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## Rapid Communications

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# Nonconfinement at high temperatures

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SU(2) lattice gauge theories in two or more space dimensions with any value of the bare coupling are shown rigorously to possess a nonconfining phase at sufficiently high temperature.

Simple physical arguments suggest that pure gauge theories possess a nonconfining high-temperature phase.[1] This may be explicitly shown in the infinite-coupling limit of lattice theories,[2] and is supported by weak-coupling perturbation theory,[1,3] and numerical Monte Carlo simulations.[4] In this Rapid Communication, we present a rigorous proof of the existence of this nonconfining phase for SU(2) lattice gauge theories. Specifically in two or more space dimensions, and for any value of the bare coupling, $g^2>0$, we find a temperature $T^*(g^2)<\infty$, above which confinement cannot occur.

Proving the absence of confinement at nonzero temperature is equivalent to demonstrating that a certain discrete global symmetry (described below) is spontaneously broken. Our method for proving this symmetry breaking is based on a modern version of the Peierls argument,[5] in which one bounds the probability of domain walls or "contours" which separate regions of differing magnetization. In standard applications of this argument to theories with discrete global symmetries, one typically shows that, as some parameter of the theory is varied, the energy per unit area of a domain wall becomes exponentially large and consequently the probability of formation of domains is exponentially suppressed. This then allows one to deduce a nonzero lower bound on the spontaneous magnetization, thereby proving spontaneous symmetry breaking. If the energy per unit area of a domain wall can be arbitrarily small, as in typically the case in theories with continuous global symmetries, then this approach cannot normally be used. Our case of SU(2) gauge theories is in between these two typical situations. As the temperature is increased, the probability of a domain wall is only suppressed by a power of the (inverse) temperature. Consequently, bounding this probability requires much more accurate estimates than is normally the case. We overcome this problem by developing what is in essence a one-dimensional renormalization-group transformation which may be rigorously controlled. We are unaware of any other examples without exponential suppression of domain walls in which Peierls arguments may be successfully applied.

An alternative proof of the absence of confinement at high temperature for three or more space dimensions has recently been given by Borg and Seiler.[6] Their proof uses the method of infrared bounds,[7] which is commonly employed in systems with spontaneous breaking of a continuous symmetry. The bounds of Borg and Seiler are in fact better at weak coupling than the bounds we are able to prove. However, the method of infrared bounds does not work in the lower critical dimension ($d=2$). Furthermore, the approach used here is, we believe, rather more direct and physically transparent and has some novel features which may be useful in other contexts.

We consider the standard (Wilson) SU(2) lattice gauge theory on a rectangular $(d+1)$-dimensional periodic hypercubic lattice $\Lambda$ of size $L_t\times L_s{}^d$. $\Lambda$ may be regarded as an anisotropic lattice with "timelike" and "spacelike" lattice spacings $a_t$ and $a_s$, respectively. The physical temperature is defined as $T=(L_t a_t)^{-1}$. The "spatial" lattice obtained as a time slice of $\Lambda$ will be denoted by $\Lambda_s$. The basic variables $U[b]\in SU(2)$ are defined on bonds $b\in\Lambda$. The partition function is given by

$$Z_\Lambda=\int\sum_{b\in\Lambda}dU[b]\exp(A_\Lambda)\quad, \tag{1}$$

where $dU[b]$ denotes normalized Haar measure on SU(2), and the action is

$$A_\Lambda=\beta_t\sum_{p_t\in\Lambda}tr(U[\partial p_t]-1)+\beta_s\sum_{p_s\in\Lambda}tr(U[\partial p_s]-1)\quad. \tag{2}$$

Here $p_t$ ($p_s$) indicates a timelike (spacelike) plaquette, and $U[\partial p]$ indicates the ordered product of link matrices around the plaquette $p$. The timelike and spacelike coupling constants are defined as $\beta_t=(2a_s{}^{d-4}/g^2)(L_t a_s T)$ and $\beta_s=(2a_s{}^{d-4}/g^2)(L_t a_s T)^{-1}$, where $g^2$ is the conventional bare coupling constant. We use anisotropic couplings in order to allow continuous independent variation of temperature and coupling.

Since our lattice is periodic, in addition to the usual topologically trivial Wilson loops, there exist topologically nontrivial loops winding around the lattice. For such paths parallel to the time axis, we define the "Wilson line":

$$L_{\vec{i}} \equiv \tfrac{1}{2}\,\mathrm{tr}\,\Omega_{\vec{i}} \equiv \tfrac{1}{2}\,\mathrm{tr}\prod_{n=1}^{L_t} U_t[\vec{i}+n\hat{t}] \quad . \tag{3}$$

(Here $\vec{i}$ labels sites in the spatial lattice $\Lambda_s$, while $U_t[s]$ indicates the timelike bond variable originating at the site $s \in \Lambda$.) The potential between a static quark and antiquark, $F^{q\bar{q}}(\vec{i})$, is determined by the correlation function of two Wilson lines,[1]

$$\exp[-F_{\Lambda}^{q\bar{q}}(\vec{i}-\vec{j})/T] \equiv \langle L_{\vec{i}} L_{\vec{j}}^{\dagger}\rangle_{\Lambda} \quad . \tag{4}$$

If the "magnetization"

$$m \equiv \lim_{|\vec{i}-\vec{j}|\to\infty}\ \lim_{L_s\to\infty} \langle L_{\vec{i}} L_{\vec{j}}^{\dagger}\rangle^{1/2} \tag{5}$$

is nonzero, then the quark-antiquark potential is bounded for all separations, and static quarks are not confined.

In addition to being invariant under the usual local gauge transformations, the action is also invariant under a global $Z(2)$ transformation which changes the sign of all timelike bond variables which emerge from the equal-time slice $\Lambda_s$. Under this transformation, topologically trivial Wilson loops remain invariant, but the Wilson line (3) changes sign. In other words, the Wilson line is an order parameter for this $Z(2)$ symmetry. Hence, if the magnetization (5) is nonzero, then the measure (1) does not describe an ergodic state and multiple phases must occur. In the pure phases (which could be selected by adding an infinitesimal "magnetic field" $h\sum_{\vec{i}\in\Lambda_s} L_{\vec{i}}$), a single Wilson line has a nonzero expectation,

$$\lim_{h\to 0}\langle L_{\vec{i}}\rangle_h = \pm m \quad ,$$

and the $Z(2)$ global symmetry is spontaneously broken. Conversely, if the $Z(2)$ symmetry is not spontaneously broken, then the magnetization must vanish and static quarks are confined.

To establish that the $Z(2)$ symmetry is in fact broken at sufficiently high temperature, we prove the following.

*Theorem:* For every coupling $g^2 > 0$ and dimension $d \geq 2$, there is a temperature $T^*(g^2)$ and a function $\mu(T)$ such that

$$\langle L_{\vec{i}} L_{\vec{j}}^{\dagger}\rangle_{\Lambda} \geq \mu(T)^2 > 0 \tag{6}$$

for all $T > T^*(g^2)$ uniformly in $|\vec{i}-\vec{j}|$ and $|\Lambda_s|$. Furthermore, $\mu(T)\to 1$ as $T\to\infty$.

We now sketch our method for proving long-range order (i.e., $\langle L_{\vec{i}} L_{\vec{j}}^{\dagger}\rangle > 0$). Define $P_{\vec{i}}^{\pm\lambda}$ to be projection operators onto all configurations satisfying $\pm L_{\vec{i}} \geq \lambda$, and

$$P_i^{(-\lambda,\lambda)} = 1 - P_{\vec{i}}^{+\lambda} - P_{\vec{i}}^{-\lambda}$$

($|\lambda| < 1$). Then one can show

$$\langle L_{\vec{i}} L_{\vec{j}}^{\dagger}\rangle \geq \lambda^2 - 2(1+\lambda^2)\langle P_{\vec{i}}^{+\lambda} P_{\vec{j}}^{-\lambda}\rangle - 2\lambda(1+\lambda)\langle P_{\vec{i}}^{(-\lambda,\lambda)}\rangle \quad . \tag{7}$$

The Peierls argument[5] now allows one to bound $\langle P_{\vec{i}}^{+\lambda} P_{\vec{j}}^{-\lambda}\rangle$ in terms of nearest-neighbor projections,

$$\langle P_{\vec{i}}^{+\lambda} P_{\vec{j}}^{-\lambda}\rangle \leq \langle P_{\vec{i}}^{+} P_{\vec{j}}^{-}\rangle \leq \sum_{\gamma\in\Lambda_s}\Big\langle \prod_{\langle kl\rangle\in\gamma} P_{\vec{k}}^{+} P_{\vec{l}}^{-}\Big\rangle \quad . \tag{8}$$

Here $P_{\vec{i}}^{+} = P_{\vec{i}}^{+0}$ is the projection onto the upper hemisphere of SU(2) (i.e., $L_{\vec{i}} > 0$), and $P_{\vec{i}}^{-}$ the projection onto the lower hemisphere. The sum is over all "contours" $\gamma$ defined as connected, coclosed sets of bonds which separate the sites $\vec{i}$ and $\vec{j}$. (In other words, if $C$ is a minimal length path of bonds from $\vec{i}$ to $\vec{j}$, then $\gamma \cap C = 1$.) $\langle kl\rangle$ denotes the bond of $\Lambda_s$ running from site $\vec{k}$ to the nearest-neighbor site $\vec{l}$. Each term in (8) may now be related to an extensive quantity by means of a chessboard estimate,[8]

$$\Big\langle \prod_{\langle kl\rangle\in\gamma} P_{\vec{k}}^{+} P_{\vec{l}}^{-}\Big\rangle \leq \Big\langle \prod_{\substack{\vec{k}\ \text{even}\\ \vec{l}\ \text{odd}}} P_{\vec{k}}^{+} P_{\vec{l}}^{-}\Big\rangle^{|\gamma|/d|\Lambda_s|} \quad . \tag{9}$$

Here $\vec{k}$ "even" ("odd") indicates spatial sites whose first coordinate ($k_1$) is even (odd). The "equatorial" projection in (7) may similarly be bounded by a chessboard estimate,

$$\langle P_{\vec{i}}^{(+\lambda,-\lambda)}\rangle \leq \Big\langle \prod_{\vec{k}} P_{\vec{k}}^{(+\lambda,-\lambda)}\Big\rangle^{1/|\Lambda_s|} \quad . \tag{10}$$

We will show that

$$\Big\langle \prod_{\substack{\vec{k}\ \text{even}\\ \vec{l}\ \text{odd}}} P_{\vec{k}}^{+} P_{\vec{l}}^{-}\Big\rangle \leq [\kappa(T)]^{d|\Lambda_s|} \tag{11}$$

and

$$\Big\langle \prod_{\vec{k}} P_{\vec{k}}^{(+\lambda,-\lambda)}\Big\rangle \leq [\eta(T)/(1-\lambda^2)^{d-1}]^{|\Lambda_s|} \quad , \tag{12}$$

where $\kappa(T)$ and $\eta(T)$ vanish as $T\to\infty$. Assuming these bounds (whose proof we will return to in a moment), we may now complete the Peierls argument. Equations (8), (9), and (11) imply that

$$\langle P_{\vec{i}}^{+\lambda} P_{\vec{j}}^{-\lambda}\rangle \leq \sum_{|\lambda|=4}^{\infty} N(|\gamma|)\,\kappa(T)^{|\gamma|} \quad , \tag{13}$$

where $N(|\gamma|)$ is the number of contours of length $|\gamma|$. A standard counting argument bounds this by $(d/18)3^{|\gamma|}|\gamma|^{1/(d-1)}$. Consequently, one finds

$$\langle P_{\vec{i}}^{+\lambda} P_{\vec{j}}^{-\lambda}\rangle \leq c\,\kappa(T)^4/[1-9\kappa(T)^2]^{d/(d-1)} \tag{14}$$

for constant $c$, provided the temperature is sufficiently large that $\kappa(T) < \frac{1}{3}$. The stated theorem is finally a consequence of inequalities (7), (10), (12), and (14).

The crux of the proof is thus the estimates of the thermodynamic expectations (11) and (12). Unfortunately, the simple energetic estimates commonly exployed[8] in conjunction with chessboard estimates turn out to be inadequate. This is because in our theory the action does not uniformly suppress the probability of a domain wall. Specifically, in the expectations (11) and (12) there are configurations of bond variables which contribute and for which the action is arbitrarily close to zero (the maximal action). However, the "phase space" of these exceptional configurations vanishes as $T\to\infty$, and this leads to power suppression of the expectations (11) and (12) as $T\to\infty$.

In order to deal with this problem, we use the following decimation procedure. Consider, for example, the numerator of the expectation (12). We may first set $\beta_s = 0$ since this only increases the numerator. Each spacelike bond variable now only interacts through the timelike plaquettes immediately above and below it. Consequently, the spacelike bond variables on every other time slice may now be integrated out (exactly). The integrand for the remaining integrations is now a product of modified Bessel functions ($I_1$'s). However, using the log convexity of Bessel functions, one may bound the integral by an integral of precisely the same form as the original except that (i) $L_t$ has been halved, (ii) the timelike coupling has been changed,

$$\beta_t \to \beta_t' = \tfrac{1}{4} \ln[I_1(2\beta_t)/\beta_t] \quad ,$$

and (iii) an overall factor of $e^{4\beta_t'}$ for each bond integrated out appears. One may show that this bound only errs in the overall constant; the exponential dependence *and* the powers of temperature are correctly reproduced. Successively iterating this procedure allows one to integrate out all spacelike bond variables. This leaves an effective theory of interacting "spins" $\Omega_{\vec{l}}$ on the spatial lattice $\Lambda_s$. Explicitly,

$$\left\langle \prod_{\vec{k}} P_{\vec{k}}^{(+\lambda,-\lambda)} \right\rangle \leq Z^{-1} \int \prod_{\vec{k}} d\Omega_{\vec{k}} P_{\vec{k}}^{(+\lambda,-\lambda)} \prod_{\langle kl \rangle \in \Lambda_s} \left| c \frac{\exp\{\tilde{\beta}[\cos(\omega_{\vec{k}} - \omega_{\vec{l}}) - 1]\}}{2\tilde{\beta} \sin\omega_{\vec{k}} \sin\omega_{\vec{l}}} \right|$$

$$\leq \left| \frac{(1-\lambda^2)^{1-d}}{(2\tilde{\beta})^2} c^d \right|^{|\Lambda_s|} Z^{-1} \int \prod_{\vec{k}} d\omega_{\vec{k}} \prod_{\langle kl \rangle} \exp\{\tilde{\beta}[\cos(\omega_{\vec{k}} - \omega_{\vec{l}}) - 1]\} \quad .$$

Here $\{e^{\pm i\omega_{\vec{k}}}\}$ are the eigenvalues of $\Omega_{\vec{k}}$, and $\tilde{\beta}$ is the effective coupling which results from the iterated decimation precedure. The remaining integral is precisely the partition function of a $d$-dimensional $x$-$y$ model. A similar decimation procedure, now applied in a spacelike direction, may be used to bound the final integrals. Finally, one must bound from below the original partition function in the denominator. This may either be done directly, or one may use a variant of the decimation technique which yields lower bounds instead of upper. Eventually, one finds the bound (12) where $\eta(T)$ is a complicated but explicit function which decreases as $T^{-1/2}$ as $T \to \infty$. An essentially identical procedure is used to evaluate (11) and one finds $\kappa(T) = O(T^{-1/2d})$ as $T \to \infty$.

Instead of using the magnetization (5) as a confinement criterion, one may alternatively choose to study the electric-flux free energy.[9] This measures the change in free energy due to a topologically nontrivial sheet of electric flux. The same technique described here can be used to bound directly the electric free energy.[10] This actually yields a slightly better bound on the critical temperature. [One can also prove in general that the electric free energy is bounded by the quark-antiquark potential, $F_\Lambda^{\mathrm{elec}}/L_s \leq F_\Lambda^{q\bar{q}}(\vec{x})/|\vec{x}|$.[10]]

For simplicity, we have restricted ourselves to the simplest non-Abelian gauge theory, SU(2). For discrete Abelian $Z(N)$ theories, a similar approach may easily be successfully applied. For general SU($N$) theories, although the physics and overall strategy are the same, technical complications arise due to the fact that a much more elaborate set of projections for decomposing the configuration space of timelike links is needed. We have not attempted to carry this out in detail.

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