Brief Reports

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The big bang in the Tolman models

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Of the two possible types of initial singularity in the Tolman model, one is merely a locus of shell-crossing points which we characterize here in terms of a surface energy tensor, with well-defined energy density, which satisfies an intrinsic strong energy condition. Thus characterized, the surface is not acceptable as an inhomogeneous generalization of the standard big-bang singularity.

Shell-crossing singularities, naked examples of which are known to develop in a wide variety of circumstances, are generally considered to be unphysical since they probably represent mere artifacts of the continuum approximation.[1] Moreover, such singularities produce negligible tidal forces on neighboring particles.[2] A locus of shell-crossing points would, therefore, not be acceptable as a singularity of the type envisioned for the big bang. It is the purpose of the present remark to point out that, of the two possible types of initial singularity encountered in the Tolman solution (the simplest solution of the Einstein equations which exhibits shell-crossing behavior, and which can be considered an inhomogeneous cosmological model), one is a locus of shell-crossing points that can in fact be described in detail by means of a surface energy three-tensor,[3] with well-defined energy density, which (just) satisfies an intrinsic strong energy condition.

Eardley and Smarr[4] have given an extensive discussion of the singularity structures in the Tolman models by use of Cauchy time functions which involves a "mean" extrinsic curvature to spacetime slices. The present remark is distinct from this analysis since we are concerned here with the detailed intrinsic structure of a specific type of singularity, the locus of shell-crossing points.

The Tolman (spherical dust) solution, in synchronous co-moving coordinates, is given by[5]

$$ds^2=\frac{R'^2}{f}dr^2+R^2d\Omega^2-dt^2 \ , \tag{1}$$

where $d\Omega^2 \equiv d\theta^2+\sin^2\theta\, d\phi^2$, $f=f(r)>0$, $R=R(t,r)$, and a prime denotes $\partial/\partial r$. R evolves according to

$$\dot{R}^2=f-1+\frac{2m}{R}+\frac{\Lambda R^2}{3} \ , \tag{2}$$

where $m=m(r)$ and an overdot denotes $\partial/\partial t$. The density is given by ρ, where

$$4\pi\rho=\frac{m'}{R^2R'} \ . \tag{3}$$

We assume that $m'(r)>0$ for $r>0$.[6] Both the surfaces $R=0$ and $R'=0$ are distinct scalar polynomial singularities[1] (away from $r=0$, where they intersect[6]). In a cosmological context, one labels these surfaces by their equation for $t(r)$ and takes the maximum, at a given r, to be the (inhomogeneous) big bang of the model.[7] Here we are interested in the (partial) three-surface (Σ):

$$R'=0, \quad R\neq 0, \quad r>0 \ , \tag{4}$$

which, according to Eq. (1), is timelike.

We treat Σ as a thin shell[3] with flat[8] interior and take (θ,ϕ,t) as the coordinates intrinsic to Σ. Following Israel,[9] the Lanczos equations, which describe the surface energy three-tensor S_{ij}, reduce to

$$8\pi S_{ij}=g_{ij}K-K_{ij} \ , \tag{5}$$

where K_{ij} is the extrinsic curvature three-tensor of Σ, g_{ij} is the metric intrinsic to Σ [given by the metric (1) with $R'=0$], and $K\equiv g^{ij}K_{ij}$. The flat interior induces on Σ the proper surface density σ with

$$4\pi\sigma=-K^{\theta}_{\theta} \tag{6}$$

which, from Eq. (1), reduces to[10]

$$4\pi\sigma=\frac{\sqrt{f}}{R} \ . \tag{7}$$

Moreover, from the intrinsic metric and Eq. (5) it follows that

$$S_{ij}u^iu^j=\tfrac{1}{2}u^iu_iS \ , \tag{8}$$

so that for an ideal fluid three-tensor S_{ij} the surface pressure P reduces to

$$P=-\sigma/2 \ . \tag{9}$$

In summary, we have shown that the surface Σ defined by (4), which is a locus of shell-crossing points, though exhibiting poor C^2 behavior (as demonstrated by the four-scalar ρ), has a well-defined C^1 structure (as shown by the behavior of the three-scalar σ). In contrast, the singularities $R=0$ do not have a well-defined C^1 structure (e.g., in terms of a surface density). In a cosmological context, Σ is not a suitable initial surface (i.e., generalized "big bang"). Rather, Σ could be interpreted as a bubble boundary (e.g., between dust and, say, vacuum) as detailed above.

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[1]See, for example, the review by F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 2.

[2]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973); H.-J. Seifert, in *Differential Geometrical Methods in Mathematical Physics*, edited by K. Bleuler and A. Reetz, Lecture Notes in Mathematics Vol. 570 (Springer, New York, 1977).

[3]See K. Lake, Phys. Rev. D 19, 2847 (1979) and references therein.

[4]D. M. Eardley and L. Smarr, Phys. Rev. D 19, 2239 (1979).

[5]See G. C. Omer, Proc. Natl. Acad. Sci. 53, 1 (1965) and references therein. The present argument, with but the change $m=m(r,t)$, holds in detail with the introduction of an isotropic pressure p (where $4\pi p=-\dot{m}/R^2\dot{R}$) as long as the pressure distribution is homogeneous ($p'=0$).

[6]See, for example, W. B. Bonnor, Mon. Not. R. Astron. Soc. 167, 55 (1974) for a discussion on this point. The global condition $m'=0$ reduces the solution to the Schwarzschild–de Sitter solution, as is well known. To verify the scalar polynomial singular character of the surfaces $R=0$ and $R'=0$ (for $m'>0$) one can calculate the Kretschmann scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=\frac{48m^2}{R^6}-\frac{32mm'}{R^5R'}+\frac{12m'^2}{R^4R'^2}+\frac{8\Lambda m'}{3R^2R'}+\frac{8\Lambda^2}{3}\ ;$$

see, for example H. Bondi, Mon. Not. R. Astron. Soc. 107, 410 (1947).

[7]For example, Ref. 6, C. C. Dyer, Mon. Not. R. Astron. Soc. 189, 189 (1979); P. Szekeres, in *Gravitational Radiation, Collapsed Objects and Exact Solutions*, Lecture Notes in Physics Vol. 124, edited by C. Edwards (Springer, Berlin, 1980).

[8]The assumption of a flat interior is not essential, it merely simplifies considerably the following discussion, and allows a detailed unambiguous discussion of the surface Σ.

[9]W. Israel, Nuovo Cimento 44B, 1 (1966); 48B, 463 (1967).

[10]It is necessary to assume that the surface Σ is C^1. That is, writing $t=g(r)$ for Σ, we assume g' is well defined over Σ.