Some general properties of the $O(\alpha)$ corrections to parity violation in atoms

W. J. Marciano
Physics Department, Brookhaven National Laboratory, Upton, New York 11973

A. Sirlin*
Department of Physics, The Rockefeller University, New York, New York 10021

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The model-dependent contributions to the parity-violating electron-nucleon interaction arising from γZ box diagrams are discussed on the basis of a formulation in which the hadronic parts of the amplitude are treated exactly *ab initio*. This leads to a derivation of general properties and more refined estimates of these effects. A method to take into account the large-distance part of the hadronic contributions to the γZ mixing diagrams using experimental information on $e^+ + e^- \rightarrow$hadrons is briefly discussed. Predictions for the C_{iN} ($i=1,2$, $N=p,n$) are updated taking into account the more detailed estimates of this paper.

I. INTRODUCTION

The investigation of atomic parity violation has been the subject of considerable interest during the last several years.[1] On the theoretical side, a number of studies have addressed the problem of evaluating the one-loop corrections to the parity-violating electron-nucleon interaction.[2–5] In particular, Ref. 5 presented detailed expressions for the radiative corrections to the effective electron-quark couplings C_{iu}, C_{id} ($i=1,2$) in terms of the μ-decay coupling constant G_μ and $\sin^2\hat{\theta}_W(m_W)$ defined by modified minimal subtraction[6] ($\overline{MS}$). We recall that the constants C_{1u}, C_{1d} govern the contributions involving vector hadronic and axial-vector leptonic currents $[(V^\mu)_h(A_\mu)_l$ for short] while C_{2u}, C_{2d} are the effective couplings of the $(A^\mu)_h(V_\mu)_l$ interactions. It was pointed out in Ref. 5 that uncertainties due to the strong interactions are expected to be small in the $(V^\mu)_h(A_\mu)_l$ terms while they are potentially significant in the $(A^\mu)_h(V_\mu)_l$ contributions. However, the analysis of Ref. 5 was based on simple quark-model calculations. While such calculations give an accurate description of the short-distance contributions, as can be verified by applying the current-algebra formulation of radiative corrections[7] in the framework of asymptotically free theories of the strong interactions, there is no particular reason why they should provide a suitable basis for discussing the long-distance, model-dependent effects.

The aim of this paper is to examine in greater generality and detail the model-dependent contributions associated with γZ box diagrams (Fig. 1) and γZ mixing graphs (Fig. 2). In particular, our discussion of the γZ box contributions is based on a formulation in which the hadronic parts of the amplitude are treated exactly *ab initio*, a fact that allows us to obtain general properties of these contributions without appealing to particular hadron models. In that way we find explicitly in Sec. II that the graphs in Fig. 1 lead to three classes of contributions: (a) long-distance Coulombic corrections, which can be disregarded because they are part of the effects associated with the bound-state wave function already included in the treatment of the lowest-order amplitude, (b) terms of $O(\alpha m_e/m_h)$ which we neglect (m_h is a hadronic mass), and (c) terms of $O(\alpha)$. For the latter contributions, we find a rather important general property: they are suppressed by a factor $\sin^2\theta_W - \frac{1}{4}$ (which is small since $\sin^2\theta_W \simeq 0.215\pm0.014$) in the $(V^\mu)_h(A_\mu)_l$ contributions but not in the $(A^\mu)_h(V_\mu)_l$ terms. This property was already noted in the quark-model calculations,[5] but it is established here without appealing to any model of hadron structure. The $O(\alpha)$ terms are then estimated in Sec. III by calculating the short-distance part and approximating the long-distance part by the Born amplitude for a physical nucleon. In Sec. IV we discuss briefly the hadronic contribution to the γZ mixing diagrams in Fig. 2. In analogy with similar discussions in Refs. 8–10, we describe a method for eliminating the dependence on light-quark masses by using experimental information regarding the hadronic vacuum-polarization functions. It is interesting to note that, aside from the contributions of Figs. 1 and 2, the $O(\alpha)$ corrections to the $(V^\mu)_h(A_\mu)_l$ terms are essentially unaffected by the strong interactions. Therefore, the more general and detailed analysis of the model-dependent terms carried out in this paper provides a stronger theoretical basis for their formulation. In Sec. V we use the estimates of the model-dependent terms to update the $SU(2)_L\times U(1)$ predictions for the C_{iN} ($i=1,2$, $N=p,n$) which parametrize atomic parity violation. Finally, we conclude in Sec. VI with some comments about our results.

II. γZ BOX DIAGRAMS: GENERAL PROPERTIES

For vanishing momentum transfer between the initial and final electrons, the sum of the amplitudes in Figs. 1(a) and 1(b) can be written in the 't Hooft–Feynman gauge as

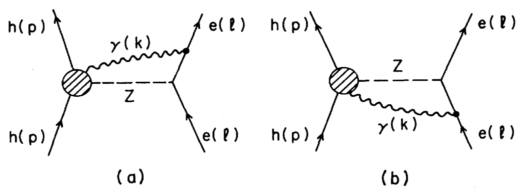

FIG. 1. γZ box diagrams contributing to the electron-nucleus interaction.

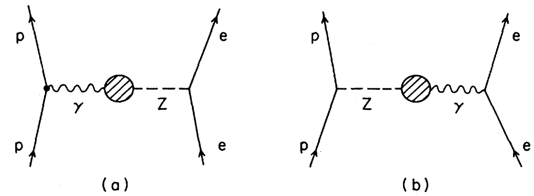

FIG. 2. γZ mixing diagrams.

$$\mathcal{M}=-i\left[\frac{g}{c}\right]^2 e^2\int\frac{d^4k}{(2\pi)^4}T^{\mu\nu}(k,p)\frac{1}{k^2}\frac{1}{(k^2-m_Z{}^2)}u'_e\left[\frac{(2l_\mu-\gamma_\mu \not{k})\gamma_\nu}{k^2-2l\cdot k}+\frac{\gamma_\nu(2l_\mu+\not{k}\gamma_\mu)}{k^2+2l\cdot k}\right](\epsilon_-a_-+\epsilon_+a_+)u_e\ , \tag{1a}$$

where l, p, and k are the lepton, hadron and photon four-momenta, $a_\pm=(1\pm\gamma_5)/2$, $\epsilon_-=\frac{1}{2}-s^2$, $\epsilon_+=-s^2$ (c^2 and s^2 are abbreviations for $\cos^2\theta_W$ and $\sin^2\theta_W$) and

$$T^{\mu\nu}(k,p)=\int d^4x\, e^{ik\cdot x}\langle p,f\,|\,T[J_\gamma^\mu(x)J_Z^\nu(0)]\,|\,p,i\rangle\ . \tag{1b}$$

In Eq. (1b) $|p,i\rangle$ and $|p,f\rangle$ stand for initial and final hadron states and J_γ^μ and J_Z^ν denote the hadronic currents coupled to A^μ and Z^ν, respectively:

$$J_\gamma^\mu=\bar\psi\gamma^\mu Q\psi\ , \tag{2a}$$

$$J_Z^\nu=\tfrac{1}{2}\bar\psi\gamma^\nu a_-C_3\psi-s^2\bar\psi\gamma^\nu Q\psi\ , \tag{2b}$$

where $\psi^T=(u\ c\ t\ d\ s\ b)$, a summation over the color degrees of freedom is understood, and

$$Q=\begin{bmatrix}\frac{2}{3}1 & 0\\ 0 & -\frac{1}{3}1\end{bmatrix} \tag{2c}$$

and

$$C_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2d}$$

are 6×6 matrices acting on the flavor degrees of freedom (1 and 0 are the unit and null 3×3 matrices, respectively). In writing down Eqs. (1a) and (1b) we have chosen a perturbative scheme in which, to zeroth order, the electron and nucleus are treated as noninteracting particles. Using the Dirac equation for the electron and the identity

$$\gamma_\mu\gamma_\rho\gamma_\nu=g_{\mu\rho}\gamma_\nu-g_{\mu\nu}\gamma_\rho+g_{\rho\nu}\gamma_\mu-i\epsilon_{\mu\rho\nu\alpha}\gamma^\alpha\gamma_5\ , \tag{3}$$

where $\epsilon_{0123}=1$, Eq. (1a) can be cast in the form

$$\begin{aligned}\mathcal{M}=-i\left[\frac{g}{c}\right]^2 e^2\int\frac{d^4k}{(2\pi)^4}T^{\mu\nu}(k,p)\frac{1}{k^2}\frac{1}{k^2-m_Z{}^2}\\ \times\bar u'_e\left[\left[\frac{1}{k^2+2l\cdot k}-\frac{1}{k^2-2l\cdot k}\right](k_\mu\gamma_\nu-\not{k}g_{\mu\nu}+k_\nu\gamma_\mu)\right.\\ \left.+\left[\frac{1}{k^2-2l\cdot k}+\frac{1}{k^2+2l\cdot k}\right](2l_\mu\gamma_\nu+i\epsilon_{\mu\rho\nu\alpha}k^\rho\gamma^\alpha\gamma_5)\right](\epsilon_-a_-+\epsilon_+a_+)u_e\ .\end{aligned} \tag{4}$$

As we will see later in greater detail, as $k\to0$, $T^{\mu\nu}(k,p)\to k^{-1}$. As a consequence, potential infrared divergences can only arise from terms with numerators of $O(k^0)$. The only such contribution in Eq. (4) is the term proportional to $2l_\mu\gamma_\nu$ and we see from the structure of its cofactor that the associated infrared-divergent contributions cancel. Thus Eq. (4) is free from infrared divergences. Combining

$$(k^2-2l\cdot k)^{-1}-(k^2+2l\cdot k)^{-1}=4l\cdot k(k^2+2l\cdot k)^{-1}(k^2-2l\cdot k)^{-1}\ ,$$

we note that the terms proportional to $2l_\mu\gamma_\nu$ and $k_\mu\gamma_\nu-\not{k}g_{\mu\nu}+k_\nu\gamma_\mu$ are formally proportional to the lepton momentum. At this stage it is convenient to separate out the term of $O(k^{-1})$ in $T^{\mu\nu}(k,p)$:

$$T^{\mu\nu}(k,p)=T^{(\mathrm{I})\mu\nu}(k,p)+T^{(\mathrm{II})\mu\nu}(k,p)\ , \tag{5a}$$

where, by definition, $T^{(\mathrm{I})\mu\nu}(k,p)$ contains all terms of $O(k^{-1})$ as $k\to0$. Elementary power counting shows that the contribution of $T^{(\mathrm{II})\mu\nu}(k,p)$ to the terms proportional to $(k_\mu\gamma_\nu-\not{k}g_{\mu\nu}+k_\nu\gamma_\mu)2l\cdot k$ and $2l_\mu\gamma_\nu$ in Eq. (4) are of order $O((m_e/m_h)\ln(m_h/m_e))$ or $O(m_e/m_h)$, where m_h is a hadronic mass. Therefore, the contribution of $T^{(\mathrm{II})\mu\nu}(k,p)$ to those terms is greatly suppressed and henceforth will be neglected. On the other hand, $T^{(\mathrm{I})\mu\nu}(k,p)$ can give non-negligible contributions to those terms because in the limit $l \to 0$ the k-integral diverges linearly in the small-k region. Thus, we may expect that the contributions of $T^{(\mathrm{I})\mu\nu}(k,p)$ to the integrals proportional to $2l_\mu\gamma_\nu$ and $(k_\mu\gamma_\nu - \not{k} g_{\mu\nu} + k_\nu\gamma_\mu)2l\cdot k$ are indeed of $O(l^0)$ and, therefore, not suppressed. We will now show, however, that if terms of $O(m_e/m_h)$ are once more neglected, these particular $T^{(\mathrm{I})\mu\nu}$ contributions can be identified (to the order of our calculation) with the long-distance Coulomb interaction between the nucleus and the electron. We recall that the terms of $O(k^{-1})$ in $T^{\mu\nu}$ arise from the Born approximation insertions of J_γ^μ and J_Z^ν in the external legs of the nucleus (Fig. 3). As a consequence, we can write

$$T^{(\mathrm{I})\mu\nu} = i2Zp^\mu \left[\frac{1}{k^2+2p\cdot k+i\epsilon} + \frac{1}{k^2-2p\cdot k+i\epsilon} \right] \times \langle p;f \,|\, J_Z^\nu \,|\, p;i\rangle + O(k^0) , \tag{5b}$$

where Z is the electric charge of the nucleus. Inserting the first term of Eq. (5b) into Eq. (4), the d^4k integration for the terms proportional to $(k_\mu\gamma_\nu - \not{k} g_{\mu\nu} + k_\nu\gamma_\mu)2l\cdot k$

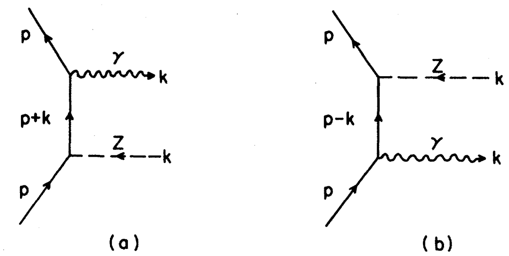

FIG. 3. Born-amplitude contribution to $T^{\mu\nu}$ in Eq. (1b)

and $2l_\mu\gamma_\nu$ can be exactly evaluated. However, since the terms of $O(l^{-1})$ in the integration arise from the region $k \ll M_N$ (M_N is the nucleus mass) it is more instructive to apply in the nucleus rest frame the approximation[11]

$$\frac{1}{k^2+2p\cdot k+i\epsilon} + \frac{1}{k^2-2p\cdot k+i\epsilon} \approx \frac{1}{2M_N k_0 + i\epsilon} + \frac{1}{-2M_N k_0 + i\epsilon} = -\frac{i\pi\delta(k_0)}{M_N} . \tag{5c}$$

Indeed, the error resulting from this approximation is of $O(m_e/m_h)$. In this way we find that, with the neglect of $O(m_e/M_N)$ terms, the contribution of $T^{(\mathrm{I})\mu\nu}(k,p)$ to the terms proportional to $(k_\mu\gamma_\nu - \not{k} g_{\mu\nu} + k_\nu\gamma_\mu)2l\cdot k$ and $2l_\mu\gamma_\nu$ in Eq. (4) is given by

$$\mathcal{M}' = -i\left[\frac{g}{c}\right]^2 \frac{1}{m_Z{}^2} \langle p;f \,|\, J_Z^\nu \,|\, p;i\rangle \frac{Z\alpha}{\pi^2} \times \int \frac{d^4k}{|\vec{k}\,|^2} \delta(k_0) \left[2l_0 \frac{1}{k^2+2l\cdot k} \bar{u}'_e \gamma^0 (\epsilon_- a_- + \epsilon_+ a_+) u_e + \bar{u}'_e \left[\frac{k_0\gamma_\nu - \not{k} g_{0\nu} + k_\nu\gamma_0}{k^2+2l\cdot k} \right] (\epsilon_- a_- + \epsilon_+ a_+) u_e \right] . \tag{5d}$$

Equation (5d) has a simple physical interpretation: if we consider the interaction mediated by Z-boson exchange between the plane wave electron and the nucleus [Fig. 4(a)], an elementary calculation shows that $\mathcal{M}'$ represents the corrections associated with the initial and final Coulomb interactions between the nucleus and the electron [Figs. 4(b) and 4(c)] in the limit of very large nuclear mass. In the usual treatment of the Z-mediated lowest-order electron-nucleus interaction, the electron is represented by a bound-state wave function rather than a plane wave. Such bound-state wave functions reflect, of course, the interaction of the electron and nucleus to all orders in the Coulomb interaction. Thus, we avoid double counting by simply disregarding the contribution of $\mathcal{M}'$. In summary, we have shown that the terms proportional to $(k_\mu\gamma_\nu - \not{k} g_{\mu\nu} + k_\nu\gamma_\mu)$ and $2l_\mu\gamma_\nu$ in Eq. (4) give rise to contributions which can be identified with the long-range Coulomb interactions already included in the bound-state wave-function description, and to terms suppressed by factors of $O(m_e/m_h)$.

On the other hand, the term proportional to $\epsilon_{\mu\rho\nu\alpha}k^\rho\gamma^\alpha\gamma_5$ in Eq. (4) represents a genuine correction of $O(\alpha)$. Denoting such terms by $\mathcal{M}''$, we see that they give rise to two types of contributions to the parity-violating electron-nucleus interaction:

$$\mathcal{M}_1'' = -\left[\frac{g}{c}\right]^2 e^2 \frac{(\epsilon_+ + \epsilon_-)}{2} \epsilon_{\mu\nu\rho\alpha} (\bar{u}'_e \gamma^\alpha\gamma_5 u_e) \int \frac{d^4k}{(2\pi)^4} \frac{k^\rho A^{\mu\nu}(k,p)}{k^2(k^2-m_Z{}^2)} \left[\frac{1}{k^2+2l\cdot k} + \frac{1}{k^2-2l\cdot k} \right] , \tag{6a}$$

$$\mathcal{M}_2'' = \left[\frac{g}{c}\right]^2 e^2 \frac{(\epsilon_- - \epsilon_+)}{2} \epsilon_{\mu\nu\rho\alpha} (\bar{u}'_e \gamma^\alpha u_e) \int \frac{d^4k}{(2\pi)^4} \frac{k^\rho V^{\mu\nu}(k,p)}{k^2(k^2-m_Z{}^2)} \left[\frac{1}{k^2+2l\cdot k} + \frac{1}{k^2-2l\cdot k} \right] , \tag{6b}$$

where $A^{\mu\nu}(k,p)$ [$V^{\mu\nu}(k,p)$] is the pseudotensor [tensor] obtained by substituting $J_Z^\nu \to A_Z^\nu$ [V_Z^ν] in Eq. (1b) and

$$A_Z^\nu = -\tfrac{1}{4}\bar{\psi}\gamma^\nu\gamma_5 C_3 \psi , \tag{7a}$$

$$V_Z^\nu = \tfrac{1}{4}\bar{\psi}\gamma^\nu (C_3 - 4s^2 Q)\psi , \tag{7b}$$

are the axial-vector and vector parts of J_Z^ν. Although Eqs. (6a) and (6b) involve two-current correlation functions in the hadron sector rather than matrix elements of simple currents, it is clear that $\mathcal{M}_1''$ and $\mathcal{M}_2''$ give contributions of the types $(V^\mu)_h(A_\mu)_l$ and $(A^\mu)_h(V_\mu)_l$, provided we general-

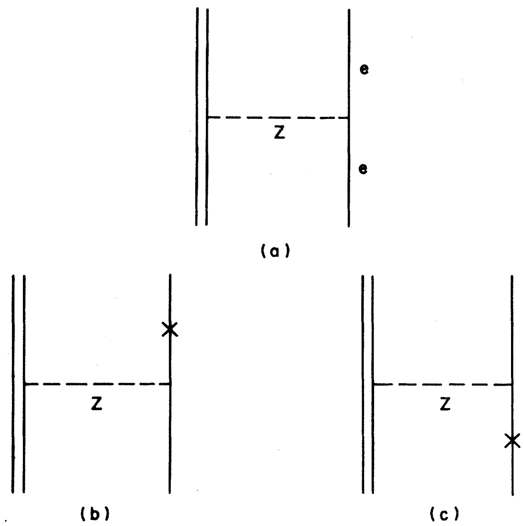

FIG. 4. Lowest-order electron-nucleus interaction mediated by the Z boson and Coulombic corrections arising from an infinitely heavy nucleus.

ize $(V^{\mu})_h$ and $(A^{\mu})_h$ to represent hadronic amplitudes transforming as vectors and axial vectors, respectively, rather than matrix elements of local vector and axial-vector currents. Recalling the values of ϵ_- and ϵ_+, we note that $(\epsilon_+ + \epsilon_-)/2 = \frac{1}{4} - s^2$, $(\epsilon_- - \epsilon_+)/2 = \frac{1}{4}$. This leads to the rather important result that $\mathscr{M}_1''$ [i.e., the $(V^{\mu})_h (A_{\mu})_l$ contributions of $O(\alpha)$ from γZ box diagrams] are suppressed by a factor $\frac{1}{4} - s^2$. Thus, this inhibition is not merely an artifact of simple quark-model calculations or other approximate descriptions of hadron structure, but emerges here as a general feature. On the other hand, we see that $\mathscr{M}_2''$ is not suppressed by a similar factor.

Considering now the specific evaluation of Eqs. (6a) and (6b), the general discussion of Ref. 7 and the calculations of Refs. 2—5 indicate that these contributions give rise to terms proportional to $\ln(m_Z{}^2/M^2)$ where M is a mass scale representing the onset of the asymptotic behavior, i.e., the regime in which the strong-interaction coupling constant $\bar{g}_s{}^2(k^2)/4\pi$ becomes small. (We assume in this discussion that the underlying theory of strong interactions is asymptotically free.) Moreover, the coefficient of $\ln(m_Z{}^2/M^2)$ is independent of the dynamical details of the strong interactions. The latter do induce contributions of $O(\bar{g}_s{}^2(k^2))$ in the Feynman integrals which lead to further corrections of $O((\ln\ln(m_Z{}^2/M^2)).$[7] On the other hand, the precise value of M^2 and the constant terms accompanying the leading-logarithm contribution are in general affected by the strong interactions. Thus, the best we can do for the low-frequency part is to provide an estimate. The above discussion suggests the following strategy: we first compute the high-frequency contributions which lead to the large logarithmic term proportional to $\ln(m_Z{}^2/M^2)$ and then approximate the low-frequency contributions by the Born approximation for the case of a physical nucleon. The latter approximation has two noteworthy features: it correctly describes the very-low-frequency contributions [terms of $O(k^{-1})$ and $O(k^0)$] and in the case of a physical nucleon, it can be evaluated in terms of known electromagnetic and weak form factors.

III. DETAILED ESTIMATES OF γZ BOX DIAGRAMS

A. Asymptotic behavior

In the limit $\bar{g}_s{}^2(k^2)=0$, the asymptotic behavior for large k of $T^{\mu\nu}$ can be obtained by an elementary application of the Bjorken-Johnson-Low limit or by carrying out short-distance expansions in the free-field theory. Either way, one obtains

$$T^{\mu\nu} \rightarrow \frac{2}{k^2}\epsilon^{\mu\nu\lambda\beta}k_\lambda \left\langle p;f \left| \bar{\psi}\gamma_\beta \left[\frac{QC_3 a_-}{2} + s^2 Q^2 \gamma_5 \right] \psi \right| p;i \right\rangle . \tag{8}$$

The corresponding asymptotic behaviors of $A^{\mu\nu}$ and $V^{\mu\nu}$ are obtained by separating out the vector and axial-vector current operators in the matrix element on the right-hand member (RHM) of Eq. (8). Thus for $A^{\mu\nu}$ and $V^{\mu\nu}$ one replaces the matrix on the RHM of Eq. (8) by

$$\tfrac{1}{4}\langle p;f \,|\, \bar{\psi}\gamma_\beta Q C_3 \psi \,|\, p;i \rangle$$

and

$$-\tfrac{1}{4}\langle p;f \,|\, \bar{\psi}\gamma_\beta\gamma_5 (QC_3 - 4s^2Q^2)\psi \,|\, p;i \rangle ,$$

respectively. Next, we insert these asymptotic expressions for $A^{\mu\nu}$ and $V^{\mu\nu}$ into Eqs. (6a) and (6b), respectively, perform a Wick rotation and restrict the range of integration to Euclidean momenta $\kappa^2 \geq M^2$ for which Eq. (8) is approximately valid. Since l_0, $|\vec{l}\,| \ll M$, we can also set $l=0$ in the integrand. In this way we obtain for the asymptotic contributions

$$(\mathscr{M}_1'')_{\text{asy}} = -i\frac{G_\mu}{\sqrt{2}}\frac{3\alpha}{4\pi}(1-4s^2)K(\bar{u}_e'\gamma^\alpha\gamma_5 u_e) \times \langle p;f \,|\, \bar{\psi}\gamma_\alpha Q C_3 \psi \,|\, p;i \rangle , \tag{9a}$$

$$(\mathscr{M}_2'')_{\text{asy}} = -i\frac{G_\mu}{\sqrt{2}}\frac{3\alpha}{4\pi}K(\bar{u}_e'\gamma^\alpha u_e) \times \langle p;f \,|\, \bar{\psi}\gamma_\alpha\gamma_5 (QC_3 - 4s^2Q^2)\psi \,|\, p;i \rangle , \tag{9b}$$

where

$$K = m_Z{}^2 \int_{M^2}^{\infty} \frac{d\kappa^2}{\kappa^2(\kappa^2 + m_Z{}^2)} \left[1 - \frac{\bar{g}_s{}^2(\kappa^2)}{4\pi^2} \right] . \tag{9c}$$

In Eq. (9c) we have included, following the discussion of Ref. 7, the lowest-order corrections induced by the strong interactions in the asymptotic domain. If the $O(\bar{\alpha}_s(\kappa^2))$ terms are neglected, then

$$K = \ln(m_Z{}^2/M^2) + O(M^2/m_Z{}^2) .$$

A method to calculate with good accuracy the integral involving $\bar{g}_s{}^2(\kappa^2)/4\pi^2$ is explained in Appendix A and we see from Table I that, for $\Lambda_{\overline{\text{MS}}} = 0.16$ GeV, it decreases K by

TABLE I. γZ box-diagram corrections as a function of M. For comparison the free-quark-model result is given in the last column.

M (GeV)	$\ln\left[\frac{m_Z^2}{M^2}\right]$	ΔK	K	$K+\frac{(\xi_1)_B^{(p)}}{(\xi_1)_{\text{asy}}^{(p)}}$	$K+\frac{(\xi_1)_B^{(n)}}{(\xi_1)_{\text{asy}}^{(n)}}$	$\ln\left[\frac{m_Z^2}{M^2}\right]+\frac{3}{2}$
0.3	11.49	−0.85	10.64	12.68	12.38	12.99
0.5	10.47	−0.65	9.82	11.86	11.56	11.97
1.0	9.08	−0.50	8.58	10.62	10.32	10.58

-7.6% if $M=0.3$ GeV and -5.7% if $M=1$ GeV. From the same table we see that K ranges from 10.6 for $M=0.3$ GeV to 8.6 for $M=1$ GeV. We will adopt $K=9.6\pm1$ as a reasonable estimate of this asymptotic contribution. We note that Eqs. (9a) and (9b) involve the matrix elements of hadronic currents constructed from the quark fields. Therefore, if the effect of strange and heavy quarks is neglected, Eqs. (9a) and (9b) may be interpreted as contributions to the parity-violating interaction between the electron and the u,d quarks. Furthermore, if corrections of $O(\bar{g}_s^2(\kappa^2)/4\pi^2)$ in Eq. (9c) are also ignored so that $K=\ln(m_Z^2/M^2)$, these contributions coincide with the terms proportional to $\ln(m_Z^2/M^2)$ in Eqs. (2a)—(2d) of Ref. 5.

B. Low-frequency part

As mentioned before, we will approximate the low-frequency contribution by the Born-approximation amplitude. We do this for a physical nucleon. The Born amplitudes for $A^{\mu\nu}$ and $V^{\mu\nu}$ are given again by Fig. 3, but this time we keep terms of $O(k^0)$ and, furthermore, we insert the appropriate weak and electromagnetic form factors at the vertices. A detailed calculation leads to

$$(\mathcal{M}_1'')_B=-\frac{G_\mu}{\sqrt{2}}16\pi\alpha(1-4s^2)(\bar{u}'_e\gamma^\alpha\gamma_5u_e)(\bar{u}'_N\gamma_\beta u_N)(g_\alpha^\beta P_\rho^\rho-P_\alpha^\beta)\,, \tag{10a}$$

$$(\mathcal{M}_2'')_B=-\frac{G_\mu}{\sqrt{2}}16\pi\alpha(\bar{u}'_e\gamma^\alpha u_e)\left[(\bar{u}'_N\gamma_\beta\gamma_5u_N)(g_\alpha^\beta Q_\rho^\rho-Q_\alpha^\beta)-\frac{p^\mu}{m_N}\epsilon_{\mu\nu\rho\alpha}\bar{u}'_N\sigma^{\lambda\nu}u_N R_\lambda^\rho\right], \tag{10b}$$

where the subscript B on the LHM reminds us that we are considering the Born amplitude and

$$P_\alpha^\beta\equiv\int\frac{d^4k}{(2\pi)^4}\frac{k^\beta k_\alpha g_1^{(N)}(k^2)G_M^{(N)}(k^2)}{(k^2+2p\cdot k)k^4}\,, \tag{10c}$$

$$Q_\alpha^\beta\equiv\int\frac{d^4k}{(2\pi)^4}\frac{k^\beta k_\alpha G_M^{(N)}(k^2)}{(k^2+2p\cdot k)k^4}[G_M^{V(N)}(k^2)-4s^2G_M^{(N)}(k^2)]\,, \tag{10d}$$

$$R_\alpha^\beta\equiv\int\frac{d^4k}{(2\pi)^4}\frac{k^\beta k_\alpha}{k^4(k^2+2p\cdot k)}\{G_M^{(N)}[G_M^{V(N)}-4s^2G_M^{(N)}]+4s^2F_1^{(N)}G_M^{(N)}-\tfrac{1}{2}[F_1^{(N)}G_M^{V(N)}+G_M^{(N)}F_1^{V(N)}]\}\,. \tag{10e}$$

In Eqs. (10c)—(10e), $G_M^{(N)}(k^2)=F_1^{(N)}(k^2)+2m_NF_2^{(N)}(k^2)$, $F_i^{(N)}(k^2)$ $(i=1,2;N=p,n)$ are the standard nucleon electromagnetic form factors, $G_M^{V(N)}(k^2)$ is the isovector contribution to $G_M^{(N)}(k^2)$ normalized so that

$$G_M^{V(p)}=-G_M^{V(n)}=G_M^{(p)}-G_M^{(n)}\,,$$

$$F_1^{V(p)}=-F_1^{V(n)}=F_1^{(p)}-F_1^{(n)}\,,$$

$$g_1^{(p)}(k^2)=-g_1^{(n)}(k^2)=g_1(k^2)\,,$$

where $g_1(k^2)$ is the usual axial-vector form factor and p^μ and m_N are the nucleon momentum and mass, respectively. For brevity we have not indicated explicitly the k^2 dependence of the form factors in Eq. (10e). Employing the dipole parametrization

$$F_i^{(N)}(k^2)=F_i^{(N)}(0)(\Lambda^2/\Lambda^2-k^2)^2$$

with $\Lambda^2=0.83m_N^2$ and approximating $g_1(k^2)=g_1(0)(\Lambda^2/\Lambda^2-k^2)^2$ with the same Λ, the integrals in Eqs. (10c)—(10e) become proportional to

$$(I_2)_\alpha^\beta\equiv\int\frac{d^4k}{(2\pi)^4}\frac{k^\beta k_\alpha}{k^4(k^2+2p\cdot k)}\left[\frac{\Lambda^2}{\Lambda^2-k^2}\right]^4 \tag{10f}$$

which is evaluated in Appendix B. Explicitly,

$$(I_2)^\beta_\alpha = \frac{i}{16\pi^2}\left[A_2(r)g^\beta_\alpha - B_2(r)\frac{p^\beta p_\alpha}{m_N{}^2}\right], \tag{10g}$$

where $r \equiv \Lambda^2/m_N{}^2$ and $A_2(r)$, $B_2(r)$ are given in Appendix B. Numerically, $A_2(0.83)=0.2436$, $B_2(0.83)=0.2006$. Inserting these expressions into Eq. (10a) and using the nonrelativistic approximation for the nucleons, we find

$$(\mathscr{M}_1'')_B = -\frac{iG_\mu}{\sqrt{2}} 3A_2(r)\frac{\alpha}{\pi}(1-4s^2)g_1^{(N)}(0)G_M^{(N)}(0)(\bar{u}_e'\gamma^\alpha\gamma_5 u_e)(\bar{u}_N'\gamma_\alpha u_N) . \tag{11a}$$

Setting

$$g_1^{(p)}(0) = -g_1^{(n)}(0) = 1.25,\quad G_M^{(p)}(0)=2.79,\quad G_M^{(n)}(0)=-1.91:$$

$$(\mathscr{M}_1'')_B = -\frac{iG_\mu}{\sqrt{2}}\frac{\alpha}{\pi}(1-4s^2)(\bar{u}_e'\gamma^\alpha\gamma_5 u_e)(\bar{u}_N'\gamma_\alpha u_N)(\xi_1)_B^{(N)} , \tag{11b}$$

where

$$(\xi_1)_B^{(p)}=2.55,\ (\xi_1)_B^{(n)}=1.74 .$$

Combining Eq. (11b) with the corresponding asymptotic contribution in Eq. (9a) we find in the case of the physical nucleon

$$(\mathscr{M}_1'')_{\rm asy} + (\mathscr{M}_1'')_B = -\frac{iG_\mu}{\sqrt{2}}\frac{\alpha}{\pi}(1-4s^2)(\bar{u}_e'\gamma^\alpha\gamma_5 u_e)(\bar{u}_N'\gamma_\alpha u_N)(\xi_1)_{\rm asy}^{(N)}\left[K + \frac{(\xi_1)_B^{(N)}}{(\xi_1)_{\rm asy}^{(N)}}\right], \tag{11c}$$

where $(\xi_1)_{\rm asy}^{(p)} = \frac{5}{4}$ and $(\xi_1)_{\rm asy}^{(n)}=1$. The contribution of Eq. (11c) to the nucleon parity-violating coupling C_{1N} is simply obtained by dropping the $-iG_\mu/\sqrt{2}$ factor. Comparison of Eq. (11c) with the results of Ref. 5 shows that in passing from the free-quark-model calculation to the more detailed analysis of the γZ box diagrams carried out in the present paper, the difference in the evaluation of $\mathscr{M}_1''$ lies in the replacement $\ln(m_Z{}^2/M^2) + \frac{3}{2} \to K + (\xi_1)_B^{(N)}/(\xi_1)_{\rm asy}^{(N)}$. These quantities are tabulated in Table I. We see that for a given value of M, the present calculation of this amplitude is very close to the free-quark-model result of Ref. 5. This is mainly due to the dominant role of the leading-logarithmic term $\ln(m_Z{}^2/M^2)$ in both calculations and the fact that, although the two approaches treat the low-frequency parts quite differently, the constant terms that emerge are not far apart. An important feature, as stressed before, is the suppression factor $(1-4s^2)$ which diminishes the effect of the uncertainties associated with this amplitude.

Performing a completely analogous analysis for $(\mathscr{M}_2'')_B$, we find

$$(\mathscr{M}_2'')_B = -\frac{iG_\mu}{\sqrt{2}}\frac{\alpha}{\pi}(\bar{u}_e'\gamma^\alpha u_e)(\bar{u}_N'\gamma_\alpha\gamma_5 u_N)\{[A_2(r)-B_2(r)]G_M^{(N)}(0)[G_M^{V(N)}(0)-4s^2G_M^{(N)}(0)]$$
$$+A_2(r)[F_1^{(N)}(0)G_M^{V(N)}(0)+G_M^{(N)}(0)F_1^{V(N)}(0)-8s^2F_1^{(N)}(0)G_M^{(N)}(0)]\} . \tag{12a}$$

Setting

$$G_M^{V(p)}(0) = -G_M^{V(n)}(0)=4.70,\quad F_1^{(p)}(0)=1,\quad F_1^{(n)}(0)=0,\quad F_1^{V(p)}(0)=-F_1^{V(n)}(0)=1$$

leads to

$$(\mathscr{M}_2'')_B = -\frac{iG_\mu}{\sqrt{2}}\frac{\alpha}{\pi}(\bar{u}_e'\gamma^\alpha u_e)(\bar{u}_N'\gamma_\alpha\gamma_5 u_N)(\xi_2)_B^{(N)} , \tag{12b}$$

where $(\xi_2)_B^{(p)}=2.389-6.776s^2=0.932$ (for $s^2=0.215$) and $(\xi_2)_B^{(n)}=0.8513-0.6275s^2=0.716$ (for $s^2=0.215$). Combination of Eqs. (9b) and (12b) leads, in the case of a physical nucleon, to

$$(\mathscr{M}_2'')_{\rm asy} + (\mathscr{M}_2'')_B = \frac{-iG_\mu}{\sqrt{2}}\frac{\alpha}{\pi}(\bar{u}_e'\gamma^\alpha u_e)(\bar{u}_N'\gamma_\alpha\gamma_5 u_N)[(\xi_2)_{\rm asy}^{(N)}K + (\xi_2)_B^{(N)}] , \tag{12c}$$

where

$$(\xi_2)_{\rm asy}^{(p)} \approx 0.935\ (\tfrac{1}{2}-\tfrac{4}{3}s^2) - 0.36\ (\tfrac{1}{4}-\tfrac{1}{3}s^2) \approx 0.135 , \tag{12d}$$

$$(\xi_2)_{\rm asy}^{(n)} \approx -0.44\ (\tfrac{1}{2}-\tfrac{4}{3}s^2) + 0.765\ (\tfrac{1}{4}-\tfrac{1}{3}s^2) \approx 0.0426 . \tag{12e}$$

As explained in Ref. 5, the numerical coefficients 0.935, 0.36, etc., in Eqs. (12d) and (12e) reflect the fact that the nucleon matrix elements of axial-vector current operators such as $\bar{u}\gamma_\alpha\gamma_5 u$ and $\bar{d}\gamma_\alpha\gamma_5 d$ involve SU(3) Clebsch-Gordan coefficients and are further affected by the contribution of the axial-current anomaly.[12] We note that in Eq. (12c) $(\xi_2)_B^{(N)}$ is considerably larger than $(\xi_2)_{\rm asy}^{(N)}$. Indeed, in this case the "low-frequency contribution" $(\xi_2)_B^{(N)}$ is of the same order of magnitude as the "leading-logarithmic term" $(\xi_2)_{\rm asy}^{(N)}K$. Thus, in the case of the $\mathscr{M}_2''$ amplitude, the present analysis leads to a result quite different from the free-quark- model calculation, in which the expression between the square brackets of Eq. (12c) is given by $(\xi_2)_{\rm asy}^{(N)}[\ln(m_Z{}^2/M^2)+\frac{5}{6}]$.[13] For practical applications, using $K\approx 9.6\pm 1$ we will adopt

$$(\xi_2)_{\rm asy}^{(p)}K+(\xi_2)_B^{(p)}\approx 2.2\pm 0.2$$

and

$$(\xi_2)_{\rm asy}^{(n)}K+(\xi_2)_B^{(n)}\approx 1.1\pm 0.1$$

as our estimates of the square brackets in Eq. (12c). As a comparison, for $M\approx 0.5$ GeV the free-quark-model calculation gives 1.5 and 0.48 for the proton and neutron cases, respectively.

In the above considerations we have combined the asymptotic contribution arising from Euclidean momenta $\kappa^2\geq M^2$ (Sec. III A) with the low-frequency part (Sec. III B), where the virtual momenta are suppressed by the factor $[\Lambda^2/(\Lambda^2+\kappa^2)]^4$ [cf. Eq. (10f)]. It is interesting to ask whether it is possible to relate the "infrared cutoff M" of the high-frequency contributions to the "ultraviolet cutoff Λ" of the low-frequency part by means of a plausible physical criterion. We first note that integration over the range $M^2\leq\kappa^2\leq\infty$ in Eq. (9c) is essentially equivalent to the substitution $\kappa^2\to\kappa^2+M^2$ in the photon propagator and integration over the full range $0\leq\kappa^2\leq\infty$. Indeed, to zeroth order in $\bar{g}_s{}^2/4\pi$, both methods of treating the asymptotic contribution differ by negligible terms of $O(\alpha M^2/m_Z{}^2)$. Now $(\kappa^2+M^2)^{-1}=(\kappa^2)^{-1}-M^2/(M^2+\kappa^2)$. Thus, our introduction of the "infrared cutoff M" is equivalent to subtracting the asymptotic integral regulated with the factor $M^2/(M^2+\kappa^2)$. A possible criterion to relate M and Λ is to demand that the slope at $\kappa^2=0$ of the factor $[\Lambda^2/(\Lambda^2+\kappa^2)]^4$, which regulates the low-frequency contribution, coincides with the corresponding slope of the factor $M^2/(M^2+\kappa^2)$, which regulates the part subtracted from the asymptotic contribution. This procedure leads to $M^{-2}=4\Lambda^{-2}$ or $M=\Lambda/2\approx 0.43$ GeV. It is a reassuring fact that this plausible criterion leads to a value for M that lies well within the range $0.3\leq M\leq 1$ GeV used in the estimates of Sec. III A.

IV. HADRONIC CONTRIBUTIONS TO γZ MIXING DIAGRAMS

At zero momentum transfer, appropriate for the present applications, the quark-model calculation of the graphs in Fig. 2 depends logarithmically on the quark masses. A method to avoid such dependence in the corrections to the tree-approximation relation $m_W\sin\theta_W=\pi\alpha/(\sqrt{2}G_\mu)$ was carried out in Ref. 8 and, independently, in Ref. 9. A similar procedure for γZ mixing diagrams was suggested in Ref. 10. In this section we briefly discuss the implementation of that idea.

The diagrams in Fig. 2 involve the tensor

$$\hat{\Pi}_{\gamma Z}^{\mu\nu}(q)=i\int d^4q\, e^{iq\cdot x}\langle 0\,|\,T^*[J_\gamma^\mu(x)J_Z^\nu(0)]\,|\,0\rangle\ . \tag{13a}$$

Since it is not possible to construct a pseudotensor out of the single four vector q^μ, only the vector part V_Z^ν of Eq. (7b) contributes. Furthermore, due to electromagnetic-current conservation, Eq. (13a) is of the form

$$\hat{\Pi}_{\gamma Z}^{\mu\nu}=\hat{\Pi}_{\gamma Z}(q^2)(q^\mu q^\nu-g^{\mu\nu}q^2)\ . \tag{13b}$$

Recalling Eq. (7b), separating out contributions which are bilinear in the $\bar{u}\gamma^\mu u$ and $\bar{d}\gamma^\mu d$ currents and using isospin invariance we have

$$\hat{\Pi}_{\gamma Z}^{\mu\nu}(q)=(\tfrac{1}{2}-s^2)\hat{\Pi}_{I=1}^{\mu\nu}(q)-\frac{s^2}{9}\hat{\Pi}_{I=0}^{\mu\nu}(q)+\hat{\Pi}'^{\mu\nu}(q)\ , \tag{13c}$$

where $\hat{\Pi}_{I=1}^{\mu\nu}$ $[\hat{\Pi}_{I=0}^{\mu\nu}]$ is the tensor obtained from Eq. (13a) by replacing both current by the isovector current $J_\mu^{I=1}=\frac{1}{2}(\bar{u}\gamma_\mu u-\bar{d}\gamma_\mu d)$ [isoscalar current $J_\mu^{I=0}=\frac{1}{2}(\bar{u}\gamma_\mu u+\bar{d}\gamma_\mu d)$]. The remaining term $\hat{\Pi}'^{\mu\nu}(q)$ involves various bilinear combinations of $J_\mu^{I=0}, \bar{s}\gamma_\mu s$, $\bar{c}\gamma_\mu c$, etc. If, following Wetzel,[9] we assume that QCD annihilation diagrams in $\hat{\Pi}'^{\mu\nu}(q)$ are unimportant:

$$\hat{\Pi}'^{\mu\nu}(q)=\frac{1}{3}\left[\frac{1}{4}-\frac{s^2}{3}\right]\hat{\Pi}_s^{\mu\nu}(q)+\frac{1}{3}\left[\frac{1}{2}-\frac{4s^2}{3}\right]\hat{\Pi}_c^{\mu\nu}(q)+\cdots\ , \tag{14a}$$

where $\hat{\Pi}_s^{\mu\nu}$ $(\hat{\Pi}_c^{\mu\nu})$ is the tensor obtained from Eq. (13a) by replacing both currents by $\bar{s}\gamma^\mu s$ $(\bar{c}\gamma^\mu c)$ and the ellipses indicate terms associated with higher mass flavors $(b,t\ldots)$. Clearly, relations analogous to Eqs. (13c) and (14a) hold under the same assumption for $\hat{\Pi}_{\gamma Z}(q^2)$. Therefore

$$\begin{aligned}\hat{\Pi}_{\gamma Z}(q^2)&=(\tfrac{1}{2}-s^2)\hat{\Pi}_{I=1}(q^2)-\frac{s^2}{9}\hat{\Pi}_{I=0}(q^2)\\&\quad+\frac{1}{3}\left[\frac{1}{4}-\frac{s^2}{3}\right]\hat{\Pi}_s(q^2)\\&\quad+\frac{1}{3}\left[\frac{1}{2}-\frac{4s^2}{3}\right]\hat{\Pi}_c(q^2)+\cdots\ .\end{aligned} \tag{14b}$$

The γZ mixing amplitudes involve the polarization functions evaluated at $q^2=0$ in some convenient renormalization scheme. Consider, for example, $\hat{\Pi}_{I=1}(0)$. To bypass the dependence on the u, d quark masses it is convenient to write

$$\hat{\Pi}_{I=1}(0)=\hat{\Pi}_{I=1}(0)-\hat{\Pi}_{I=1}(-s)+\hat{\Pi}_{I=1}(-s)\ , \tag{15a}$$

where we choose $-s$ to be a very large spacelike momentum. For the first two terms we have the dispersion relation

$$\hat{\Pi}_{I=1}(-s)-\hat{\Pi}_{I=1}(0)=\frac{-s}{12\pi^2}\int\frac{R_{I=1}(s')ds'}{s'(s'+s)}\ , \tag{15b}$$

where $R(s')$ is related to $\sigma(e^+e^-\to\text{hadrons})$ by $\sigma(s')=R(s')4\pi\alpha^2/3s'$. The next step is to split the integral from the two-pion threshold up to s_1 and from s_1 to ∞, choosing s_1 sufficiently large so that perturbation theory can be applied in the range $s'\geq s_1$ and s sufficiently large so that $s\gg s_1$. This leads to

$$\hat{\Pi}_{I=1}(-s)-\hat{\Pi}_{I=1}(0) = -\frac{1}{12\pi^2}\int_0^{s_1}\frac{R_{I=1}(s')ds'}{s'} - \frac{1}{12\pi^2}R_{I=1}(s_1)\ln\left[\frac{s}{s_1}\right] . \tag{15c}$$

Computing $\hat{\Pi}_{I=1}(-s)$ in perturbation theory in the $\overline{MS}$ scheme with $\mu=m_W$ (this corresponds to the renormalization scheme in Ref. 5):

$$\hat{\Pi}_{I=1}(-s)=\frac{-3}{4\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{sx(1-x)}{m_W{}^2}\right] . \tag{15d}$$

Combining Eqs. (15c) and (15d)

$$\hat{\Pi}_{I=1}(0)=\frac{-9}{12\pi^2}\left[\frac{1}{6}\ln\left[\frac{s}{m_W{}^2}\right]-\frac{5}{18}\right] + \frac{1}{12\pi^2}R_{I=1}(s_1)\ln(s/s_1) + \frac{1}{12\pi^2}\int_0^{s_1}\frac{R_{I=1}(s')}{s'}ds' . \tag{15e}$$

Clearly, $R_{I=1}(s_1)=\frac{3}{2}$. From the experimental data on e^+e^- annihilation, Wetzel has estimated that

$$\int_0^{s_1} ds' R_{I=1}(s')/s'=8$$

for $s_1=9$ GeV^2 with the same result for

$$\int_0^{s_1} ds' R_{I=0}(s')/s' .$$

Setting $s_1=9$ GeV^2 in Eq. (15e), using Wetzel's estimate and including the QCD corrections discussed in Appendix C, we find

$$\hat{\Pi}_{I=1}(0)=\hat{\Pi}_{I=0}(0)=0.178 . \tag{15f}$$

Note that in Eqs. (15e) and (15f) we have circumvented the dependence on m_u, m_d by using experimental $e^+e^-\rightarrow$hadrons data. Similarly, using Wetzel's estimate

$$\int_0^{s_1} ds' R^{(s)}(s')/s'=9.5$$

for $s_1=(3.5\text{ GeV})^2$ one finds by the same procedure

$$\hat{\Pi}_s(0)=0.292 . \tag{15g}$$

In the free-quark-model (FQM) calculation

$$[\hat{\Pi}_{\gamma Z}(0)]_{\rm FQM}=\frac{1}{8\pi^2}\sum_f (C_{3f}Q_f-4s^2Q_f{}^2)\ln(m_W/m_f) , \tag{16a}$$

where the sum is over flavors, $Q_f\equiv$electric charge of flavor f (in units of $|e|$), $C_{3f}\equiv$twice its weak isospin (i.e., $C_{3u}=-C_{3d}=1$) and m_f the corresponding effective mass. Comparing the contributions of the u, d, and s quarks to Eq. (16a) with Eqs. (14b), (15f), and (15g) one finds that the dispersive analysis corresponds in this case to effective masses $m_u=m_d\approx 75$ MeV and $m_s\approx 250$ MeV, or, equivalently, $\ln(m_W/m_u)=\ln(m_W/m_d)=7.0$ and $\ln(m_W/m_s)=5.8$. The important point, however, is that we can circumvent the uncertainty related to the values of the light-quark masses m_u, m_d, m_s by replacing Eq. (16a) by Eqs. (14b), (15f), and (15g). On the other hand, for the large-mass flavors one expects perturbation theory to provide a suitable basis, since the corresponding contributions $\hat{\Pi}^{(i)}(0)$ are dominated by distances considerably smaller than $(\Lambda_{\overline{\rm MS}})^{-1}$.

V. EXPERIMENTAL IMPLICATIONS

Several experimental groups have already detected or are preparing to search for parity-violating effects in atomic transitions.[1] Ongoing heavy-atom experiments with bismuth,[14] thallium,[15] and cesium[16] have observed atomic parity violation at a level consistent with the $SU(2)_L\times U(1)$ model and attempts to measure such effects in hydrogen and deuterium are currently underway.[1,17] The latter because of their single-electron structure have little uncertainty in their atomic-physics calculations. Hence, very precise measurements of parity violation in those systems may allow the observation of higher-order electroweak radiative corrections. The analysis of the model dependent parts of the γZ box diagrams and hadronic contribution to γZ mixing presented in the preceding sections when used in conjunction with our earlier work[5] provide rather precise $SU(2)_L\times U(1)$-model predictions for certain atomic parity-violating effects with little uncertainty. In this section we present the $O(\alpha)$ corrections to the electron-nucleon parity-violating interaction, provide numerical predictions, and comment on some implications of our results for the various ongoing experiments.

The electron-nucleon parity-violating (PV) Hamiltonian (at zero momentum transfer) is conventionally parametrized as follows[2,18,19]:

$$H_{\rm PV}=\frac{G_\mu}{\sqrt{2}}(C_{1N}\bar{u}_e\gamma_\mu\gamma_5u_e\bar{u}_N\gamma^\mu u_N + C_{2N}\bar{u}_e\gamma_\mu u_e\bar{u}_N\gamma^\mu\gamma_5u_N) , \tag{17}$$

$$N=p,n ,$$

where $G_\mu=(1.66\,32\pm0.000\,02)\times10^{-5}$ GeV^{-2} is the muon decay constant. The C_{iN}'s in Eq. (17) can be obtained to $O(\alpha)$ by combining our previous calculation in Ref. 5 with the new model-dependent estimates given in Sec. III for the γZ box diagrams and the phenomenological determination of the hadronic contribution to γZ mixing made in Sec. IV.

In the case of C_{1p} and C_{1n}, we find

$$C_{1p}=\tfrac{1}{2}\rho'_{\rm PV}[1-4\kappa'_{\rm PV}(0)\sin^2\hat{\theta}_W(m_W)] , \tag{18a}$$

$$C_{1n}=-\tfrac{1}{2}\rho'_{\rm PV}-\frac{\alpha}{\pi}(1-4s^2)[\tfrac{4}{5}(\xi_1)_B^p-(\xi_1)_B^n] , \tag{18b}$$

where

$$\rho'_{\rm PV}=1+\frac{\alpha}{2\pi}\left\{\frac{3}{8s^4}\ln c^2-\frac{7}{8s^2}+\frac{3}{8s^2}\frac{m_t{}^2}{m_W{}^2}+\frac{3\xi}{8s^2}\left[\frac{\ln(c^2/\xi)}{c^2-\xi}+\frac{1}{c^2}\frac{\ln\xi}{1-\xi}\right]\right.$$
$$\left.-1-\frac{1}{s^2}-4(1-4s^2)[K+\tfrac{4}{5}(\xi_1)_B^p]-\frac{9}{16s^2c^2}(1-\tfrac{16}{9}s^2)[1+(1-4s^2)^2]\right\}, \tag{19a}$$

$$\kappa'_{\rm PV}(0)=1-\frac{\alpha}{2\pi s^2}\left\{\frac{7}{9}-\frac{s^2}{3}+\tfrac{1}{6}\sum_i(C_{3i}Q_i-4s^2Q_i{}^2)\ln\frac{m_i{}^2}{m_W{}^2}+\frac{9-8s^2}{8s^2}\right.$$
$$-\frac{(1-4s^2)}{6}\left[\ln\frac{m_Z{}^2}{m_e{}^2}+\frac{1}{6}\right]+(\tfrac{9}{4}-4s^2)(1-4s^2)[K+\tfrac{4}{5}(\xi_1)_B^p]$$
$$\left.+\frac{9}{16s^2c^2}(\tfrac{1}{2}-2s^2+\tfrac{16}{9}s^4)[1+(1-4s^2)^2]\right\}. \tag{19b}$$

In the above expressions $\alpha=1/137.036$, $s^2\equiv\sin^2\hat{\theta}_W(m_W)$ (defined by $\overline{\rm MS}$), $c^2=1-s^2$, $\xi=m_\phi{}^2/m_Z{}^2$ (m_ϕ=Higgs-scalar mass), m_t=top-quark mass, Q_i=fermion electric charge, C_{3i}=twice the weak isospin, K is the short-distance contribution to the γZ box diagram, and $(\xi_1)_B^{p,n}$ denotes the long-distance γZ box-diagram contribution estimated using the Born approximation. The theoretical uncertainties in the above formulas were discussed in Secs. III and IV; they are numerically insignificant. Other (presumably temporary) uncertainties reside in the precise values of s^2, m_t, and ξ that should be employed. Fortunately, our results are not very sensitive to the probable range $0.01\le\xi\le 100$ and m_t becomes important only if $m_t>m_W$. Of course, the presence of additional fermions or Higgs scalars would modify the $O(\alpha)$ corrections; however, their effect can be easily included if required.

To obtain numerical predictions we employ the values $K=9.6\pm1$, $(\xi_1)_B^p=2.55$, and $(\xi_1)_B^n=1.74$ obtained in Sec. III and the "effective" light-quark masses $m_u=m_d\simeq75$ MeV, $m_s=250$ MeV determined in Sec. IV. In addition we use $m_c=1.5$ GeV, $m_b=4.5$ GeV, and for definiteness take $m_t=36$ GeV for the other quark masses in $\kappa'_{\rm PV}$ (the lepton masses m_e, m_μ, and m_τ are fixed). Then, employing $\xi=1$, $m_W=38.5$ GeV$/\sin\hat{\theta}_W(m_W)$, and the experimental range[20]

$$\sin^2\hat{\theta}_W(m_W)=0.215\pm0.014 \tag{20}$$

allowed by deep-inelastic neutrino scattering, we find

$$\rho'_{\rm PV}=0.9764\pm0.0057\ , \tag{21a}$$

$$\kappa'_{\rm PV}(0)=1.0055\pm0.0030\ , \tag{21b}$$

$$C_{1p}=0.066\pm0.028\ , \tag{21c}$$

$$C_{1n}=-0.4883\pm0.0030\ . \tag{21d}$$

The only large uncertainty resides in C_{1p} and that comes about primarily because of the 6.5% uncertainty in $\sin^2\hat{\theta}_W(m_W)$. Eventually, measurements of m_W, m_Z, and Z^0 decay asymmetries should determine $\sin^2\hat{\theta}_W(m_W)$ to within a few tenths of a percent; then even C_{1p} will be very precisely predicted by Eq. (18a).

The quantities C_{1p} and $C_{1d}\equiv C_{1p}+C_{1n}=-0.422\pm0.030$ may eventually be measured by experiments employing hydrogen and deuterium[17]; however, such measurements are not yet in progress. At present, only heavy-atom experiments have observed atomic parity violation. They measure a coherent effect which is proportional to the so-called weak charge Q_W where[18,19]

$$Q_W(A,Z)=2(A-Z)C_{1n}+2ZC_{1p}\ . \tag{22}$$

For the three elements so far examined, our results in Eq. (21) imply

$$Q_W({}^{209}_{83}{\rm Bi})=-112.1\pm5.3\ , \tag{23a}$$

$$Q_W({}^{205}_{81}{\rm Tl})=-110.4\pm5.2\ , \tag{23b}$$

$$Q_W({}^{203}_{81}{\rm Tl})=-108.4\pm5.2\ , \tag{23c}$$

$$Q_W({}^{133}_{55}{\rm Cs})=-68.9\pm3.5\ , \tag{23d}$$

where again most of the uncertainty comes from the allowed range in $\sin^2\hat{\theta}_W(m_W)$ [see Eq. (20)]. Eventually, these theoretical predictions will be fine tuned to have an error less than 1%; but how well can experiment do? [We note that the contribution of C_{2p} and C_{2n} to parity violation in heavy atoms is considerably less than 1% in the $SU(2)_L\times U(1)$ model because it is incoherent and in addition suppressed by a $(1\text{-}4s^2)$ factor.]

Presently completed experimental runs have found[14–16]

$$Q_W({\rm Bi})=-140\pm40\ , \tag{24a}$$

$$Q_W({\rm Tl})=-155\pm63\ , \tag{24b}$$

$$Q_W({\rm Cs})=-57.1\pm9.4\pm4.7\pm8.2\ , \tag{24c}$$

where the uncertainties are statistical, systematic, and in some cases an estimate of the atomic-physics theory uncertainty. To get some feeling as to how precise a comparison of theory and experiment can be accomplished, let us focus on cesium. The first results from cesium became available last year.[16] One would expect that the ±9.4 statistical error quoted above may be substantially reduced by much longer runs and perhaps the systematic uncertainty of ±4.7 will be lowered in the process; but what about the approximate 15% uncertainty (± 8.2) in the atomic theory? That uncertainty stems from a lack of precise knowledge regarding cesium's electronic wave function; it would have to be lowered considerably before parity violation in cesium could possibly be used to test the $SU(2)_L \times U(1)$ at the level of its radiative corrections. If a significant improvement in the wave function calculation were possible it would be well worth the undertaking. In that event, $Q_W(Cs)$ might eventually determine ρ'_{PV} with high precision. At the level of a few percent, ρ'_{PV} is sensitive not only to the higher-order corrections we have included, but also to potential new phenomena such as heavy fermions, Higgs scalars, dynamical symmetry breaking, etc. At present the cesium result implies

$$\rho'_{PV}=0.83^{+0.38}_{-0.35} \text{ (cesium)} . \tag{25}$$

(Thallium and bismuth give larger values for ρ'_{PV}, but also with big errors.)

We now go on to examine the theoretical predictions for C_{2p} and C_{2n}. Again we rely primarily on the analysis in Ref. 5 and supplement it with our new estimates for the γZ box-diagram and hadronic contributions to γZ mixing. It should be pointed out that in the case of C_{2p} and C_{2n} there are other sources of rather large uncertainty due to strong interactions which we have not tried to improve on. Those include long-distance contributions to the charge radii[21] and axial-vector-current renormalization[4] as well as QCD-induced axial-vector isoscalar neutral-current effects.[12,22]

We find up to $O(\alpha)$,

$$\begin{aligned} C_{2p}=&0.676\rho_{PV}[1-4\kappa_{PV}(0)s^2] \\ &+\frac{\alpha}{\pi}\left[-0.114(1-4s^2)+0.184\left[\ln\frac{m_W{}^2}{m^2}+\frac{1}{6}\right]-(0.144-0.330s^2)\left[\ln\frac{m_W{}^2}{m^2}+\frac{1}{6}\right]\right. \\ &\left.\quad+\frac{0.5125}{s^2}+\frac{1}{s^2c^2}(1-4s^2)(0.054-0.189s^2+0.282s^4)+(0.135K+2.389-6.776s^2)\right] , \end{aligned} \tag{26a}$$

$$\begin{aligned} C_{2n}=&-0.6025\rho_{PV}[1-4\kappa_{PV}(0)s^2] \\ &+\frac{\alpha}{\pi}\left[0.070(1-4s^2)-0.219\left[\ln\frac{m_W{}^2}{m^2}+\frac{1}{6}\right]+(0.0914-0.187s^2)\left[\ln\frac{m_Z{}^2}{m^2}+\frac{1}{6}\right]\right. \\ &\left.\quad-\frac{0.316}{s^2}+\frac{1}{s^2c^2}(1-4s^2)(0.030+0.0144s^2-0.083s^4)+(0.0426K+0.8513-0.6275s^2)\right] , \end{aligned} \tag{26b}$$

where

$$\rho_{PV}=1+\frac{\alpha}{4\pi}\left[\frac{3}{4s^4}\ln c^2-\frac{7}{4s^2}+\frac{3}{4s^2}\frac{m_t{}^2}{m_W{}^2}+\frac{3}{4}\frac{\xi}{s^2}\left[\frac{\ln(c^2/\xi)}{c^2-\xi}+\frac{1}{c^2}\frac{\ln\xi}{1-\xi}\right]\right] , \tag{26c}$$

$$\kappa_{PV}(0)=1-\frac{\alpha}{2\pi s^2}\left[\frac{7}{9}-\frac{s^2}{3}+\tfrac{1}{6}\sum_i(C_{3i}Q_i-4s^2Q_i{}^2)\ln(m_i{}^2/m_W{}^2)\right] , \tag{26d}$$

and m is a charge-radius cutoff that we take here to be ≈ 0.5 GeV. Using the parameter values employed for C_{1N} above and the range of $\sin^2\hat{\theta}_W(m_W)$ in Eq. (20), we obtain

$$\rho_{PV}=0.993\pm0.004 , \tag{27a}$$

$$\kappa_{PV}(0)=1.040\pm0.007 , \tag{27b}$$

$$C_{2p}=0.082\pm0.030 , \tag{27c}$$

$$C_{2n}=-0.068\pm0.030 , \tag{27d}$$

$$C_{2D}\equiv C_{2p}+C_{2n}=0.014^{+0.001}_{-0.002} . \tag{27e}$$

Again, the uncertainty in C_{2p} and C_{2n} is mainly due to the allowed spread in $\sin^2\hat{\theta}_W(m_W)$ and their sensitive dependence on that parameter. However, these quantities unfortunately also suffer from rather intrinsic strong-interaction uncertainties which are present at least at the 5% level. Ongoing experiments[17] with hydrogen are trying to measure C_{2p}; as yet no results are available.

The quantity C_{2D} in Eq.(27e) is a measure of the axial-vector isoscalar neutral current which is zero in lowest order for the $SU(2)_L \times U(1)$ model. The prediction given is entirely induced by higher-order electroweak and QCD corrections. In addition to the error quoted in Eq. (27e), there are QCD uncertainties at the level of $\pm 0.002 \sim 0.003$. Unlike the other parameters that we have discussed, an experimental measurement of C_{2D} would directly detect a higher-order effect. For that reason, C_{2D} seems to be the most interesting parity-violating parameter accessible to atomic-physics experiments.

VI. CONCLUSION

We have estimated the model-dependent corrections to the electron-nucleon γZ box diagrams and the hadronic contribution to γZ mixing. Taken together with our previous calculation of the short-distance radiative corrections, we now have rather precise predictions for the electron-nucleon parity-violating interaction. For the case when the nucleon amplitude is vector (rather than axial-vector) our results are very insensitive to strong interactions and thus provide the possibility of experimental probes of higher-order effects. To avoid atomic-physics complications, hydrogen and deuterium seem to be the likely candidates for atomic experiments dedicated to observing such higher-order corrections. Heavy atoms because of coherent enhancements and much larger wave function overlaps are potentially interesting, but only if theoretical uncertainties in the atomic-physics calculations can be overcome. In the case of the nucleon axial-vector amplitude, our results are somewhat clouded by strong-interaction uncertainties. In any case, the induced axial-vector isoscalar coupling arises completely from higher-order effects and it may be measurable in atomic deuterium experiments. The direct detection of this effect with the predicted sign and magnitude would represent a triumph for the standard $SU(2)_L \times U(1)$ model at the level of its quantum corrections.

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APPENDIX A

In this appendix we discuss a method for evaluating to good accuracy the contribution involving $\bar{g}_s{}^2(\kappa^2)$ in Eq. (9c). The running strong-interaction coupling constant $\hat{\alpha}_3(\kappa^2) \equiv (\kappa^2)/4\pi$ is given by[22a]

$$\hat{\alpha}_3(\kappa^2) = -\frac{2}{b_3 \ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)} - \frac{4b_{33} \ln\ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)}{b_3{}^3 \ln^2(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)} , \tag{A1}$$

where

$$b_3 = -\frac{1}{2\pi}(11 - 2N_F/3) , \tag{A2}$$

$$b_{33} = -\frac{1}{8\pi^2}(102 - 38N_F/3) , \tag{A3}$$

are the coefficients of the one- and two-loop contributions to the QCD β function and N_F is the number of quark flavors with mass $\leq \kappa$. We wish to evaluate the second term of Eq. (9c):

$$\Delta K \equiv \frac{m_Z{}^2}{\pi} \int_{M^2}^{\infty} \frac{d\kappa^2}{\kappa^2(\kappa^2 + m_Z{}^2)} \hat{\alpha}_3(\kappa^2) , \tag{A4}$$

where M is a mass of 0(1 GeV), larger than $\Lambda_{\overline{\rm MS}}$. We split the integral into two regions, $M^2 \leq \kappa^2 \leq m_t{}^2$ and $m_t{}^2 \leq \kappa^2 \leq \infty$. In the first region we approximate $(\kappa^2 + m_Z{}^2)^{-1} \to m_Z{}^{-2}$ and the integral splits into a sum of the form

$$\frac{1}{\pi} \sum_{i=1}^{N} \int_{m_i{}^2}^{m_{i+1}{}^2} \frac{d\kappa^2}{\kappa^2} \left[\frac{c_0^{(i)}}{\ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)} - \frac{c_1^{(i)} \ln\ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)}{\ln^2(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)} \right] , \tag{A5}$$

where $m_1 = M$, $m_2 =$ mass of next heavier quark, $m_{N+1} = m_t$ and $c_0^{(i)}$ and $c_1^{(i)}$ are evaluated for $N_F =$ number of flavors of mass less then m_{i+1}. Introducing $z = \ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)$, the integral over this region becomes

$$\frac{1}{\pi} \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} dz \left[\frac{c_0^{(i)}}{z} - \frac{c_1^{(i)} \ln z}{z^2} \right] = \frac{1}{\pi} \sum_{i=1}^{N} \left[c_0^{(i)} \ln \left[\frac{z_{i+1}}{z_i} \right] + c_1^{(i)} \left[\frac{\ln z_{i+1} + 1}{z_{i+1}} - \frac{\ln z_i + 1}{z_i} \right] \right] , \tag{A6}$$

where $z_i \equiv \ln(m_i{}^2/\Lambda_{\overline{\rm MS}}{}^2)$.

In the region $m_t{}^2 \leq \kappa^2 \leq \infty$ we evaluate c_0 and c_1 for the maximum number of flavors (6 in the standard model). Introducing again $z = \ln(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)$ and defining $e^{\nu} \equiv m_Z{}^2/\Lambda_{\overline{\rm MS}}{}^2 \gg 1$, the contribution from $m_t{}^2 \leq \kappa^2 \leq \infty$ is

$$\begin{aligned} \frac{1}{\pi} \int_{\ln(m_t{}^2/\Lambda_{\overline{\rm MS}}{}^2)}^{\infty} \frac{dz}{(1+e^{z-\nu})} \left[\frac{c_0}{z} - \frac{c_1 \ln z}{z^2} \right] &= \frac{1}{\pi(1 + m_t{}^2/m_Z{}^2)} \left[-c_0 \ln\ln \left[\frac{m_t{}^2}{\Lambda_{\overline{\rm MS}}{}^2} \right] - c_1 [\ln\ln \left[\frac{m_t{}^2}{\Lambda_{\overline{\rm MS}}{}^2} \right] + 1]/\ln \left[\frac{m_t{}^2}{\Lambda_{\overline{\rm MS}}{}^2} \right] \right] \\ &\quad + \frac{1}{\pi} \int_{\ln(m_t{}^2/\Lambda_{\overline{\rm MS}}{}^2)}^{\infty} \frac{e^{z-\nu} dz}{(1+e^{z-\nu})^2} [c_0 \ln z + c_1(\ln z + 1)/z] , \end{aligned} \tag{A7}$$

where we have performed a partial integration and retained terms of $O(m_t{}^2/m_Z{}^2)$. Next we note that the factor $e^{z-\nu}/(1+e^{z-\nu})^2$ is strongly peaked at $z = \nu$. The idea is then to expand the relatively slowly varying factor $c_0 \ln z + c_1(\ln z + 1)/z$ about $z = \nu$. For our purposes, it is sufficient to keep the first term in such an expansion (i.e., we set $z = \nu$ in the square brackets within the integrand) in which case the remaining integral in Eq. (A7) simply reduces to $(1/\pi)(1 + m_t{}^2/m_Z^2)^{-1}[c_0 \ln\nu + c_1(\ln\nu + 1)/\nu]$. Inserting m_c, m_b, m_t (see Sec. V) and the appropriate N_F values in Eqs. (A1)–(A7) leads to the determination of ΔK. Values of ΔK and $K = \ln(m_Z{}^2/M^2) - \Delta K$ as functions of M are given in Table I for $\Lambda_{\overline{\rm MS}} = 0.16$ GeV.

APPENDIX B

In this appendix we evaluate the integral $(I_2)_\alpha^\beta$ defined in Eq. (10f), for arbitrary values of $r \equiv \Lambda^2/m_N{}^2$. It is convenient to first study the simpler integral

$$(I_1)^{\alpha\beta} \equiv \int \frac{d^4k}{(2\pi)^4} \frac{k^\alpha k^\beta}{k^4(k^2+2p\cdot k)} \left[\frac{\Lambda^2}{\Lambda^2-k^2} \right] . \tag{B1}$$

Combining denominators, shifting the variable of integration, and performing the d^4k integration in the usual manner,

$$(I_1)^{\alpha\beta} = \frac{i\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^1 y\, dy \left[\frac{g^{\alpha\beta}}{2} \ln \left[\frac{m_N{}^2x^2y^2+\Lambda^2(1-y)}{m_N{}^2x^2y^2} \right] - \frac{p^\alpha p^\beta}{m_N{}^2} \frac{\Lambda^2(1-y)}{[m_N{}^2x^2y^2+\Lambda^2(1-y)]} \right] . \tag{B2}$$

Performing the x integration

$$(I_1)^{\alpha\beta} = \frac{i\pi^2}{(2\pi)^4} \int_0^1 dy \left[\frac{g^{\alpha\beta}}{2} y \ln \left[\frac{y^2+r(1-y)}{y^2} \right] + [r(1-y)]^{1/2} \tan^{-1} \left[\frac{y}{[r(1-y)]^{1/2}} \right] \left[g^{\alpha\beta} - \frac{p^\alpha p^\beta}{m_N{}^2} \right] \right] . \tag{B3}$$

By judicious changes of variables and partial integrations Eq. (B3) leads to

$$(I_1)^{\alpha\beta} = \frac{i}{16\pi^2} \left[A_1(r) g^{\alpha\beta} - B_1(r) \frac{p^\alpha p^\beta}{m_N{}^2} \right] , \tag{B4}$$

where

$$A_1(r) = (r-4)^2 Q_0/24 + \frac{r}{4} \left[1 - \frac{r}{6} \right] \ln r + \frac{r}{12} , \tag{B5}$$

$$B_1(r) = (r-4)(r-1) Q_0/6 + \frac{(3-r)}{6} r \ln r + \frac{r}{3} , \tag{B6}$$

$$Q_0 = \int_0^1 \frac{dx}{x^2/r+1-x} . \tag{B7}$$

Explicitly,

$$Q_0 = \begin{cases} \dfrac{2}{X} \left[\tan^{-1} \left[\dfrac{2/r-1}{X} \right] + \tan^{-1} \left[\dfrac{1}{X} \right] \right] & (r<4) \\ 2 \quad (r=4) & \\ \dfrac{1}{X} \left[\ln r + 2 \ln \left[\dfrac{1-2/r+X}{1+X} \right] \right] & (r>4) \end{cases} \tag{B8}$$

where $X=[(4/r)-1]^{1/2}$ when $r<4$ and $X=[1-(4/r)]^{1/2}$ when $r>4$.

The integral $(I_2)^{\alpha\beta}$ of Eq. (10f) is related to $(I_1)^{\alpha\beta}$ by

$$\begin{aligned} I_2^{\alpha\beta} &= -\tfrac{1}{6}(\Lambda^2)^4 \frac{\partial^3}{\partial(\Lambda^2)^3} \left[\frac{1}{\Lambda^2} I_1^{\alpha\beta} \right] \\ &= -\frac{1}{6} r^4 \frac{\partial^3}{\partial r^3} \left[\frac{1}{r} I_1^{\alpha\beta} \right] . \end{aligned} \tag{B9}$$

Expressing back the derivatives of Q_0 in terms of Q_0 and more elementary functions, Eqs. (B9), (B5), (B6), and (B7) lead to the desired results

$$(I_2)^{\alpha\beta} = \frac{i}{16\pi^2} \left[A_2(r) g^{\alpha\beta} - B_2(r) \frac{p^\alpha p^\beta}{m_N{}^2} \right] , \tag{B10}$$

$$A_2(r) = \frac{1}{(r-4)} \left[-\frac{r}{12} + \frac{Q_0}{6} \left[\frac{3r}{2} - 5 \right] \right] , \tag{B11}$$

$$B_2(r) = \frac{1}{(r-4)^2} \left[r \left[\frac{r}{2} - 1 \right] + \frac{2Q_0}{3} (5-2r) \right] . \tag{B12}$$

It is useful to verify the asymptotic behavior for large r

$$Q_0 \to \left[1 + \frac{2}{r} + \frac{6}{r^2} \right] \ln r - \frac{1}{r} \left[2 + \frac{7}{r} \right] + O \left[\frac{1}{r^3} \ln r \right] , \tag{B13}$$

which when substituted in Eqs. (B5), (B6), (B11), and (B12) leads to

$$A_1(r) \to \tfrac{1}{4}(\ln r + \tfrac{3}{2}), \quad (r \gg 1) \tag{B14}$$

$$B_1(r) \to \tfrac{1}{2}, \quad (r \gg 1) \tag{B15}$$

$$A_2(r) \to \tfrac{1}{4}(\ln r + \tfrac{3}{2} - 1 - \tfrac{1}{2} - \tfrac{1}{3}), \quad (r \gg 1) \tag{B16}$$

$$B_2(r) \to \tfrac{1}{2}, \quad (r \gg 1) . \tag{B17}$$

In the usual treatment of the Feynman integrals in which $\Lambda^2 \gg m_N{}^2$, the asymptotic behaviors of A_1, B_1 are obtained directly from Eq. (B2) and those for A_2 and B_2 follow by applying Eq. (B9). Thus Eqs. (B14)–(B17), obtained here from the exact expressions of Eqs. (B5), (B6), (B11), and (B12) constitute a welcome check. Using the exact expressions, we find for $r=0.83$ corresponding to the dipole fit to the electromagnetic form factors: $Q_0(0.83)=1.1235$, $A_2(0.83)=0.243\,63$, $B_2(0.83)=0.200\,63$ and, for comparison, $A_1(0.83)=0.506\,27$, $B_1(0.83)=0.321\,64$. Analogous methods can be used to evaluate the traditional Feynman integrals

$$J_{1;2;3} = 8i \int \frac{[1;k_\sigma;k_\sigma k_\tau]}{(k^2+2p_1\cdot k)(k^2+2p_2\cdot k)k^2} \times \left[\frac{\Lambda^2}{\Lambda^2-k^2} \right] \frac{d^4k}{(2\pi)^2} \tag{B18}$$

when the cutoff Λ is not large with respect to the masses.[23]

APPENDIX C

For simplicity, in the various formulas of Sec. IV, perturbative corrections induced by strong interactions in the functions $\hat{\Pi}(-s)$ and in the contributions to Eq. (15b) arising from the asymptotic region $s' \geq s_1$, were not explicitly exhibited. In this appendix we briefly explain how the leading effects due to these corrections can be incorporated. Consider, for example, $\hat{\Pi}_{I=1}(-s) - \hat{\Pi}_{I=1}(0)$. As in Sec. IV we split the integral [see Eq. (15b)] for this quantity into two intervals: $0 \leq s' \leq s_1$ and $s_1 \leq s' \leq \infty$. In the latter we write $R_{I=1}(s') = \frac{3}{2}[1+\hat{\alpha}_3(s')/\pi]$ where the $\frac{3}{2}$ represents the free-field-theory result (see Sec. IV) and $1+\hat{\alpha}_3(s')/\pi$ is the QCD correction factor induced by strong interactions for $\sigma(e^+e^- \to \text{hadrons})$. Neglecting terms of $O(s_1/s)$ we find

$$\hat{\Pi}_{I=1}(-s) - \hat{\Pi}_{I=1}(0) = -\frac{1}{12\pi^2}\int_0^{s_1} \frac{R_{I=1}(s')ds'}{s'} - \frac{1}{8\pi^2}\ln\left[\frac{s}{s_1}\right] - \frac{1}{8\pi^2} Z(s,s_1) \;, \tag{C1}$$

where

$$Z(s,s_1) = \frac{s}{\pi}\int_{s_1}^{\infty} \frac{\hat{\alpha}_3(s')ds'}{s'(s'+s)} \;. \tag{C2}$$

Comparison with Eq. (A4) shows that the integral in Eq. (C2) can be obtained from ΔK by substituting $m_Z{}^2 \to s$, $M^2 \to s_1$. Thus $Z(s,s_1)$ can be evaluated with the methods of Appendix A. Retaining only the leading-logarithmic contributions arising from the first term in Eq. (A1) and neglecting terms of $O(m_t{}^2/s)$ we find for $m_c{}^2 \leq s_1 \leq m_b{}^2$:

$$\begin{aligned} Z(s,s_1) = {} & \frac{12}{25}\left[\ln\ln\left[\frac{m_b{}^2}{\Lambda_{\overline{\rm MS}}{}^2}\right] - \ln\ln\left[\frac{s_1}{\Lambda_{\overline{\rm MS}}{}^2}\right]\right] \\ & + \frac{12}{23}\left[\ln\ln\left[\frac{m_t{}^2}{\Lambda_{\overline{\rm MS}}{}^2}\right] - \ln\ln\left[\frac{m_b{}^2}{\Lambda_{\overline{\rm MS}}^2}\right]\right] \\ & + \frac{12}{21}\left[\ln\ln\left[\frac{s}{\Lambda_{\overline{\rm MS}}{}^2}\right] - \ln\ln\left[\frac{m_t{}^2}{\Lambda_{\overline{\rm MS}}{}^2}\right]\right] \\ & + \cdots \;, \end{aligned} \tag{C3}$$

where the $\cdots$ represent nonleading contributions. While the leading terms given in Eq. (C3) behave as ln ln functions of large ratios, it is easy to see that the nonleading contributions behave as ln ln/ln functions of the same ratios. It is also interesting to note that, for fixed s, s_1, and $m_i{}^2 (i=b,t\cdots)$, $Z(s,s_1) \to 0$ as $\Lambda_{\overline{\rm MS}} \to 0$. In that limit all the relevant mass scales become infinitely large in comparison with the QCD mass scale and the perturbative corrections induced by strong interactions vanish. Using Eqs. (C1) and (C3) we incorporate the leading perturbative corrections induced by strong interactions in Eq. (15c) for large s. On the other hand, the long-distance effects are included in the integral on the RHM of Eq. (15c) and that contribution does not require theoretical corrections since it is extracted from experiment. In order to obtain $\hat{\Pi}_{I=1}(0)$ we must subtract $\hat{\Pi}_{I=1}(-s)$. To zeroth order in $\hat{\alpha}_3$, that function is given in Eq. (15d). Under the assumption that $m_i{}^2/m_W{}^2 \ll 1$ the leading corrections to $\hat{\Pi}_{I=1}(-s)$ can be obtained from the following considerations. (i) Because the function is evaluated at the nonexceptional momentum $-s$, it does not involve mass singularities and the various quark masses may be set to zero. In that limit $\hat{\Pi}_{I=1}(-s)$ depends on three mass scales: $\sqrt{s}$, the 't Hooft mass unit μ which we have set equal to m_W, and $\Lambda_{\overline{\rm MS}}$. (ii) The s dependence of $\hat{\Pi}(-s)$ must cancel that of $Z(s,s_1)$ for all values of s. As a consequence, for large s the leading corrections to $\hat{\Pi}(-s)$ must be of the form

$$-\frac{12}{21}\left[\frac{1}{8\pi^2}\right][\ln\ln(s/\Lambda_{\overline{\rm MS}}{}^2) - f(m_W{}^2/\Lambda_{\overline{\rm MS}}{}^2)] \;.$$

(iii) For fixed s, $m_W{}^2$ the corrections cancel as $\Lambda_{\overline{\rm MS}}{}^2 \to 0$. Therefore, $f(m_W{}^2/\Lambda_{\overline{\rm MS}}{}^2) = \ln\ln(m_W{}^2/\Lambda_{\overline{\rm MS}}{}^2)$ plus terms that vanish as $m_W{}^2/\Lambda_{\overline{\rm MS}}{}^2 \to \infty$, i.e., plus nonleading functions of large ratios. Combining these observations with Eqs. (15d) and (C1) we obtain

$$\hat{\Pi}_{I=1}(0) = \frac{1}{8\pi^2}\left[\frac{2}{3}\int_0^{s_1}\frac{R_{I=1}(s')ds'}{s'} + \ln\left[\frac{m_W{}^2}{s_1}\right] + \frac{5}{3} + Z(m_W{}^2,s_1)\right] + \cdots \;. \tag{C4}$$

Inserting Wetzel's evaluation of the integral for $s_1 = 9$ GeV^2 (see Sec. IV), the appropriate mass values and $\Lambda_{\overline{\rm MS}} = 0.16$ GeV we find that $Z(m_W{}^2,s_1)$ induces a small correction of $+2.9\%$. The analogous analysis for $\hat{\Pi}_s(0)$ shows that the corresponding correction is $+3.3\%$. To evaluate the radiative corrections all one needs are the functions $\hat{\Pi}_{I=1}(0)$, $\hat{\Pi}_{I=0}(0)$, $\hat{\Pi}_s(0)$, $\cdots$ or, equivalently, the effective values of $\ln(m_W/m_f)$ in Eq. (16a). As we have seen, the effect of $Z(m_W{}^2,s_1)$ on these quantities is quite small. On the other hand, the determination of "effective" light-quark masses is more sensitive to $Z(m_W{}^2,s_1)$ on account of the exponential dependence.

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*Permanent address: Department of Physics, New York University, New York, NY 10003.

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$$\frac{G_\mu}{\sqrt{2}}\left[\frac{i}{2m_N}C_{3N}q_\nu\bar{u}_e\gamma_\mu\gamma_5u_e\bar{u}_N\sigma^{\mu\nu}u_N + \frac{i}{2m_e}C_{4N}q_\nu\bar{u}_e\sigma^{\mu\nu}u_e\bar{u}_N\gamma_\mu\gamma_5u_N\right],$$

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[22a]*Note added in proof.* If a single value of $\Lambda_{\overline{\rm MS}}$ is employed for all N_F, terms of the form $C/\ln^2(\kappa^2/\Lambda_{\overline{\rm MS}}{}^2)$, judiciously chosen to ensure the continuity of $\hat{\alpha}_3(\kappa^2)$, should be added to Eq. (A1) for $N_F \neq 4$. We have checked that the inclusion of such terms is not numerically significant for the analysis in this paper.
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