q^2 dependence of the photon structure functions are obtainable from perturbative QCD, i.e., that the knowledge of the structure functions at some reference value $Q_0{}^2$ is not required to obtain them at values of Q^2 large enough. It has, however, been noted[7] that the quantities $\pi_m(n)$ of Eq. (2.9) have simple poles at those values n_m of n where $d_{\rm NS}^n=m-1$, so that their inverse Mellin transform $\pi_m(x)$,

$$\pi_m(n)=\int_0^1 dx\,x^{n-1}\pi_m(x)$$

exhibits at the origin singularities of the form $(1/x)^{n_m}$. Similar singularities affect the singlet analogs of $\pi_m(x)$. In fact, since $d_-^n>d_{\rm NS}^n$, the $x\to 0$ behavior of $a(x)$, $b(x)$, and $r_l(x)$ is determined by the zeros of (d_-^n-m+1). In particular, for $f=4$ one has (see Table I)

$$a(x)\propto(1/x)^{n=1.596},\quad b(x)\propto(1/x)^{n=2}\ ,$$

and $r_l(x)=(1/x)^{n_l}$, where n_l is the value of n at which $d_-^n=l$.

At each successive order the exponents n_l grow larger so that the power singularities at the origin become increasingly severe. At any experimentally foreseeable Q^2 their effects cannot be significantly damped by the additional powers of $\alpha(Q^2)$. The presence of these singularities compromises the convergence of the perturbation expansion for the pointlike part at small x and as higher orders are taken into account larger and larger values of x (those controlled by $n<n_l$) are affected.

From Eqs. (2.8)–(2.10) one sees that the singularities of the pointlike part are in fact canceled by corresponding

TABLE I. Values of the quantities d^n_{NS}, d^n_-, and d^n_+ (four flavors).

n	d^n_{NS}	d^n_-	d^n_+
2	0.427	0	0.747
3	0.667	0.609	1.386
4	0.837	0.817	1.852
5	0.971	0.960	2.192
6	1.080	1.074	2.460
	$d_{NS}(n=0.3099)=-1$	$d_-(n=1.596)=-1$	
	$d_{NS}(n=1)=0$	$d_-(n=2)=0$	
	$d_{NS}(n=5.250)=1$	$d_-(n=5.326)=1$	$d_+(n=2.386)=1$
	$d_{NS}(n=26.58)=2$	$d_-(n=26.59)=2$	$d_+(n=4.402)=2$

singularities in the hadronlike terms since

$$\lim_{A\to 0}\frac{1}{A}\left[1-\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^A\right]=-\ln\left[\frac{\bar g^2}{\bar g_0{}^2}\right] .$$

Therefore even for values of Q^2 well above the range (Q^2 of the order of 10 GeV^2) at which current experimental data are taken, hadronlike contributions must be included in the theoretical analysis to avoid unphysical singular predictions at $x=0$. At each successive order in perturbation theory the effect of hadronlike terms becomes relevant at larger and larger values of x.

In leading order only the region of very small x is affected: hadronlike terms cancel from the nonsinglet and singlet parts of $a(x)$ spikes of the form $(1/x)^{0.3099}$ and $(1/x)^{1.596}$ which are present, for example, in the plots obtained for these quantities in Ref. 4. In next-to-leading order taking hadronlike terms into account eliminates the unphysical negative $(1/x)$ behavior of the pointlike part of $F_2^\gamma(x,Q^2)$ obtained by Duke and Owens[2] for the region $x \lesssim 0.25$. At the next order $O(\alpha)$, one can expect significant modifications of the pointlike result in the whole range controlled by $n \lesssim 5.3$ (i.e., $x \lesssim 0.8$). It should, however, be noted that even for Q^2 in the range at which current experiments are performed there might well be no strong need of pursuing the theoretical analysis beyond the next-to-leading order [in particular, higher twists and quark-mass effects might well yield corrections more sizable than those due to $O(\alpha)$ terms]. It might therefore be argued that there is some region of x ($0.5 \lesssim x \lesssim 0.8$) where, truncating at the next-to-leading level the perturbation expansion for the pointlike part alone, one obtains an approximate prediction for the structure function.

The treatment used for the Bjorken-limit structure function can be repeated for the structure functions in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$. As noted above, the matrix elements of hadronic operators can be calculated for this region and the additional scale P^2 can be used to fix the subtraction point. Therefore, one obtains a complete prediction for both the x and Q^2 behavior which is free of power singularities at $x=0$.[5]

One can see from (2.7) and (2.10) that the part of $F_2^{NS}(n,Q^2)$ containing $A^n_{NS}(Q_0{}^2)$ has the same form as the complete nonsinglet part of a hadronic structure function. In fact, Eq. (2.10) can be rewritten as

$$F_2^{NS}(n,Q^2)=\frac{\alpha_{EM}}{4\pi}C^n_{NS}(1,\bar g^2)f(n,\bar g)\sum_{m=0}^{\infty}\pi_m(n)\left[\left[\frac{\bar g^2}{16\pi^2}\right]^{m-1}-\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^{d^n_{NS}}\left[\frac{\bar g_0{}^2}{16\pi^2}\right]^{m-1}\right]$$
$$+\frac{C^n_{NS}(1,\bar g^2)f(n,\bar g)}{C^n_{NS}(1,\bar g_0{}^2)f(n,\bar g_0)}\left[\frac{\bar g^2}{\bar g_0{}^2}\right]^{d^n_{NS}}F_2^{NS}(n,Q_0{}^2) . \quad (2.12)$$

The last term which contains $A^n_{NS}(Q_0{}^2)$ contributes to $F_2^{NS}(n,Q^2)$ already in leading order and of course it is the only term which survives at $Q_0{}^2=Q^2$. On the other hand, in the region $\Lambda^2 \ll P^2 \ll Q^2$ the matrix elements of Eq. (2.3) which enter the analog of Eq. (2.10) are of order $O(1)$ and begin contributing to the structure function in next-to-leading order.

In practice, perturbative expansions such as (2.10) can be carried out explicitly only up to a certain order: At present results are available up to next-to-leading order. For the region $\Lambda^2 \ll P^2 \ll Q^2$ these can be written in the form

$$F_2^\gamma(n,Q^2,P^2)=\frac{\alpha_{EM}}{4\pi}\left\{\frac{4\pi}{\beta_0\alpha}\frac{1}{2}\sum_i L_i(n)\left[1-\left[\frac{\alpha}{\alpha_p}\right]^{1+d^n_i}\right]+\sum_i\frac{A_i(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_p}\right]^{d^n_i}\right]\right.$$
$$\left.+\sum_i\frac{B_i(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_p}\right]^{i+d^n_i}\right]+\tilde D_\gamma(n)\right\},\quad i=\mathrm{NS},\pm . \quad (2.13)$$

Here $\alpha_p = \bar{g}^2(P^2)/4\pi$; the explicit form of $L_i(n)$, $A_i(n)$, and $B_i(n)$ can be found in Ref. 5 and

$$\tilde{D}_\gamma(n) = 3f\langle e^4\rangle(B_\gamma^n + 2\tilde{A}_{nG}^{(2)\psi}) , \tag{2.14}$$

where B_γ^n is the one-loop photon coefficient function.

In the Bjorken limit, one can write the next-to-leading-order structure function in analogy to (2.12) as the sum of two parts: The first is proportional to $\langle\gamma\,|\,O_\gamma^n\,|\,\gamma\rangle$ and has the form (2.13) with α_p substituted by $\alpha_0 = \bar{g}_0{}^2/4\pi$ and with terms containing $\tilde{A}_{nG}^{(2)\psi}$ in $A_i(n)$ and $\tilde{D}_\gamma(n)$ set equal to zero; the second part contains matrix elements of hadronic operators and its form is the same as that of structure functions of hadrons.[13] Here we report the result for the valence part (the part proportional to $\langle e^4\rangle$) only:

$$F_2^{\rm val}(n,Q^2) = \frac{\alpha_{\rm EM}}{4\pi}\frac{\langle e^4\rangle}{\langle e^4\rangle - \langle e^2\rangle^2}\left\{\frac{4\pi}{\beta_0\alpha}\frac{1}{2}L_{\rm NS}(n)\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{1+d_{\rm NS}^n}\right] + \frac{A_{\rm NS}^{\rm Bj}(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{d_{\rm NS}^n}\right]\right.$$
$$\left. + \frac{B_{\rm NS}(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{1+d_{\rm NS}^n}\right]\right\}$$
$$+3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}B_\gamma^n + \left[1+\frac{\alpha-\alpha_0}{4\pi}R_{\rm NS}^n\right]\left[\frac{\alpha}{\alpha_0}\right]^{d_{\rm NS}^n}\frac{\langle e^4\rangle}{\langle e^4\rangle-\langle e^2\rangle^2}F_2^{\rm NS}(n,Q_0{}^2) . \tag{2.15}$$

Here $A_{\rm NS}^{\rm Bj}(n)$ is obtained from $A_{\rm NS}(n)$ of Eq. (2.13) dropping terms containing $\tilde{A}_{nG}^{(2)\psi}$ and

$$R_{\rm NS}^n = B_{\rm NS}^n + \frac{\gamma_{NS}^{1,n}}{2\beta_0} - \frac{\beta_1}{2\beta_0{}^2}\gamma_{\rm NS}^{0,n} . \tag{2.16}$$

The different way in which matrix elements of hadronic operators enter (2.13) and (2.15) is attributable to the different physical significance of the scales P^2 and $Q_0{}^2$. In (2.15), $Q_0{}^2$ is an arbitrary scale (within the perturbative range) at which data are taken. From these data, using (2.15) and its analog for the sea part, one can predict the form of the structure function at different values of Q^2. Within the precision afforded by perturbation theory, these predictions must be $Q_0{}^2$ independent if the theory is to be consistent. On the other hand, P^2 is a parameter determined from experiments; for any P^2 and Q^2 which satisfy the condition $\Lambda^2 \ll P^2 \ll Q^2$, Eq. (2.13) predicts completely (up to next-to-leading order) the form of $F_2^\gamma(x,Q^2,P^2)$: these predictions depend on both Q^2 and P^2.

It should be noted that in Eq. (2.15) (and in its analog for the sea part), terms containing matrix elements of hadronic operators are renormalization-convention dependent in next-to-leading order: Since the whole valence part must be convention independent and since at $Q^2 = Q_0{}^2$ one has

$$F_2^{\rm val}(n,Q^2=Q_0{}^2) = 3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}B_\gamma^n + \frac{\langle e^4\rangle}{\langle e^4\rangle-\langle e^2\rangle^2}F_2^{\rm NS}(n,Q^2=Q_0{}^2) , \tag{2.17}$$

so that the scheme dependence of the next-to-leading part of $F_2^{\rm NS}(n,Q^2=Q_0{}^2)$ must be such as to cancel the scheme dependence of B_γ^n. At $Q^2 \neq Q_0{}^2$, the convention dependences of B_γ^n and $F_2^{\rm NS}(n,Q^2)$ are canceled by the convention dependence of the term containing $A_{\rm NS}^{\rm Bj}(n)$.

Vector-meson-dominance (VMD) arguments have often been used in the literature to obtain estimates of the hadronlike contributions. It is apparent that the VMD model can account for only part of the hadronlike terms since it cannot reproduce the detailed singularity structure of the perturbation expansion discussed above. In attempts to identify those terms of the structure function which can be accounted for by the VMD model, one should notice that such parts must be convention independent. Since the VMD model can account for the $\gamma\gamma$ total cross section up to $Q^2 \approx 1$ GeV^2, it is possible to obtain VMD predictions for $F_2^{\rm val}(x,Q^2 \simeq 1$ GeV) and its analog for the sea part; on the other hand, since at these low values of Q^2, perturbative expansions such as the one in the RHS of (2.17) are unreliable, one is left with the problem of evaluating the quantity $F_2^{\rm NS}(n,Q^2=1$ GeV) that might be used as a boundary condition in (2.15).

The quantities d_i^n which enter the exponents in (2.13) and (2.15) are proportional to (ln n) at large n; as a result, in the region $x \to 1$, which is controlled by large n, the terms containing $(\alpha/\alpha_p)^{d_i}$ and $(\alpha/\alpha_0)^{d_i}$ become less and less relevant. The limiting behavior has been studied in detail in Ref. 14. It turns out that the predictions of perturbation theory are unreliable in this region: in fact, the leading term is suppressed by a $\ln(1-x)$ with respect to the next-to-leading term, which is negative. As a result an unphysical negative result for the structure function is obtained for $x \simeq 1$ at finite Q^2.

III. x-SPACE FORMALISM

Use of the Altarelli-Parisi equations in the context of photon-photon scattering has first been discussed by the authors of Ref. 4. The difference between these equations and the ones for hadronic deep-inelastic scattering is that in addition to the quark $q^i(x,t)$ and gluon $G(x,t)$ densities, one has to consider the elementary photon density $\Gamma(x,t)$ in the target photon. The densities $q^i(x,t)$ and $G(x,t)$ are $O(\alpha_{\rm EM})$, $\Gamma(x,t)$ is simply

$$\Gamma(x,t)=\delta(x-1)+O(\alpha_{\rm EM})$$

(this is the translation in parton language of $\langle\gamma\,|\,O_n^\gamma\,|\,\gamma\rangle=1$). Therefore the Altarelli-Parisi equations at lowest order in $\alpha_{\rm EM}$ and in the effective strong coupling constant $\alpha(t)$ are

$$\begin{aligned}\frac{dq^i}{dt}(x,t)&=\frac{1}{2\pi}\int_x^1\frac{dy}{y}\left\{\alpha(t)\left[P_{qq}^0\left[\frac{x}{y}\right]q^i(y,t)+P_{qG}^0\left[\frac{x}{y}\right]G(y,t)\right]+\alpha_{\rm EM}e_i{}^2P_{q\gamma}^0\left[\frac{x}{y}\right]\Gamma(y,t)\right\},\\ \frac{dG}{dt}(x,t)&=\frac{\alpha(t)}{2\pi}\int_x^1\frac{dy}{y}\left[P_{Gq}^0\left[\frac{x}{y}\right]\sum_{J=1}^{2f}q^J(y,t)+P_{GG}^0\left[\frac{x}{y}\right]G(y,t)\right]\end{aligned}\tag{3.1}$$

with the photon structure function $F_2^\gamma(x,t)$ given by

$$F_2^\gamma(x,t)=\sum_{i=1}^{2f}e_i{}^2xq^i(x,t)\ .\tag{3.2}$$

Here $t=\ln Q^2/\Lambda^2$, $\alpha(t)=4\pi/\beta_0 t$, and $P_{qq}^0(z)$, $P_{qG}^0(z)$, $P_{Gq}^0(z)$, and $P_{GG}^0(z)$ are the ordinary Altarelli-Parisi fragmentation functions.[8] The new fragmentation $P_{q\gamma}^0(z)$ is given by[4]

$$P_{q\gamma}^0(z)=3a^0(z)=6p_{qG}^0(z)\ ,\tag{3.3}$$

where $a^0(z)=2z^2-2z+1$ and the factor 3 corresponds to quark densities summed over colors.

Ordinarily the equations are rewritten in terms of singlet and nonsinglet quark densities. Letting

$$q^{\rm S}(x,t)=\frac{1}{2f}\sum_{i=1}^{2f}q^i(x,t),\quad q^{\rm NS}(x,t)=\tfrac{1}{2}[q^u(x,t)-q^d(x,t)]\tag{3.4}$$

(q^u and q^d refer to densities of quarks with the same charge assignment of the up and down quark, respectively), one gets

$$\frac{dq^{\rm NS}}{dt}(x,t)=3(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}\frac{\alpha_{\rm EM}}{2\pi}a^0(x)+\frac{2}{\beta_0 t}\int_x^1\frac{dy}{y}P_{qq}^0\left[\frac{x}{y}\right]q^{\rm NS}(y,t)\ ,\tag{3.5}$$

$$\frac{d}{dt}q^{\rm S}(x,t)=3\langle e^2\rangle\frac{\alpha_{\rm EM}}{2\pi}a^0(x)+\frac{2}{\beta_0 t}\int_x^1\frac{dy}{y}\left[P_{qq}^0\left[\frac{x}{y}\right]q^{\rm S}(y,t)+P_{qG}^0\left[\frac{x}{y}\right]G(y,t)\right]\ ,\tag{3.6a}$$

$$\frac{d}{dt}G(x,t)=\frac{2}{\beta_0 t}\int_x^1\frac{dy}{y}\left[2fP_{Gq}^0\left[\frac{x}{y}\right]q^{\rm S}(y,t)+P_{GG}^0\left[\frac{x}{y}\right]G(y,t)\right]\ ,\tag{3.6b}$$

where

$$F_2^\gamma(x,t)=F_2^{\rm NS}(x,t)+F_2^{\rm S}(x,t)=2f[(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}xq^{\rm NS}(x,t)+\langle e^2\rangle xq^{\rm S}(x,t)]\ .\tag{3.7}$$

Due to the new fragmentation function $P_{q\gamma}^0(z)$, (3.5) and (3.6a) contain inhomogeneous terms which are absent in the corresponding equations for deep-inelastic scattering off hadrons.

Using the convolution properties of the Mellin transform

$$\int_0^1 dx\,x^{n-1}\int_x^1\frac{dy}{y}P\left[\frac{x}{y}\right]h(y)=P(n)h(n)\ ,$$

where $f(n)=\int_0^1 dx\,x^{n-1}f(x)$, one obtains the n-space version of the equations

$$\frac{dq^{\rm NS}}{dt}(n,t)=a_{\rm NS}^0(n)-\frac{1}{t}d_{\rm NS}^nq^{\rm NS}(n,t)\ ,\tag{3.8}$$

$$\frac{dq^{\rm S}}{dt}(n,t)=a_{\rm S}^0(n)-\frac{1}{t}[d_{qq}^nq^{\rm S}(n,t)+d_{qG}^nG(n,t)]\ ,\tag{3.9a}$$

$$\frac{dG}{dt}(n,t)=-\frac{1}{t}[2fd_{Gq}^nq^{\rm S}(n,t)+d_{GG}^nG(n,t)]\ ,\tag{3.9b}$$

where

$$\begin{aligned}a_{\rm NS}^0(n)&=\frac{3\alpha_{\rm EM}}{2\pi}(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}a^0(n)\ ,\\ a_{\rm S}^0(n)&=\frac{3\alpha_{\rm EM}}{2\pi}\langle e^2\rangle a^0(n)\ ,\end{aligned}\tag{3.10}$$

and where the quantities d_{ij}^n are related to the Mellin transform of $P_{ij}^0(z)$ and to the usual one-loop anomalous dimensions $\gamma_{ij}^{0,n}$ by

$$P^0_{qq}(n)=-\frac{\gamma^{0,n}_{\psi\psi}}{4}=-\frac{\beta_0}{2}d^n_{qq}=-\frac{\beta_0}{2}d^n_{\rm NS}\ ,$$
$$P^0_{qG}(n)=-\frac{\gamma^{0,n}_{\psi G}}{8f}=-\frac{\beta_0}{2}d^n_{qG}\ ,$$
$$P^0_{Gq}(n)=-\frac{\gamma^{0,n}_{G\psi}}{4}=-\frac{\beta_0}{2}d^n_{Gq}\ ,\qquad (3.11)$$
$$P^0_{GG}(n)=-\frac{\gamma^{0,n}_{GG}}{4}=-\frac{\beta_0}{2}d^n_{GG}\ .$$

Once a set of equations such as (3.5) and (3.6) or (3.8) and (3.9) is written down one must specify an appropriate set of boundary conditions for the t evolution. In practice, one must get insight from other methods such as the OPE. This is illustrated by the following example. Consider (3.8). In the Bjorken limit, following Ref. 4, one may require

$$q^{\rm NS}(n,t)=h(n)t\left[1+O\left[\frac{1}{t}\right]\right]\ .\qquad (3.12)$$

Then from (3.8) one gets

$$h(n)=\frac{a^0_{\rm NS}(n)}{1+d^n_{\rm NS}}\ .\qquad (3.13)$$

On the other hand, one may require

$$q^{\rm NS}(n,t)=\sum_A h_A(n)t\left[1-\left[\frac{t_0}{t}\right]^A\right]\ .\qquad (3.14)$$

Substituting (3.14) in (3.8), one gets

$$\sum_A h_A\left[1+(A-1)\left[\frac{t_0}{t}\right]^A\right]=a^0_{\rm NS}-\sum_A h_A d_{\rm NS}\left[1-\left[\frac{t_0}{t}\right]^A\right]\ ,$$

and separating different t dependences one obtains

$$A=1+d_{\rm NS}\ ;$$

that is, there is one value of A for each different n and the corresponding $h_A(n)$ is given by (3.13) so that (3.14) becomes

$$q^{\rm NS}(n,t)=\frac{a^0_{\rm NS}}{1+d_{\rm NS}}t\left[1-\left[\frac{t_0}{t}\right]^{1+d_{\rm NS}}\right]\ .\qquad (3.15)$$

Letting $t_0=\ln Q_0{}^2/\Lambda^2$, Eq. (3.15) reproduces the complete leading result of OPE for the part of (2.4) containing the matrix element of the photon operator. On the other hand, Eqs. (3.12) and (3.13) give the result in the formal limit $Q^2\to\infty$: i.e., when powers of α/α_0 are dropped and only the pointlike part is retained. The results of Ref. 4, which are obtained solving the equations in x space with boundary conditions of the form (3.12), indeed exhibit at $x\to 0$ the singular behavior of the pointlike part.

The Altarelli-Parisi formulation does not itself provide any reason to choose (3.12) or (3.14) as boundary conditions. By the same token it should be noted that the part of $F_2^{\rm NS}(x,Q^2)$ containing matrix elements of $O_n^{\rm NS}$ has not been accounted for in the above treatment. To do so one can treat such parts independently with ordinary homogeneous hadronic Altarelli-Parisi equations[15] and add the two parts at the end of the calculation. Equivalently,[16] one can include the solution of the homogeneous equations directly in the boundary conditions and use

$$q^{\rm NS}(n,t)=\sum_A h_A t\left[1-\left[\frac{t_0}{t}\right]^A\right]+q^{\rm NS}(n,t_0)\left[\frac{t}{t_0}\right]^{d_{\rm NS}}\qquad (3.16)$$

in place of (3.14).

The leading-order results in the region $\Lambda^2\ll P^2\ll Q^2$ can be obtained from Eqs. (3.8) and (3.9), requiring

$$q^{\rm NS}(n,t,t_p)=\sum_A h_A^{\rm NS}(n)t\left[1-\left[\frac{t_p}{t}\right]^A\right]\ ,$$
$$q^{\rm S}(n,t,t_p)=\sum_A h_A^{\rm S}(n)t\left[1-\left[\frac{t_p}{t}\right]^A\right]\ ,\qquad (3.17)$$
$$G(n,t,t_p)=\sum_A g_A(n)t\left[1-\left[\frac{t_p}{t}\right]^A\right]\ ,$$

where $t_p=\ln P^2/\Lambda^2$. The procedure used in the first of Ref. 5 is equivalent to these prescriptions. Contrary to what happens for the Bjorken limit [see (3.16)], no term containing matrix elements of hadronic operators enters the leading-order result in this case. This is in agreement with the fact discussed in the previous section, that for this region such terms are $O(1)$ in $\alpha(Q^2)$, so that they start contributing to the structure function in the next-to-leading order.

Extension of the Altarelli-Parisi formalism to next-to-leading order for hadronic deep-inelastic scattering has been discussed by several authors.[17–20] One major novel feature is that there is no unique way to define parton densities:[17] the most common approach, which will be adopted below, can be illustrated considering formula (2.4) for the nonsinglet portion of $F_2^\gamma(n,Q^2)$. One can define $q^{\rm NS}(x,Q^2)$ as the inverse Mellin transform of

$$q^{\rm NS}(n,Q^2)=q^{{\rm NS},H}(n,Q^2)+q^{{\rm NS},\gamma}(n,Q^2)\ ,\qquad (3.18)$$

where

$$\delta'_{\rm NS}q^{{\rm NS},H}(n,Q^2)=\langle\gamma\mid O_n^{\rm NS}\mid\gamma\rangle M^n_{\rm NS}(\bar g^2(Q^2),\bar g^2(Q_0{}^2))\ ,\qquad (3.19a)$$

$$\delta'_{\rm NS}q^{{\rm NS},\gamma}(n,Q^2)=\langle\gamma\mid O_n^\gamma\mid\gamma\rangle X^n_{\rm NS}(\bar g^2(Q^2),\bar g^2(Q_0{}^2),\alpha_{\rm EM})\qquad (3.19b)$$

[the charge factor $\delta'_{\rm NS}=2f(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}$ has been introduced to ensure consistency with (3.4) and (3.7)]. The structure function is then obtained as

$$F_2^{\rm NS}(x,Q^2)=\int_x^1\frac{dy}{y}C^{\rm NS}\left[\frac{x}{y},Q^2\right]q^{\rm NS}(y,Q^2)\delta'_{\rm NS}\ .$$

Here $C^{\rm NS}(z,Q^2)$ is the inverse Mellin transform of

$C_n^{\rm NS}(1,\bar g^2(Q^2))$ and can be interpreted as an elementary cross section for the scattering of a current off a quark with an effective Q^2-dependent distribution $q^{\rm NS}(x,Q^2)$. Alternatively, one can, for example, express the structure function completely in terms of a quark density $\tilde q^{\rm NS}(x,Q^2)$ defined by

$$F_2^{\rm NS}(x,Q^2)=x\tilde\delta_{\rm NS}\tilde q^{\rm NS}(x,Q^2)\ ,$$

where $\tilde\delta_{\rm NS}$ is some charge factor. These two definitions are both equivalent in leading order to (3.7). The advantage of the first one is that the structure-function-dependent part of (2.4) is separated from the universal quark density: for example, to obtain the nonsinglet part $F_L^{\rm NS}(x,Q^2)$ of the longitudinal structure function $F_L^\gamma(x,Q^2)$, one only has to convolute $q_{\rm NS}(x,Q^2)$ defined by (3.18) with the appropriate coefficient function $C_L^{\rm NS}(z,Q^2)$. The disadvantage is that the parton densities and the elementary parton cross sections defined in this way are separately regularization-prescription dependent and, in general, gauge dependent: therefore, to compare corresponding parton distributions for different processes, one has to make sure that they are defined consistently in the same renormalization scheme. We shall use the $\overline{\rm MS}$ scheme throughout. Incidentally, the two definitions above can be related, respectively, to the two-step and one-step procedures of Ref. 20.

Several sources of higher-order contributions must be taken into account to write the Altarelli-Parisi equations in next-to-leading order. For hadronic deep-inelastic scattering there are higher-order terms in the coefficient functions, in the fragmentation functions, and in the β function. To deal with the latter it is convenient to write the equations using α itself, instead of t, as the evolution parameter: this is done by observing that the derivatives with respect to t can be rewritten as

$$\frac{d}{dt}\to-\left[\frac{\beta_0}{4\pi}\alpha^2+\frac{\beta_1}{(4\pi)^2}\alpha^3\right]\frac{d}{d\alpha}\ . \tag{3.20}$$

For two-photon processes it is also necessary to take into consideration higher-order corrections to the inhomogeneous terms. In particular, since $P_{G\gamma}$ becomes relevant at this order inhomogeneous terms are now present in all the equations.

Let us first consider the kinematical configuration $\Lambda^2\ll P^2\ll Q^2$. As remarked above, for this region terms containing the matrix elements of hadronic operators first appear in the next-to-leading order. The way such terms are included in the equations below is equivalent to solving separately a set of homogeneous Altarelli-Parisi equations for the hadronic term with boundary conditions of the form

$$\delta'_{\rm NS}q^{{\rm NS},H}(n,\alpha=\alpha_p)=A_n^{(2)\rm NS}$$

and to add these terms to the photonic contribution obtained solving inhomogeneous equations. Since the procedure used below accounts for the hadronic parts through an alteration of the inhomogeneous terms and the photon coefficient function only, in the diagrammatic formulation of Ref. 3 one may interpret such modifications as a change in the coupling between the photon of momentum p and the fermions.

One obtains

$$-\left[\frac{\beta_0}{4\pi}\alpha^2+\frac{\beta_1}{(4\pi)^2}\alpha^3\right]\frac{dq^{\rm NS}}{d\alpha}(x,\alpha,\alpha_p)=\frac{\alpha_{\rm EM}}{4\pi}6(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}\times\left[a^0(x)-\frac{\alpha}{4\pi}[C_FF_{qG}^2(x)-I_1(x)]\right]+\frac{\alpha}{2\pi}\int_x^1\frac{dy}{y}P_{\rm NS}\left[\frac{x}{y},\alpha\right]q^{\rm NS}(y,\alpha,\alpha_p)\ , \tag{3.21}$$

$$-\left[\frac{\beta_0}{4\pi}\alpha^2+\frac{\beta_1}{(4\pi)^2}\alpha^3\right]\frac{dq^s}{d\alpha}(x,\alpha,\alpha_p)=\frac{\alpha_{\rm EM}}{4\pi}6\langle e^2\rangle\left[a^0(x)-\frac{\alpha}{4\pi}[C_FF_{qG}^2(x)-I_1(x)]\right]+\frac{\alpha}{2\pi}\int_x^1\frac{dy}{y}\left[P_{qq}\left[\frac{x}{y},\alpha\right]q^s(y,\alpha,\alpha_p)+P_{qG}\left[\frac{x}{y},\alpha\right]G(y,\alpha,\alpha_p)\right]\ , \tag{3.22a}$$

$$-\left[\frac{\beta_0}{4\pi}\alpha^2+\frac{\beta_1}{(4\pi)^2}\alpha^3\right]\frac{dG}{d\alpha}(x,\alpha,\alpha_p)=4f\frac{\alpha_{\rm EM}}{4\pi}\langle e^2\rangle\frac{\alpha}{4M}[C_FF_{GG}^2(x)+J_1(x)]+\frac{\alpha}{2\pi}\int_x^1\frac{dy}{y}\left[2fP_{Gq}\left[\frac{x}{y},\alpha\right]q^{\rm S}(y,\alpha,\alpha_p)+P_{GG}\left[\frac{x}{y},\alpha\right]G(y,\alpha,\alpha_p)\right]\ , \tag{3.22b}$$

$$F_2^{\rm NS}(x,\alpha,\alpha_p)=2f(\langle e^4\rangle-\langle e^2\rangle^2)^{1/2}x\left[q^{\rm NS}(x,\alpha,\alpha_p)+\frac{\alpha}{4\pi}\int_x^1\frac{dy}{y}B_{\rm NS}\left[\frac{x}{y}\right]q^{\rm NS}(y,\alpha,\alpha_p)\right]\ , \tag{3.23}$$

$$F_2^{\rm S}(x,\alpha,\alpha_p)=2f\langle e^2\rangle x\left[q^{\rm S}(x,\alpha,\alpha_p)+\frac{\alpha}{4\pi}\int_x^1\frac{dy}{y}B_\psi\left[\frac{x}{y}\right]q^s(y,\alpha,\alpha_p)\right]+3\langle e^2\rangle x\frac{\alpha}{4\pi}\int_x^1\frac{dy}{y}B_G\left[\frac{x}{y}\right]G(y,\alpha,\alpha_p)\ , \tag{3.24}$$

$$F_2^{\gamma}(x,\alpha,\alpha_p)=F_2^{\rm NS}(x,\alpha,\alpha_p)+F_2^{S}(x,\alpha,\alpha_p)+\frac{\alpha_{\rm EM}}{4\pi}x\widetilde{D}_{\gamma}(x)\ . \tag{3.25}$$

In these equations the fragmentation functions are expanded according to

$$P_{\rm NS}(z,\alpha)=P_{qq}^{0}(z)+\frac{\alpha}{2\pi}P_{\rm NS}^{1}(z)\ , \tag{3.26a}$$

$$P_{ij}(z,\alpha)=P_{ij}^{0}(z)+\frac{\alpha}{2\pi}P_{ij}^{1}(z)\ . \tag{3.26b}$$

The two-loop fragmentation functions are related to the analytic continuation of the even-n two-loop anomalous dimensions by

$$P_{\rm NS}^{1}(n)=-\frac{\gamma_{\rm NS}^{1,n}}{8}\ ,$$

$$P_{qq}^{1}(n)=-\frac{\gamma_{\psi\psi}^{1,n}}{8},\quad P_{qG}^{1}=-\frac{1}{2f}\frac{\gamma_{\psi G}^{1,n}}{8}\ , \tag{3.27}$$

$$P_{Gq}^{1}(n)=-\frac{\gamma_{G\psi}^{1,n}}{8},\quad P_{GG}^{1}=-\frac{\gamma_{GG}^{1,n}}{8}\ .$$

The new quantities which appear in the inhomogeneous terms are related to the matrix elements of hadronic operators and to the two-loop anomalous dimensions which represent mixing between the photon and hadronic operators by

$$I_1(n)=-\tfrac{1}{4}\gamma_{\psi\psi}^{0,n}\widetilde{A}_{nG}^{(2)\psi}\ , \tag{3.28a}$$

$$J_1(n)=-\tfrac{1}{4}\gamma_{G\psi}^{0,n}\widetilde{A}_{nG}^{(2)\psi}\ , \tag{3.28b}$$

$$K_{\rm NS}^{1,n}=-24f(\langle e^4\rangle-\langle e^2\rangle^2)C_F F_{qG}^{2}(n)\ , \tag{3.29a}$$

$$K_{\psi}^{1,n}=-24f\langle e^2\rangle C_F F_{qG}^{2}(n)\ , \tag{3.29b}$$

$$K_{G}^{1,n}=24f\langle e^2\rangle C_F F_{GG}^{2}(n)\ . \tag{3.30}$$

The explicit x-space expressions for these quantities and for the coefficient functions $B_{\rm NS}(z)$, $B_{\psi}(z)$, and $B_G(z)$ are collected in the Appendix.

One can use the relations (3.26)—(3.30) to obtain the n-space version of the equations. Once appropriate boundary conditions for the α and α_p (or α_0) dependences of the solutions are specified, such equations can be treated in the way used to obtain (3.15); in this way the moments results of the OPE approach can be reproduced solving a set of simple algebraic equations.[21] For example, the results of Ref. 5 for the quantities L_i, A_i, and B_i which enter Eq. (2.13) can be reobtained using

$$q^{\rm NS}(n,\alpha,\alpha_p)=\frac{4\pi}{\beta_0\alpha}\sum_A h_A^{\rm NS}(n)\left[1-\left[\frac{\alpha}{\alpha_p}\right]^A\right]+\sum_E S_E^{\rm NS}(n)\left[1-\left[\frac{\alpha}{\alpha_p}\right]^E\right] \tag{3.31}$$

and similar boundary conditions for the singlet and gluon densities.

Because of our treatment of the hadronic part, the set of equations (3.21)—(3.25) is appropriate only for the kinematical configuration $\Lambda^2 \ll P^2 \ll Q^2$. As pointed out above, a simple way to deal with the Bjorken limit consists in writing two separate sets of equations: an inhomogeneous set for the parts containing the matrix element of the photon operator such as $q^{{\rm NS},\gamma}(x,Q^2)$ and its singlet and gluon analogs, and a homogeneous set for densities such as $q^{{\rm NS},H}(x,Q^2)$ which contain the matrix elements of hadronic operators. One obtains the appropriate inhomogeneous equations from (3.21) and (3.22), dropping terms containing $I_1(x)$ and $J_1(x)$; the equations for the hadronic densities are exactly the same as those used for hadronic deep-inelastic scattering,[12] i.e., one retains only the homogeneous terms in (3.21) and (3.22). The two sets of solutions must then be added together and substituted in the coefficient equations (3.23) and (3.24). In place of (3.25) one has to use

$$F_2^{\gamma}(x,Q^2)=F_2^{\rm NS}(x,Q^2)+F_2^{\rm S}(x,Q^2)+3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}xB_{\gamma}(x)\ , \tag{3.32}$$

where $B_\gamma(x)$ is the inverse Mellin transform of the one-loop photon coefficient function.

In agreement with the discussion of the previous section, the solutions of the inhomogeneous and homogeneous sets of equations are, when considered separately, renormalization-prescription dependent in next-to-leading order. Only when the two sets of solutions are added together does one obtain a result which is prescription independent and ${Q_0}^2$ independent within the precision afforded by perturbation theory in next-to-leading order.

To solve the homogeneous equations one needs to know the form of the densities at some reference value ${Q_0}^2$ in the perturbative range. These should be obtained from experiments and present data are too inaccurate to allow a quantitative analysis. In the next section we solve numerically the equations for the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$ where no such knowledge is required. We have also solved the inhomogeneous set of equations for the Bjorken limit in the $\overline{\rm MS}$ scheme. The quantity obtained in this case is the inverse Mellin transform of the part of the structure function proportional to the matrix element of the photon operator [the last term in the RHS of (2.1)]. In the next-to-leading order it can be written as

$$F_2^{\rm gh}(n,Q^2,Q_0{}^2)=\frac{\alpha_{\rm EM}}{4\pi}\left\{\frac{4\pi}{\beta_0\alpha}\frac{1}{2}\sum_i L_i(n)\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{1+d_i^n}\right]\right.$$

$$\left.+\sum_i\frac{A_i^{\rm Bj}(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{d_i^N}\right]+\sum_i\frac{B_i(n)}{2\beta_0}\left[1-\left[\frac{\alpha}{\alpha_0}\right]^{1+d_i^n}\right]+3f\langle e^4\rangle B_\gamma^n\right\},\quad i={\rm NS},+\ ,\qquad(3.33)$$

where $A_i^{\rm Bj}(n)$ are the quantities obtained from the $A_i(n)$ of Eq. (2.13) letting $\widetilde{A}_{nG}^{(2)\psi}$ equal to zero and are therefore renormalization-convention dependent. Note that the sum

$$\sum_i\frac{A_i^{\rm Bj}}{2\beta_0}+3f\langle e^4\rangle B_\gamma^n\ ,\quad i={\rm NS},+,-$$

is convention independent so that the next-to-leading-order pointlike contribution $b(n)$ of Eq. (2.11) is prescription independent. On the other hand, the term

$$-\sum_i\frac{A_i^{\rm Bj}(n)}{2\beta_0}\left[\frac{\alpha}{\alpha_0}\right]^{d_i}$$

in (3.33) is convention dependent and such dependence must be canceled by the convention dependence of the solutions of the homogeneous equations.

One can see from (3.33) that $F_2^{\rm gh}(x,Q^2,Q_0{}^2)$ is free of power singularities at $x=0$ and it is completely calculable for any given Q^2 and $Q_0{}^2$. The hadronlike terms in Eq. (3.33) cannot be obtained from a homogeneous Altarelli-Parisi equation; essentially this is because such terms contain matrix elements of the photon operators and, as noted at the beginning of this section, this involves the presence of inhomogeneous terms. Of course, to reproduce such hadronlike terms correctly one has to use boundary conditions of the form (3.31) with α_p replaced by α_0.

IV. SOLUTIONS OF THE EQUATIONS IN x SPACE

To solve in x space Altarelli-Parisi equations such as (3.21)–(3.25), the α and α_p dependences have first to be separated from the x dependence: it is at this stage that the particular forms of the boundary conditions are taken into account. After the separation one is ordinarily left with a set of integral equations for the x-dependent parts which can be solved by standard methods.

The authors of Ref. 4 solved the leading equations (3.5) and (3.6) using boundary conditions of the form (3.12). In this case the separation procedure is very simple. One has

$$q^{\rm NS}(x,t)=h_{\rm NS}(x)t\ ,$$
$$q^{\rm S}(x,t)=h_{\rm S}(x)t\ ,\qquad(4.1)$$
$$G(x,t)=h^G(x)t\ .$$

Substituting (4.1) in (3.5) and (3.6), one immediately gets a set of integral equations for $h_{\rm NS}(x)$, $h_{\rm S}(x)$, and $h_G(x)$. As remarked in the previous section, using boundary conditions such as (4.1) reproduces only the pointlike part of the structure function. Indeed, the solutions obtained in this way exhibit at $x=0$ the singular behavior discussed in Sec. II.

If one has to implement in x space conditions of the form (3.31) where the exponents A and E are themselves n-dependent, the separation procedure is slightly less straightforward. To accomplish the separation in the kinematical region $\Lambda^2\ll P^2\ll Q^2$, we rewrite (3.31) as

$$q^{\rm NS}(n,\alpha,\alpha_p)=-\frac{4\pi}{\beta_0\alpha}\sum_{m=1}^{\infty}\frac{H_m}{m!}(n)\ln^m\frac{\alpha}{\alpha_p}-\sum_{m=1}^{\infty}\frac{R_m(n)}{m!}\ln^m\frac{\alpha}{\alpha_p}\ .\qquad(4.2)$$

Next, isolating a trivial charge factor, we write the inverse Mellin transform of (4.2) as

$$q^{\rm NS}(x,\alpha,\alpha_p)=\frac{\alpha_{\rm EM}}{4\pi}6(\langle e^4\rangle-\langle e^2\rangle^2)\times\left[-\frac{4\pi}{\beta_0\alpha}\sum_{m=1}^{\infty}\frac{\widetilde{H}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}-\sum_{m=1}^{\infty}\frac{\widetilde{R}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}\right].\qquad(4.3)$$

Substituting this expression in (3.21), dropping terms of order $O(\alpha^2)$, and equating terms with the same α and α_p dependences, we get the recurrence relations

$$\widetilde{H}_1(x)=a^0(x)\ ,\qquad(4.4a)$$

$$\widetilde{H}_{m+1}(x)=\widetilde{H}_m(x)-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}P_{qq}^0\left[\frac{x}{y}\right]\widetilde{H}_m(y)\ ,\qquad(4.4b)$$

$$\widetilde{R}_1(x)=\frac{1}{\beta_0}[I_1(x)-C_FF_{qG}^2(x)]-\frac{\beta_1}{\beta_0{}^2}\widetilde{H}_1(x)\ ,\qquad(4.5a)$$

$$\widetilde{R}_{m+1}(x)=\frac{\beta_1}{\beta_0{}^2}[\widetilde{H}_m(x)-\widetilde{H}_{m+1}(x)]-\frac{4}{\beta_0{}^2}\int_x^1\frac{dy}{y}P_{\rm NS}^1\left[\frac{x}{y}\right]\widetilde{H}_m(y)-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}P_{qq}^0\left[\frac{x}{y}\right]\widetilde{R}_m(y)\ .\qquad(4.5b)$$

Similarly, substituting in the singlet equations the expressions

$$q^{S}(x,\alpha,\alpha_p)=\frac{\alpha_{EM}}{4\pi}6\langle e^2\rangle\left[-\frac{4\pi}{\beta_0\alpha}\sum_{m=1}^{\infty}\frac{\tilde{K}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}-\sum_{m=1}^{\infty}\frac{\tilde{S}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}\right], \quad (4.6)$$

$$G(x,\alpha,\alpha_p)=\frac{\alpha_{EM}}{4\pi}4f\langle e^2\rangle\left[-\frac{4\pi}{\beta_0\alpha}\sum_{m=1}^{\infty}\frac{\tilde{G}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}-\sum_{m=1}^{\infty}\frac{\tilde{Q}_m(x)}{m!}\ln^m\frac{\alpha}{\alpha_p}\right], \quad (4.7)$$

one obtains the relations

$$\tilde{K}_1(x)=a^0(x)\,, \quad (4.8a)$$

$$\tilde{G}_1(x)=0\,, \quad (4.8b)$$

$$\tilde{K}_{m+1}(x)=\tilde{K}_m(x)-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}\left[P^0_{qq}\left[\frac{x}{y}\right]\tilde{K}_m(y)+2fP^0_{qG}\left[\frac{x}{y}\right]\tilde{G}_m(y)\right], \quad (4.8c)$$

$$\tilde{G}_{m+1}(x)=\tilde{G}_m(x)-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}\left[P^0_{Gq}\left[\frac{x}{y}\right]\tilde{K}_m(y)+P^0_{GG}\left[\frac{x}{y}\right]\tilde{G}_m(y)\right], \quad (4.8d)$$

and

$$\tilde{S}_1(x)=\frac{1}{\beta_0}[I_1(x)-C_FF^2_{qG}(x)]-\frac{\beta_1}{\beta_0^2}\tilde{K}_1(x)\,, \quad (4.9a)$$

$$\tilde{Q}_1(x)=\frac{1}{\beta_0}[J_1(x)+C_FF^2_{GG}(x)]\,, \quad (4.9b)$$

$$\tilde{S}_{m+1}(x)=\frac{\beta_1}{\beta_0^2}[\tilde{K}_m(x)-\tilde{K}_{m+1}(x)]-\frac{4}{\beta_0^2}\int_x^1\frac{dy}{y}\left[P^1_{qq}\left[\frac{x}{y}\right]\tilde{K}_m(y)+2fP^1_{qG}\left[\frac{x}{y}\right]\tilde{G}_m(y)\right]-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}\left[P^0_{qq}\left[\frac{x}{y}\right]\tilde{S}_m(y)+2fP^0_{qG}\left[\frac{x}{y}\right]\tilde{Q}_m(y)\right], \quad (4.9c)$$

$$\tilde{Q}_{m+1}(x)=\frac{\beta_1}{\beta_0^2}[\tilde{G}_m(x)-\tilde{G}_{m+1}(x)]-\frac{4}{\beta_0^2}\int_x^1\frac{dy}{y}\left[P^1_{Gq}\left[\frac{x}{y}\right]\tilde{K}_m(y)+P^1_{GG}\left[\frac{x}{y}\right]\tilde{G}_m(y)\right]-\frac{2}{\beta_0}\int_x^1\frac{dy}{y}\left[P^0_{Gq}\left[\frac{x}{y}\right]\tilde{S}_m(y)+P^0_{GG}\left[\frac{x}{y}\right]\tilde{Q}_m(y)\right]. \quad (4.9d)$$

Inserting the solutions of (4.4) and (4.5) and (4.8) and (4.9) in (3.23)—(3.25), one finally gets for $F_2^{\gamma}(x,\alpha,\alpha_p)$

$$F_2^{\gamma}(x,\alpha,\alpha_p)=\frac{\alpha_{EM}}{4\pi}12f\langle e^4\rangle x\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\ln^m\frac{\alpha}{\alpha_p}\left[-\frac{4\pi}{\beta_0\alpha}\tilde{H}_m(x)-\tilde{R}_m(x)-\frac{1}{\beta_0}\int_x^1\frac{dy}{y}B_{NS}\left[\frac{x}{y}\right]\tilde{H}_m(y)\right]+D_\gamma(x)\right\}$$
$$+\frac{\alpha_{EM}}{4\pi}12f\langle e^2\rangle^2x\sum_{m=1}^{\infty}\frac{1}{m!}\ln^m\frac{\alpha}{\alpha_p}\left\{\frac{4\pi}{\beta_0\alpha}[\tilde{H}_m(x)-\tilde{K}_m(x)]+\tilde{R}_m(x)-\tilde{S}_m(x)+\frac{1}{\beta_0}\int_x^1\frac{dy}{y}\left[B_{NS}\left[\frac{x}{y}\right][\tilde{H}_m(y)-\tilde{K}_m(y)]-B_G\left[\frac{x}{y}\right]\tilde{G}_m(y)\right]\right\}. \quad (4.10)$$

Here we have separated the contribution proportional to $\langle e^4\rangle^2$ (valence part) from the contribution proportional to $\langle e^2\rangle$ (sea part). In practice the sea part turns out to be of some relevance only for x very close to zero and this is a consequence of the fact that the nondiagonal terms in the singlet Altarelli-Parisi equations mix effectively the fermion singlet and gluon sectors only at small x: i.e., in formal operator language, mixing of gluon and fermion operators of spin $n>2$ is very weak: as a result, outside the small-x region, one has $\tilde{H}_m(x)\approx\tilde{K}_m(x)$ and $\tilde{R}_m(x)\approx\tilde{S}_m(x)$.

Using the results from the leading-order recurrence relations (4.4) and (4.8), one can obtain the longitudinal structure function $F_L^{\gamma}(x,\alpha,\alpha_p)$ by means of a simple quadrature with the appropriate coefficient functions

$$F_L^\gamma(x,\alpha,\alpha_p)=\frac{\alpha_{EM}}{4\pi}12f\langle e^4\rangle x\left[\tfrac{1}{4}B_{\gamma,L}(x)-\frac{1}{\beta_0}\sum_{m=1}^{\infty}\frac{1}{m!}\ln^m\left[\frac{\alpha}{\alpha_p}\right]\int_x^1\frac{dy}{y}B_{NS,L}\left[\frac{x}{y}\right]\widetilde{H}_m(y)\right]$$

$$+\frac{\alpha_{EM}}{4\pi}12f\langle e^2\rangle^2 x\frac{1}{\beta}\sum_{m=1}^{\infty}\frac{1}{m!}\ln^m\left[\frac{\alpha}{\alpha_p}\right]\int_x^1\frac{dy}{y}\left[B_{\psi,L}\left[\frac{x}{y}\right][\widetilde{H}_m(y)-\widetilde{K}_m(y)]-B_{G,L}\left[\frac{x}{y}\right]\widetilde{G}_m(y)\right], \tag{4.11}$$

where[17] [for color group SU(3)]

$$B_{\gamma,L}(z)=\frac{2}{f}B_{G,L}(z)=16z(1-z)\ , \tag{4.12}$$

$$B_{NS,L}(z)=B_{\psi,L}(z)=\tfrac{16}{3}z\ . \tag{4.13}$$

The procedure described above can be used to solve the inhomogeneous set of equations appropriate for the Bjorken limit to obtain the quantity $F_2^{ph}(x,Q^2,Q_0{}^2)$ of Eq. (3.33), where

$$F_2^{ph}(n,Q^2,Q_0{}^2)=\int_0^1 dx\, x^{n-2}F_2^{ph}(x,Q^2,Q_0{}^2)\ .$$

In leading order the two quantities $F_2^{ph}(X,Q^2,Q_0{}^2)$ and $F_2^\gamma(x,\alpha,\alpha_p)$ coincide if $Q_0{}^2=P^2$. Of course, the physical interpretation for the two quantities is different: $F_2^\gamma(x,\alpha,\alpha_p)$ gives a complete prediction for the structure functions in the kinematical region $\Lambda^2\ll P^2\ll Q^2$; $F_2^{ph}(x,Q^2,Q_0{}^2)$ accounts only for the portion of the structure function $F_2^\gamma(x,Q^2)$ proportional to the matrix elements of the photon operators. To obtain a complete $Q_0{}^2$-independent prediction for the structure function $F_2(x,Q^2)$, one must add to $F_2^{ph}(x,Q^2,Q_0{}^2)$ the solution of a set of homogeneous equations obtained using data taken at $Q_0{}^2$ as boundary conditions.

Uematsu and Walsh[5] have stressed that as P^2 grows closer to Q^2, the expressions for F_2^γ and F_L^γ relative to the kinematical region $\Lambda^2\ll P^2\ll Q^2$ approach the corresponding parton-model results. In fact, since

$$\ln\frac{\alpha}{\alpha_p}=-\frac{\ln Q^2/P^2}{\ln Q^2/\Lambda^2}-\frac{1}{\ln^2 Q^2/\Lambda^2}\left[\frac{\beta_1}{\beta_0{}^2}\ln\frac{Q^2}{P^2}\left[1-\ln\ln\frac{Q^2}{\Lambda^2}\right]-\tfrac{1}{2}\ln^2\frac{Q^2}{P^2}\right]+\cdots\ , \tag{4.14}$$

only the first few terms in the expansions (4.3), (4.6), and (4.7) are relevant when P^2 gets close to Q^2, and one immediately obtains

$$F_2^\gamma(x,\alpha,\alpha_p)=\frac{\alpha_{EM}}{4\pi}12f\langle e^4\rangle x\left[a^0(x)\ln\frac{Q^2}{p^2}+D_\gamma(x)+O\left[\frac{1}{\ln Q^2/\Lambda^2}\right]\right]. \tag{4.15}$$

The first two terms in square brackets in (4.15) are indeed the same which enter the parton-model result for $m_i{}^2\ll P^2\ll Q^2$ (m_i stands for the ith-quark mass).[21,22] One should, however, notice that the term of order $O(1/\ln Q^2/\Lambda^2)$ contains parts proportional to $\ln(1-x)$ which at $x\approx 1$ are competitive with the $\ln Q^2/\Lambda^2$ in the denominator: for this reason, in the large-x region there are sensible differences with the parton-model expression even for large P^2.

Our results for $F_2^\gamma(x,\alpha,\alpha_p)$ are shown in Figs. 1–4. The normalization is that of Ref. 5. All these results refer to $f=4$: in the normalization used the corresponding curves for $f=3$ and $f=5$ differ by a few percent at most from those presented here, except for small x ($x\lesssim 0.15$) where the sea contributions become relevant (note that a factor

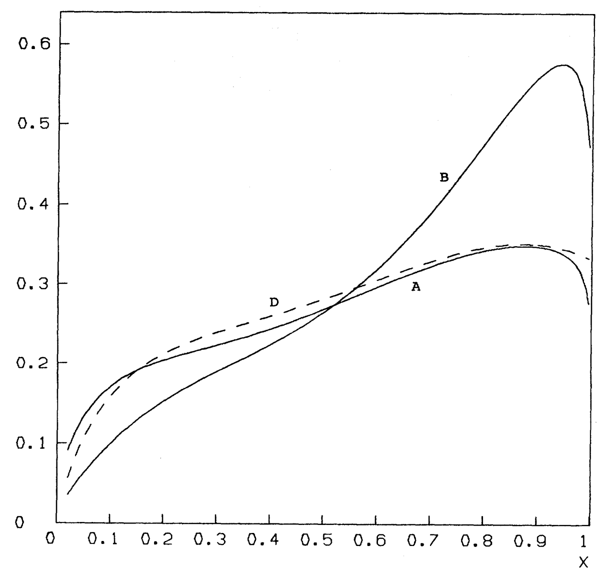

FIG. 1. The quantity plotted is

$$F_2^\gamma(x,Q^2,P^2)\Big/\left[3f\langle e^4\rangle\frac{\alpha_{EM}}{\pi}\ln\frac{Q^2}{P^2}\right]$$

with $Q^2=20$ GeV^2, $P^2=1$ GeV^2, and $\Lambda_{\overline{MS}}=200$ MeV. Curve A is the next-to-leading order QCD result. Curve B is the leading-order QCD result ($4\pi/\beta_0\alpha=t$, $4\pi/\beta_0\alpha_p=t_p$). Curve D is the parton-model result for $m_i{}^2\ll P^2\ll Q^2$.

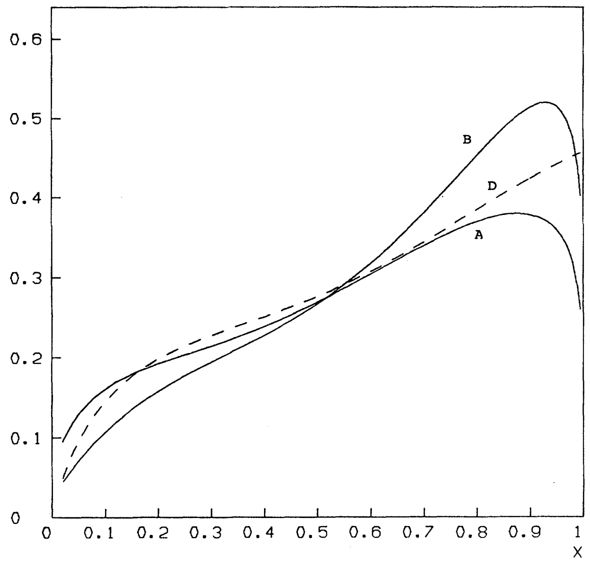

FIG. 2. Same as Fig. 1 with $Q^2=20$ GeV^2, $P^2=0.5$ GeV^2, and $\Lambda_{\overline{\rm MS}}=200$ MeV.

$$f\langle e^4\rangle=\sum_{i=1}^{f} e_i{}^4$$

is present in the denominator of the quantities shown in the figures). Here no attempt has been made to keep into account effects due to the quark masses and massive quark production thresholds. The latter should manifest themselves with increases in the measured structure function at x smaller than certain threshold values x_M for the production of massive quark pairs of mass M. Figure 5 shows the result for the longitudinal structure function obtained from (4.11). In this case the parton-model result for $Q^2 \gg P^2$ is independent of both Q^2 and P^2. Note that

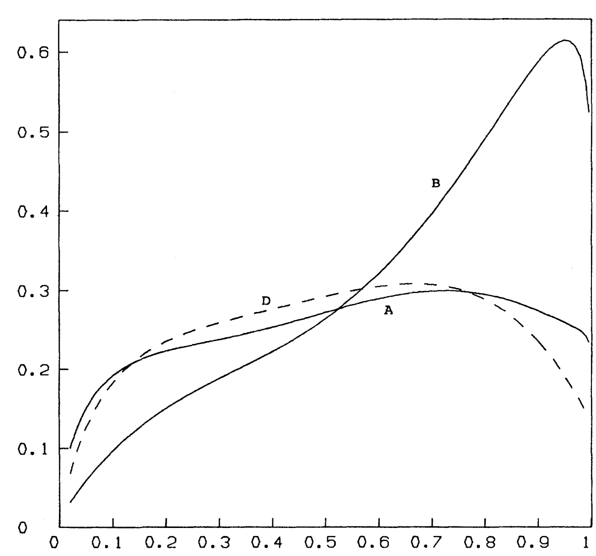

FIG. 3. Same as Fig. 1 with $Q^2=10$ GeV^2, $P^2=1$ GeV^2, and $\Lambda_{\overline{\rm MS}}=200$ MeV.

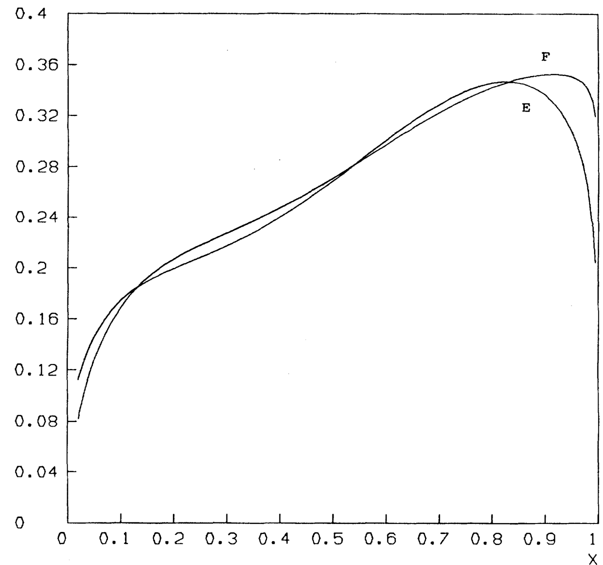

FIG. 4. The quantity plotted is the same as in curve A of Fig. 1, but with different values of $\Lambda_{\overline{\rm MS}}$. Curve E refers to $Q^2=20$ GeV^2, $P^2=1$ GeV^2, and $\Lambda_{\overline{\rm MS}}=400$ MeV. Curve F refers to $Q^2=20$ GeV^2, $P^2=1$ GeV^2, and $\Lambda_{\overline{\rm MS}}=100$ MeV.

in the figures the variable x is assumed modified to cover the range $0\leq x\leq 1$ (strictly speaking for $P^2\neq 0$ one has $x\leq 1-P^2/2p\cdot q$). We use $\Lambda_{\overline{\rm MS}}=200$ MeV^2 and obtain α and α_p from (2.4). One can see from Fig. 4 that the results are quite insensitive to the value of $\Lambda_{\overline{\rm MS}}$. As one

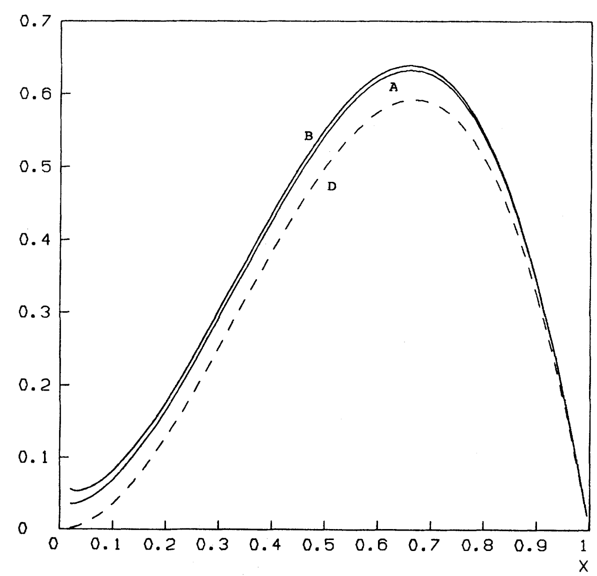

FIG. 5. The quantity plotted is

$$F_L^\gamma(x,Q^2,P^2)\Big/\left[3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{\pi}\right]$$

at lowest order. Curve A refers to $Q^2=20$ GeV^2, $P^2=1$ GeV^2, and $\Lambda_{\overline{\rm MS}}=200$ MeV. Curve B refers to $Q^2=20$ GeV^2, $P^2=0.5$ GeV^2, and $\Lambda_{\overline{\rm MS}}=200$ MeV. Curve D is the parton-model result.

might expect from the discussion at the end of the second section, next-to-leading order corrections turn out to be especially relevant at large x. In practice, from a physical point of view, one should take with due skepticism those among the results which refer to $P^2=0.5$ GeV^2 because of possible breakdown in the perturbation expansion, and those for which $P^2/Q^2=0.1$ because $O(P^2/Q^2)$ corrections may prove sizable.

A few comments about our computations are in order. The $\delta(1-x)$ present in $F_{GG}^2(x)$, namely in the inhomogeneous term of the singlet equation for the gluon distribution, has been treated using the ordinary Gaussian form

$$\delta(1-x)\approx\frac{2}{\epsilon\sqrt{\pi}}\exp\left[-\frac{(x-1)^2}{\epsilon^2}\right]$$

(the factor 2 is in accordance with the definition

$$\int_0^1 dx\,\delta(1-x)=1) .$$

The choice of the parameter ϵ depends on the type of mesh of data points in x. Due to weak mixing between gluon and single fermion sectors at large x, the results for $F_2^\gamma(x,\alpha,\alpha_p)$ are not very sensitive to the type of approximation used. Note that $\tilde{Q}_m(x)$ does not enter $F_2^\gamma(x,\alpha,\alpha_p)$ in next-to-leading order.

The expansions (4.3) and (4.6) and (4.7) are series of the exponential and therefore they are always convergent; in practice if $|\ln(\alpha/\alpha_p)|$ is large the expansions fluctuate wildly before starting to converge: as a result, one is limited by the number of significant digits one's computer can handle. At any rate, this is not a major problem: our double-precision program on a VAX 11/780 computer (16 significant digits) could treat values of $|\ln(\alpha/\alpha_p)|\leq 1$ and all physically relevant values of P^2, Q^2, and Λ^2 fall within this range. With a mesh of 69 data points in x it took approximately 15 min central-processing-unit time to obtain the set of curves relative to each set of Q^2, P^2, and Λ^2 values. We checked our results by comparing their moments, computed numerically with the actual moments obtained from the OPE results: the agreement is quite good (within a few tenths of a percent for $2\leq n\leq 60$).

Results for the quantity $F_2^{\rm ph}(x,Q^2,Q_0{}^2)$, which is the inverse Mellin transform of the part of the structure function proportional to the matrix element of the photon operators, are shown in Figs. 6–8. As discussed above, this quantity is renormalization-convention dependent. Our results are obtained in the $\overline{\rm MS}$ scheme, solving the inhomogeneous equations relative to the Bjorken limit. As noted at the end of the last section, there is no power singularity at $x=0$. As Q^2 approaches $Q_0{}^2 F_2^{\rm ph}(x,Q^2,Q_0{}^2)$ is expected to differ more and more from the complete structure function $F_2(x,Q^2)$. Essentially this is because those terms in $F_2^{\rm ph}(x,Q^2,Q_0{}^2)$ which depend on $Q_0{}^2$ become more relevant [note that the complete structure function $F_2(x,Q^2)$ should be $Q_0{}^2$ independent]. In the limit $Q^2=Q_0{}^2$, the leading part vanishes and in next-to-leading order one has

$$F_2^{\rm ph}(x,Q^2=Q_0{}^2)=3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}xB_\gamma(x) .$$

Since one is dealing with quantities which are $Q_0{}^2$ and

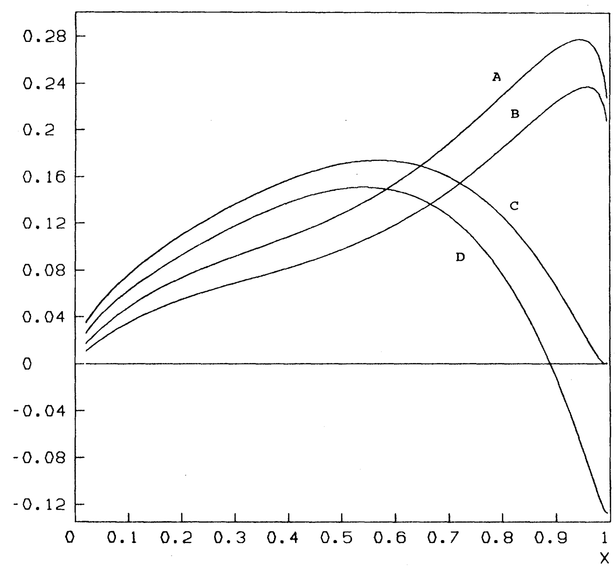

FIG. 6. The quantity plotted is

$$F_2^{\rm ph}(x,Q^2,Q_0{}^2)/3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}\ln\frac{Q^2}{\Lambda^2}$$

in the $\overline{\rm MS}$ scheme, with $Q^2=20$ GeV^2 and $\Lambda_{\overline{\rm MS}}=200$ MeV. Curve A is the leading-order result for $Q_0{}^2=1$ GeV^2. Curve B is the leading-order result for $Q_0{}^2=2$ GeV^2. Curve C is the next-to-leading-order result for $Q_0{}^2=1$ GeV^2. Curve D is the next-to-leading-order result for $Q_0{}^2=2$ GeV^2.

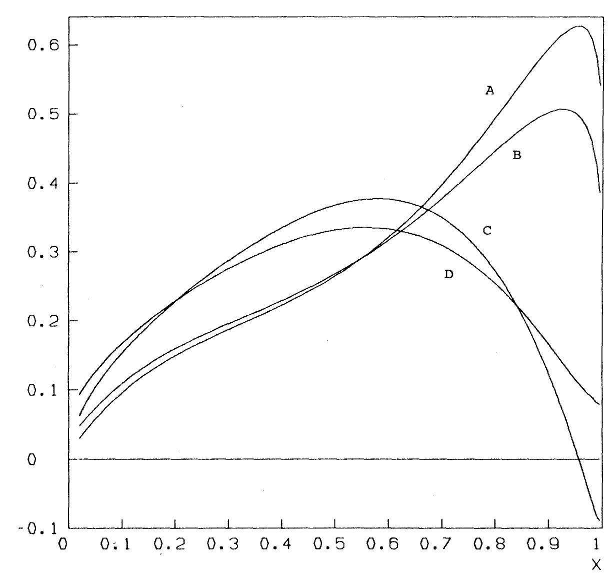

FIG. 7. The quantity plotted is

$$F_2^{\rm ph}(x,Q^2,Q_0{}^2)/3f\langle e^4\rangle\frac{\alpha_{\rm EM}}{4\pi}\ln\frac{Q^2}{Q_0{}^2}$$

in the $\overline{\rm MS}$ scheme with $Q^2=20$ GeV^2 and $Q_0{}^2=1$ GeV^2. Curve A is the leading-order result for $\Lambda_{\overline{\rm MS}}=100$ MeV. Curve B is the leading-order result for $\Lambda_{\overline{\rm MS}}=400$ MeV. Curve C is the next-to-leading-order result for $\Lambda_{\overline{\rm MS}}=100$ MeV. Curve D is the next-to-leading-order result for $\Lambda_{\overline{\rm MS}}=400$ MeV.

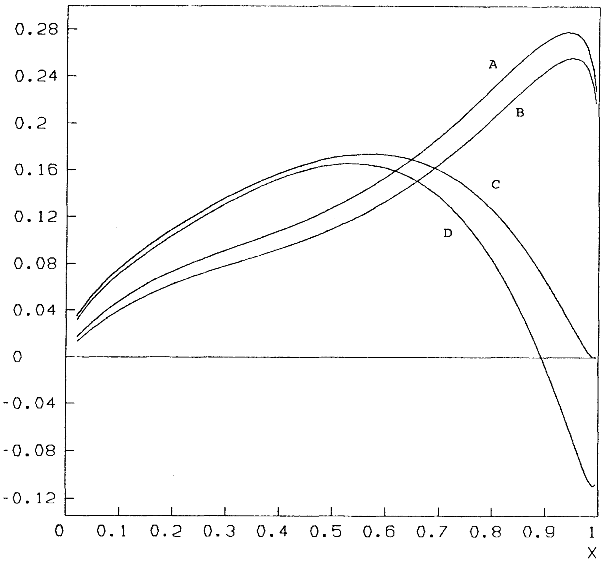

FIG. 8. The quantity plotted is the same as the one in Fig. 6 with ${Q_0}^2=1$ GeV^2 and $\Lambda_{\overline{\rm MS}}=200$ MeV. Curve A is the leading-order result for $Q^2=20$ GeV^2. Curve B is the leading-order result for $Q^2=10$ GeV^2. Curve C is the next-to-leading-order result for $Q^2=20$ GeV^2. Curve D is the next-to-leading-order result for $Q^2=10$ GeV^2.

convention dependent, one should be very cautious in drawing conclusions from the results shown in the figures. We only remark that the poor sensitivity of the results on the value of Λ (see Fig. 7) is not very encouraging for the prospect of obtaining Λ from $\gamma\gamma$ experiments, since presumably the solution of the homogeneous equations will not be any more sensitive to the value of Λ than hadronic structure functions.

The x-space formulation presented here has, over the ordinary moments inversion techniques,[23] the advantages of increased accuracy and reliability in dealing with the end-point ($x \approx 0$, $x \approx 1$) behavior. Present experimental results are affected by large uncertainties and the predictions of the theory are not dependable at large x. Therefore the increased precision may not prove very significant at present. However, this formulation is no more complicated than the usual one; it makes it possible to work directly in x space and, as has been stressed by the authors of Ref 16, its reliability at small x should make it useful in future analysis of experimental results.

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APPENDIX

In the $\overline{\rm MS}$ scheme the coefficient functions are given by [for color group SU(3) $C_F=\frac{4}{3}$, $C_A=3$]

$$B_{\rm NS}(z)=B_\psi(z)=C_F\left[2(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+-\frac{3}{2}\frac{1+z^2}{(1-z)_+}-\frac{1+z^2}{1-z}\ln z+\tfrac{1}{2}(9+5z)-\left[9+\frac{2\pi^2}{3}\right]\delta(1-z)\right], \tag{A1}$$

$$B_G(z)=2f\left[(2z^2-2z+1)\ln\frac{1-z}{z}-1+8z-8z^2\right], \tag{A2}$$

$$B_\gamma(z)=\frac{2}{f}B_G(z), \tag{A3}$$

and $\tilde{D}_\gamma(z)$ is the inverse Mellin transform of the quantity introduced in (2.14),

$$\tilde{D}_\gamma(z)=12f\langle e^4\rangle D_\gamma(z)=12f\langle e^4\rangle[-2(2z^2-2z+1)\ln z-6z^2+6z-2]. \tag{A4}$$

For the fragmentation functions we use expressions consistent with the results of Refs. 18, 12, and 19 [the authors of Ref. 11 report a different result for the part of $P^1_{GG}(z)$ proportional to ${C_A}^2$]. For $P^1_{\rm NS}(z)$ we use Refs. 18, 20, and 19,

$$P^1_{\rm NS}(z)=({C_F}^2-\tfrac{1}{2}C_FC_A)\left[P_F(z)+P_A(z)+\delta(1-z)\left[\frac{3}{8}-\frac{\pi^2}{2}+2\zeta(3)\right]\right]$$
$$+\tfrac{1}{2}C_FC_A\left[P_F(z)+P_G(z)+\delta(1-z)\left[\frac{43}{24}+\frac{13\pi^2}{18}-4\zeta(3)\right]\right]$$
$$+C_F\frac{f}{2}\left[P_{N_f}-\delta(1-z)\left[\frac{1}{6}+\frac{2\pi^2}{9}\right]\right], \tag{A5}$$

where

$$P_F(z)=-2(1+z^2)\left[\frac{\ln z\ln(1-z)}{1-z}\right]_+ -3\frac{\ln z}{1-z}-2z\ln z-\tfrac{1}{2}(1+z)\ln^2 z-5(1-z)\ , \tag{A6}$$

$$P_A(z)=-4\frac{1+z^2}{1+z}\mathrm{Li}_2(-z)+\frac{1+z^2}{1+z}\left[\ln^2 z-4\ln z\ln(1+z)-\frac{\pi^2}{3}\right]+2(1+z)\ln z+4(1-z)\ , \tag{A7}$$

$$P_G(z)=\left[\frac{67}{9}-\frac{\pi^2}{3}\right]\frac{1+z^2}{(1-z)_+}+\frac{1+z^2}{1-z}(\ln^2 z+\tfrac{11}{3}\ln z)+2(1+z)\ln z+\tfrac{40}{3}(1-z)\ , \tag{A8}$$

$$P_{N_f}(z)=-\frac{10}{9}\frac{1+z^2}{(1-z)_+}-\tfrac{2}{3}\ln z\frac{1+z^2}{1-z}-2(1-z)\ . \tag{A9}$$

Here $\mathrm{Li}_2(z)$ is the ordinary dilogarithm

$$\mathrm{Li}_2(z)=-\int_0^z\frac{dt}{t}\ln(1-t) \tag{A10}$$

and $\zeta(x)$ is the Riemann ζ function [$\zeta(3)=1.2020569$]. The distribution $[f(x)]_+$ is used to regularize singularities at $x=1$ and it is defined by

$$[f(x)]_+=f(x)-\delta(1-x)\int_0^1 dt\,f(t)\ . \tag{A11}$$

This formula is used to regularize nonintegrable singularities before inserting the coefficient functions or the fragmentation functions in a computer program. It is convenient to use it also for the singularity of $P_F(z)$ in (A6) even if such singularity is integrable. Of course,

$$\frac{\ln z\ln(1-z)}{1-z}=\left[\frac{\ln z\ln(1-z)}{1-z}\right]_+ +\zeta(3)\delta(1-z)\ . \tag{A12}$$

The remaining four fragmentation functions can be written as

$$P_{qq}^1(z)=P_{\rm NS}^1(z)+C_F fF_{qq}(z)\ , \tag{A13}$$

$$P_{qG}^1(z)=-\frac{C_A}{4}F_{qG}^1(z)-\frac{C_F}{4}F_{qG}^2(z)\ , \tag{A14}$$

$$P_{Gq}^1(z)=-C_F{}^2F_{Gq}^1(z)-C_FC_AF_{Gq}^2(z)-C_F\frac{f}{2}F_{Gq}^3(z)\ , \tag{A15}$$

$$P_{GG}^1(z)=C_A\frac{f}{2}F_{GG}^1(z)+C_F\frac{f}{2}F_{GG}^2(z)+C_A{}^2F_{GG}^3(z)\ . \tag{A16}$$

We checked that the expressions for $F_{ij}^N(z)$ of Ref. 20 are correct. We use such expressions, except in the case of F_{GG}^3: In this case to be consistent with Refs. 18, 12, and 19, we use

$$\begin{aligned}F_{GG}^3(x)=&-\tfrac{1}{18}(25+109x)-\tfrac{1}{3}(25-11x+44x^2)\ln x+\frac{\pi^2}{3}\left[4+2x^2-\frac{1}{1+x}\right]\\&+\ln^2 x\left[\frac{1}{1-x}+4x-2x^2+\frac{1}{1+x}\right]-4\left[\frac{1}{x}-2+x-x^2\right]\ln x\ln(1-x)\\&+4\left[\frac{1}{x}+2+x+x^2-\frac{1}{1+x}\right][\mathrm{Li}_2(-x)+\ln x\ln(1-x)]\\&-4\left[\frac{\ln x\ln(1-x)}{(1-x)}\right]_+ +\tfrac{1}{3}(\tfrac{67}{3}-\pi^2)\frac{1}{(1-x)_+}+[\tfrac{8}{3}-\zeta(3)]\delta(1-x)\ .\end{aligned} \tag{A17}$$

The functions $F_{qG}^2(x)$ and $F_{GG}^2(x)$ which appear in the inhomogeneous terms of (3.21) and (3.22) are, of course, the same as those which enter (A14) and (A16).

Finally, we report the explicit expressions for the new quantities $I_1(x)$ and $J_1(x)$ which account for the hadronic parts: for color group SU(3) one has

$$\begin{aligned}I_1(x)=\int_x^1\frac{dy}{y}P_{qq}^0\left[\frac{x}{y}\right]\tilde{A}_G^{(2)\psi}(y)=&(\tfrac{32}{3}x^2-16x+8)\mathrm{Li}_2(x)-\frac{4\pi^2}{9}(1-2x)\\&+\tfrac{8}{3}\{(4x^2-4x+2)\ln(1-x)[\ln x-\ln(1-x)]+(2x^2-x+\tfrac{1}{2})\ln^2 x\\&-(4x^2-4x-\tfrac{1}{2})\ln x+(4x^2-6x-\tfrac{3}{2})\ln(1-x)-(5x^2-6x+\tfrac{5}{2})\}\ ,\end{aligned} \tag{A18}$$

$$J_1(x)=\int_x^1 \frac{dy}{y} P_{Gq}^0 \frac{x}{y} \widetilde{A}_G^{(2)\psi}(y)$$

$$=\tfrac{8}{3}\left[2(1-x)\left[\mathrm{Li}_2(x)-\frac{\pi^2}{6}\right]+\left[\tfrac{4}{3}x^2+x-1-\frac{4}{3x}\right]\ln(1-x)\right.$$

$$\left.+(\tfrac{4}{3}x^2+3x-1)\ln x-(1-x)\ln^2 x-\left[\tfrac{38}{9}x^2-\tfrac{4}{3}x-\tfrac{8}{3}-\frac{2}{9x}\right]\right], \quad \text{(A19)}$$

where we have used

$$\widetilde{A}_G^{(2)\psi}(x)=-2\{(2x^2-2x+1)\ln[x(1-x)]-2x^2+2x+1\}, \quad \text{(A20)}$$

which is the inverse Mellin transform of

$$\widetilde{A}_{nG}^{(2)\psi}=2\left[\left[\frac{2}{n+2}-\frac{2}{n+1}+\frac{1}{n}\right]S_1(n)+\frac{4}{(n+2)^2}-\frac{4}{(n+1)^2}+\frac{1}{n^2}-\frac{1}{n}\right]. \quad \text{(A21)}$$

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