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Dynamical gauge-symmetry breaking and left-right asymmetry in higher-dimensional theories

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We examine dynamical gauge-symmetry breaking and left-right asymmetry in higher-dimensional theories by taking non-Abelian gauge theory on a manifold $M^2\times S^2$, where M^2 and S^2 are a two-dimensional Minkowski space and a two-sphere, respectively. It is shown that a shift in fermion zero-point energies due to the compactness of the extra-dimensional space S^2 induces dynamical gauge-symmetry breaking, provided that there exist many heavy fermions. With additional Weyl fermions incorporated in $M^2\times S^2$ we obtain left-right—asymmetric massless fermions in M^2. The effective Lagrangian in M^2 is given. A relationship between four-dimensional and two-dimensional anomalies is also established.

I. INTRODUCTION

In this paper we examine the problems of dynamical gauge-symmetry breaking and left-right asymmetry in higher-dimensional theories by taking a gauge theory on a manifold $M^2\times S^2$, where M^2 and S^2 are a two-dimensional Minkowski space and a two-sphere, respectively.

Generally speaking, higher-dimensional theories unify particles with different spins in lower dimensions in one multiplet, very similar to supersymmetric theories in four dimensions. In the original Kaluza-Klein approach[1–6] one starts from higher-dimensional Riemannian geometry with the Einstein-Hilbert action to unify gravitation (spin 2), gauge fields (spin 1) associated with isometry of the extra-dimensional space, and scalar fields (spin 0) corresponding to deformation of the extra-dimensional space. It explains the origin of gauge invariance to yield a relationship between the Newtonian constant of gravitation and the gauge coupling constant. On the other hand, gauge theory in higher dimensions[7–9] unifies gauge fields (spin 1) and scalar fields (spin 0). Although the origin of gauge invariance is left unexplained, it has been shown recently that a class of theories in this category exhibit dynamical gauge-symmetry breaking by quantum corrections,[9] which could replace the Higgs mechanism in the standard unified theory of electroweak and strong interactions. Finally, higher-dimensional supergravity theories[10–12] unify more. With supersymmetry incorporated they unify particles with spin 2, $\frac{3}{2}$, 1, $\frac{1}{2}$, and 0. In particular, in 11-dimensional supergravity, in short, geometry determines everything.

Higher-dimensional theories are efficient and attractive due to their nature as unified theories, containing more symmetry and fewer arbitrary parameters. An unsatisfactory point is that so far none of them are realistic. There are many problems to be solved to construct a realistic theory. We briefly discuss them below, simultaneously to explain the necessity of introducing at least some of the gauge fields as external matter fields (sources to the energy-momentum tensor $T_{\mu\nu}$) rather than as a part of a metric $g_{\mu\nu}$.

(a) The Einstein equation for gravity must admit a solution with extra dimensions being compactified. A ground state must be a product of four-dimensional Minkowski space (M^4) and a compact extra-dimensional space with tiny size.[6,13]

(b) In the reduced lower dimensions, namely in M^4, we need almost massless fermions.

(c) The theory must admit left-right asymmetry in M^4.

(d) We need SU(3)×SU(2)×U(1) gauge symmetry at Weinberg-Salam energies (∼300 GeV).

All these requirements are apparently trivial, but indeed appear as severe problems in constructing realistic higher-dimensional theories.

Problem (a) implies that we need matter fields giving rise to nonvanishing $T_{\mu\nu}$, unless extra dimensions are flat. Candelas and Weinberg[6] have discussed that quantum corrections due to quark and lepton loops are responsible for the compactification. We take a viewpoint that external gauge fields also are responsible for that.

Problems (b) and (c) are more serious. In general, massless fermions in higher dimensions do not yield massless fermions in lower dimensions. For a spin-$\frac{1}{2}$ spinor on a compact manifold with no other matter fields present, there is a simple mathematical theorem[14] that if the scalar curvature R is positive definite everywhere, the associated Dirac operator has no zero-eigenvalue mode. It means that if an extra-dimensional space is a positively curved compact space there are no massless fermions at low energies. All fermions have masses of $O(M)$, where M is a typical energy scale (∼10^{17} GeV) characterizing size of an extra-dimensional space.

It is very difficult to get left-right asymmetry, if one starts from a system consisting of gravity and spinors only. In even dimensions Weyl spinors can be introduced. But a higher-dimensional spinor of positive (or negative) chirality always contains lower-dimensional spinors of both positive and negative chirality so that one usually ends up with left-right—symmetric theories in lower dimensions.

Problems (b) and (c) are interrelated to each other, and can be avoided if there exist external gauge fields in a theory. The main purpose of this paper is to show that dynamical gauge-symmetry breaking induced by quantum effects leads to left-right—asymmetric massless-fermion content in lower dimensions, thus solving the problems (b) and (c).

In a previous paper[9] we showed that a gauge theory with fermions on $M^n \times S^2$ (M^n=n-dimensional Minkowski space) exhibits dynamical gauge-symmetry breaking by quantum effects for $n=4p+3$ ($p=0,1,2,\ldots$). The analysis was limited to odd n because of divergences encountered. To handle the fermion problems discussed above, we have to consider even-dimensional theories. In four dimensions renormalization is well defined. For this reason we investigate a gauge theory on $M^2 \times S^2$ as a toy model. It is a four-dimensional theory, but reduces at low energies to a two-dimensional gauge theory. We will see how gauge-symmetry breaking is induced by quantum corrections due to the compactness of the extra-dimensional space S^2, and how it leads to left-right-asymmetric massless fermions in M^2. As a by-product we establish a relationship between anomalies in four and two dimensions, analogous to the 't Hooft condition[15] relating anomalies in preons and composite particles.

We summarize the results in Ref. 9 in Sec. II. Renormalization is carried out in $M^2 \times S^2$ in Sec. III to see that gauge symmetry is dynamically broken under some conditions. In the following sections we write the effective Lagrangian in M^2, analyze anomaly equations in both $M^2 \times S^2$ and M^2, and show that left-right asymmetry really arises at low energies. The final section is devoted to summary and discussions.

II. ZERO-POINT ENERGIES

As was shown in a previous paper[9] gauge symmetry can be dynamically broken on a manifold $M^n \times S^2$ by fermion one-loop corrections. It is a phenomenon caused by a shift in fermion zero-point energies due to the compactness of the extra-dimensional space S^2, which is very similar to the Casimir effect[16] in electrodynamics. We summarize the results of a previous paper in this section to apply them to the case $n=2$ in later sections.

We first consider SU(2) gauge theory ($A_\mu = \vec{A}_\mu \cdot \vec{\tau}/2$) with doublet fermions (Ψ) on $M^n \times S^2$. Gravity is neglected. We look for a gauge field configuration minimizing the effective potential $V_{\rm eff}[A]$. We denote coordinates of M^n and S^2 by x_m and polar coordinates (θ,ϕ), respectively. Accordingly A_μ splits into A_m and (A_θ, A_ϕ). (A_θ, A_ϕ) play the role of effective Higgs fields in the adjoint representation in lower dimensions M^n. Unlike the standard unified theory of strong and electroweak interactions there are no arbitrary parameters associated with these effective Higgs fields, all coupling constants being uniquely fixed by gauge invariance.

The effective potential is evaluated for two typical configurations, a pure gauge configuration ($A_\mu=0$) and a monopole configuration ($A_\mu = A_\mu^{\rm mon}$). The latter[17] is given by

$$\begin{aligned} &A_m^{\rm mon}(x,\theta,\phi)=0 \,, \\ &A_\theta^{\rm mon}(x,\theta,\phi)=-\frac{1}{2gr}\vec{e}_\phi\cdot\vec{\tau} \,, \\ &A_\phi^{\rm mon}(x,\theta,\phi)=+\frac{1}{2gr}\vec{e}_\theta\cdot\vec{\tau} \,, \end{aligned} \tag{2.1}$$

which yields nonvanishing field strengths

$$F_{\theta\phi}^{\rm mon}(x,\theta,\phi)=-\frac{1}{2gr^2}\vec{e}_r\cdot\vec{\tau} \,. \tag{2.2}$$

Here g, r, and $(\vec{e}_r,\vec{e}_\theta,\vec{e}_\phi)$ are the SU(2) gauge coupling constant, radius of S^2, and unit vectors on S^2 in r, θ, and ϕ directions. $A_\mu^{\rm mon}$ solves the equation of motion, though classically unstable against small fluctuations.[18] We will see that the configuration $A_\mu^{\rm mon}$ can be stabilized by quantum effects.

It has also been argued in Ref. 9 that $A_\mu=0$ and $A_\mu=A_\mu^{\rm mon}$ represent two extremes of rotationally symmetric configurations on S^2, justifying to particularly pick up the two configurations.

The effective potential is evaluated by first integrating fermion fields Ψ. This amounts to evaluating eigenvalues of a Dirac operator $D(A)$, since fermion contributions to the effective potential are summarized by

$$\begin{aligned} V_{\rm eff}[A]^F &= i\,{\rm Tr}\ln[D(A)-m] \\ &= \frac{i}{2}{\rm Tr}\ln[-D(A)^2+m^2] \,. \end{aligned} \tag{2.3}$$

For the two special configurations,

$$-D(A)^2=[\partial^2]_n+D_{S^2}(A)^2 \,,$$

where $[\partial^2]_n$ is the d'Alembertian operator in M^n and $D_{S^2}(A)$ is the Dirac operator on S^2 given by

$$\begin{aligned} D_{S^2}^{(0)}(A)=\sigma_1&\left[\frac{1}{r}\left[\frac{\partial}{\partial\theta}+\tfrac{1}{2}\cot\theta\right]-igA_\theta\right] \\ &+\sigma_2\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}-igA_\phi\right] \,, \end{aligned} \tag{2.4}$$

or in a rotationally symmetric representation by

$$\begin{aligned} D_{S^2}(A)&=\Omega D_{S^2}^{(0)}(A)\Omega^\dagger \\ &=\vec{e}_\theta\cdot\vec{\sigma}\left[\frac{1}{r}\frac{\partial}{\partial\theta}-igA_\theta\right] \\ &\quad+\vec{e}_\phi\cdot\vec{\sigma}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}-igA_\phi\right]-\frac{1}{r}\vec{e}_r\cdot\vec{\sigma} \,, \\ \Omega&=\exp\left[-\frac{i}{2}\phi\sigma_3\right]\exp\left[-\frac{i}{2}\theta\sigma_2\right] . \end{aligned} \tag{2.5}$$

Eigenvalues of $D_{S^2}(A)$ are easily found, since

$$\begin{aligned} &D_{S^2}(A=0)^2=\frac{1}{r^2}[(\vec{l}+\tfrac{1}{2}\vec{\sigma})^2+\tfrac{1}{4}] \,, \\ &D_{S^2}(A^{\rm mon})^2=\frac{1}{r^2}(\vec{l}+\tfrac{1}{2}\vec{\sigma}+\tfrac{1}{2}\vec{\tau})^2 \,. \end{aligned} \tag{2.6}$$

Here $\vec{l}$ is the orbital angular momentum on S^2 expressed in terms of θ and ϕ. $D_{S^2}(A=0)^2$ has eigenvalues $(j+\frac{1}{2})^2/r^2$ $(j=\frac{1}{2},\frac{3}{2},\ldots)$ with multiplicity $4(2j+1)$, whereas $D_{S^2}(A^{\rm mon})^2$ has eigenvalues $j(j+1)/r^2$ $(j=0,1,2,\ldots)$ with multiplicity 2 for $j=0$ and $4(2j+1)$ otherwise.

To evaluate $V_{\rm eff}[A]$ we employ the dimensional regularization of 't Hooft and Veltman,[19] to find for a Dirac fermion

$$V_2^F=V_{\rm eff}[A=0;M^n\times S^2]^F$$
$$=-2^{[n/2]}\int\frac{d^np}{(2\pi)^n}\frac{1}{4\pi r^2}\sum_{j=1}^{\infty}4j\ln\left[p^2+\frac{j^2}{r^2}+m^2\right]$$
$$=\frac{2^{[n/2]-n-2}}{\pi^{n/2+1}}\Gamma\left[-\frac{n}{2}\right]\frac{1}{r^{n+2}}\sum_{j=1}^{\infty}4j(j^2+b^2)^{n/2}\,, \qquad (2.7)$$

$$V_3^F=V_{\rm eff}[A^{\rm mon};M^n\times S^2]^F$$
$$=\frac{2^{[n/2]-n-2}}{\pi^{n/2+1}}\Gamma\left[-\frac{n}{2}\right]\frac{1}{r^{n+2}}\left[b^n+\sum_{j=1}^{\infty}2(2j+1)[j(j+1)+b^2]^{n/2}\right]\,, \qquad (2.8)$$

where $b=mr$. In the flat-space M^{n+2} fermion contributions to $V_{\rm eff}$ are given by

$$V_1^F=V_{\rm eff}[A=0;M^{n+2}]$$
$$=-\frac{2^{[n/2]-n}}{\pi^{n/2+1}}\Gamma\left[-\frac{n}{2}\right]\frac{m^{n+2}}{n+2}\,. \qquad (2.9)$$

The sums over j in Eqs. (2.7) and (2.8) have to be first done for $n<-2$ and to be defined for positive n by analytic continuation. Detailed calculations are given in the Appendix. The results are

$$V_a^F=-\frac{2^{[n/2]-n}}{\pi^{n/2+1}}\Gamma\left[-\frac{n}{2}\right]\frac{1}{r^{n+2}}f_a(n,b)\quad(a=2,3)\,, \qquad (2.10)$$
$$f_2(n,b)=\frac{b^{n+2}}{n+2}+2\int_0^b dx\frac{x(b^2-x^2)^{n/2}}{e^{2\pi x}-1}+2\cos\frac{n\pi}{2}\int_b^{\infty}dx\frac{x(x^2-b^2)^{n/2}}{e^{2\pi x}-1}\,,$$
$$f_3(n,b)=\frac{b^{n+2}}{n+2}+\int_0^{\infty}dx\frac{R^{n/2}}{e^{2\pi x}-1}\left[\sin\frac{nw}{2}+2x\cos\frac{nw}{2}\right]\,,$$
$$R=[(x^2-b^2)^2+x^2]^{1/2},\quad w=\tan^{-1}\frac{x}{b^2-x^2}\quad(0\le w\le\pi)\,.$$

For even n, $f_a(n,b)$ can be evaluated in a closed form:

$$f_2(2,b)=\tfrac{1}{4}b^4+\tfrac{1}{12}b^2-\tfrac{1}{120},\quad f_3(2,b)=\tfrac{1}{4}b^4+\tfrac{1}{12}b^2+\tfrac{1}{30}\,,$$
$$f_2(4,b)=\tfrac{1}{6}b^6+\tfrac{1}{12}b^4-\tfrac{1}{60}b^2+\tfrac{1}{252},\quad f_3(4,b)=\tfrac{1}{6}b^6+\tfrac{1}{12}b^4+\tfrac{1}{15}b^2-\tfrac{4}{315},\ \text{etc}\,. \qquad (2.11)$$

In our approximation the difference between total effective potentials for $A_\mu=0$ and $A_\mu=A_\mu^{\rm mon}$ is

$$\Delta V_{\rm eff}^{\rm tot}=V_{\rm eff}^{\rm tot}[A^{\rm mon};M^n\times S^2]-V_{\rm eff}^{\rm tot}[A=0;M^n\times S^2]$$
$$={\rm Tr}(F_{\theta\phi}^{\rm mon})^2+\sum_i[V_3^F(m_i)-V_2^F(m_i)]$$
$$=\frac{1}{2g^2r^4}-\frac{2^{[n/2]-n}}{\pi^{n/2+1}}\Gamma\left[-\frac{n}{2}\right]\frac{1}{r^{n+2}}\sum_i[f_3(n,m_ir)-f_2(n,m_ir)]\,. \qquad (2.12)$$

For odd n the correction term in (2.12) is finite. Since

$$f_3(n,mr)-f_2(n,mr)\sim\frac{n}{48}(mr)^{n-2}>0$$

for $m\gg r^{-1}$, we conclude that gauge symmetry is dynamically broken ($\Delta V_{\rm eff}^{\rm tot}<0$) for $n=4p+3$ $(p=0,1,2,\ldots)$, provided that there exists a very heavy fermion. For even n the correction term in (2.12) diverges and has to be properly renormalized. We discuss this problem in the next section for $n=2$.

So far we have calculated only fermion-loop corrections. If one calculates gauge-boson-loop contributions, one would find an imaginary part, since A^{mon} is a saddle point of the action. This imaginary part, however, is a fictitious one and should have vanished if dressed gauge-boson propagators with fermion loops were used in the calculations. This implies that the loop expansion is not very good.

III. RENORMALIZATION

To extract physically meaningful finite results, the effective potential must be renormalized in even dimensions. It is not very clear if this can be done consistently for general n and m even at the one-loop level. In four dimensions, namely for $n=2$, renormalization of gauge theory on a curved manifold is well defined. In our approximation in which only fermion one-loop corrections are evaluated, renormalization of g, A_μ, Λ (cosmological constant), and coupling constants associated with R, R^2, ${R_{\mu\nu}}^2$, and ${R_{\mu\nu\rho\sigma}}^2$ is enough to render the theory finite.[20] (Here R, $R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma}$ are scalar curvature, Ricci tensor, and Riemann tensor, respectively.) Arguments simplify for the difference $\Delta V_{\mathrm{eff}}^{\mathrm{tot}}$ in (2.12), since divergences associated with gravity ($\Lambda, R, R^2, \ldots$) cancel.

Divergent parts in Eqs. (2.9) and (2.10) are given by

$$\begin{aligned} V_1^{F(\mathrm{div})} &= \frac{1}{4\pi^2} m^4 \frac{1}{\epsilon} , \\ V_2^{F(\mathrm{div})} &= \left[\frac{1}{4\pi^2} m^4 + \frac{1}{12\pi^2}\frac{m^2}{r^2} - \frac{1}{120\pi^2}\frac{1}{r^4} \right] \frac{1}{\epsilon} , \\ V_3^{F(\mathrm{div})} &= \left[\frac{1}{4\pi^2} m^4 + \frac{1}{12\pi^2}\frac{m^2}{r^2} + \frac{1}{30\pi^2}\frac{1}{r^4} \right] \frac{1}{\epsilon} , \end{aligned} \tag{3.1}$$

where $\epsilon = 2-n$. In particular

$$\Delta V^{F(\mathrm{div})} = V_3^{F(\mathrm{div})} - V_2^{F(\mathrm{div})} = \frac{1}{24\pi^2}\frac{1}{r^4}\frac{1}{\epsilon} . \tag{3.2}$$

To carry out renormalization with the monopole background field it is most convenient to take the background-field gauge. We introduce a dimensionless coupling constant by substituting g in the preceding formulas by $g\mu^{\epsilon/2}$, where μ is a scale parameter. In the background gauge $Z_g {Z_3}^{1/2} = 1$,[20,21] where $A_\mu^{(0)} = {Z_3}^{1/2} A_\mu$ and $g^{(0)} = Z_g g \mu^{\epsilon/2}$. If only fermion one-loop corrections are taken into account,

$$\begin{aligned} Z_3 &= 1 - N_f \frac{g^2}{6\pi^2} T_f \frac{1}{\epsilon} , \\ Z_g &= 1 + N_f \frac{g^2}{12\pi^2} T_f \frac{1}{\epsilon} , \end{aligned} \tag{3.3}$$

for SU(N) gauge theory, where $T_f = \frac{1}{2}$ for fermions in the fundamental representation and N_f is the number of fermion species. In this gauge

$$\tfrac{1}{2}\,\mathrm{Tr} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} = Z_3 \tfrac{1}{2}\,\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} . \tag{3.4}$$

Note that the renormalization constants in (3.3) are determined in the flat spacetime M^4.

Now we are ready to show that the renormalization (3.3) removes the divergence (3.2):

$$\begin{aligned} \Delta V_{\mathrm{eff}}^{(\mathrm{ren})} &= Z_3 \mathrm{Tr}(F_{\theta\phi}^{\mathrm{mon}})^2 + \Delta V^F \\ &= \frac{1}{2g^2\mu^\epsilon r^4}\left[1 - \frac{N_f g^2}{12\pi^2}\frac{1}{\epsilon} \right] + \Delta V^{F(\mathrm{div})} + \Delta V^{F(\mathrm{finite})} \\ &= \frac{1}{2g^2 r^4} + \frac{N_f}{24\pi^2 r^4}\left[\ln 2\sqrt{\pi}\mu r + \frac{1-\gamma}{2} \right] - \frac{1}{\pi^2 r^4}\sum_i f'(2, m_i r) + O(\epsilon) , \end{aligned} \tag{3.5}$$

where $f(n,b) = f_3(n,b) - f_2(n,b)$. By choosing a scale $\mu = e^{(\gamma-1)/2}/2\sqrt{\pi} r = 0.23/r$ we have

$$\Delta V_{\mathrm{eff}}^{(\mathrm{ren})} = \frac{1}{2g^2 r^4} - \frac{1}{\pi^2 r^4}\sum f'(2, m_i r) . \tag{3.6}$$

Here the coupling constant g should be defined at the energy scale μ. In passing,

$$\frac{1}{g^2(\mu)} + \frac{N_f}{12\pi^2}\ln\mu$$

is a scale-invariant quantity in our approximation. $f'(2,b)$ is given by

$$f'(2,b) = \int_0^\infty dx \frac{1}{e^{2\pi x}-1}\left\{ \tfrac{1}{2}x[1+2(b^2-x^2)]\ln R + \tfrac{1}{2}(b^2-3x^2)w - x(b^2-x^2)\ln|b^2-x^2| \right\} . \tag{3.7}$$

For large b

$$f'(2,b) \sim \tfrac{1}{24}\ln b + \tfrac{1}{48} + O\left[\frac{1}{b^2}\right] . \tag{3.8}$$

Numerically, $f' = -7.8\times10^{-2}$, 0, 1.5×10^{-2}, 1.2×10^{-1}, and 1.9×10^{-1} for $b = mr = 0$, 0.75, 1, 10, and 100, respectively. Unlike higher-dimensional theory ($n>2$), the dependence of f' on m is very weak. To have dynamical gauge-symmetry breaking in four dimensions, namely, to get $\Delta V_{\rm eff}^{(\rm ren)}<0$, we need a large number of heavy fermions. For instance, if $g^2/4\pi\sim\frac{1}{10}$ and $mr\sim 10$, we need ~ 30 fermions. Consequences of dynamical gauge-symmetry breaking are examined in the following sections.

In actual fact all divergences in (3.1) can be removed to define a renormalized effective potential for each case. Coefficients in counterterms in gauge theory on a curved spacetime have been determined in Ref. 20. By using the results there and noting that $R=2/r^2$, $R_{\mu\nu}R^{\mu\nu}=2/r^4$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=4/r^4$ for $M^2\times S^2$, one can confirm that the divergences (3.1) are precisely canceled by the counterterms.

IV. EFFECTIVE LOW-ENERGY THEORY

Let us assume that there exist both massless and massive fermions in $M^2\times S^2$, say, one massless fermion and a large number of heavy fermions ($m>r^{-1}$) so that gauge symmetry is broken by the mechanism discussed in the previous section. The aim of this section is to clarify the particle content and their interactions at low energies, namely, in the reduced two-dimensional space M^2.

The monopole background field $A_\mu^{\rm mon}$ breaks SU(2) symmetry down to U(1). The associated two-dimensional U(1) gauge field $a_m(x)$ ($m=0,1$) is related to $A_\mu(x,\theta,\phi)$ by

$$\begin{aligned}&A_m=\frac{1}{4\sqrt{\pi}r}a_m(x)\vec{e}_r\cdot\vec{\tau}\ ,\\&A_{\theta,\phi}=A_{\theta,\phi}^{\rm mon}(\theta,\phi)\ .\end{aligned}\tag{4.1}$$

Indeed

$$\int r^2 d\Omega\, d^2x\tfrac{1}{2}\,{\rm Tr}F_{\mu\nu}F^{\mu\nu}=\int d^2x\left[\frac{2\pi}{g^2r^2}+\tfrac{1}{4}(\partial_m a_n-\partial_n a_m)^2\right]\ ,\tag{4.2}$$

where $d\Omega=\sin\theta\, d\theta\, d\phi$.

Heavy fermions are relevant for inducing dynamical gauge-symmetry breaking, but contain no light-particle components in M^2. Only the massless fermion in $M^2\times S^2$, which we denote by Ψ, contains massless particles in M^2.

We introduce two-dimensional and four-dimensional Dirac matrices γ^a and Γ^a by

$$\begin{aligned}&\gamma^0=\rho_1\ ,\\&\gamma^1=-i\rho_2\ ,\\&\gamma_5=\gamma^0\gamma^1=\rho_3\ ,\end{aligned}$$

and

$$\begin{aligned}&\Gamma^0=\rho_1\ ,\\&\Gamma^1=-i\rho_2\ ,\\&\Gamma^2=i\rho_3\otimes\sigma_1\ ,\\&\Gamma^3=i\rho_3\otimes\sigma_2\ ,\\&\Gamma_5=i\Gamma^0\Gamma^1\Gamma^2\Gamma^3=\rho_3\otimes\sigma_3\ .\end{aligned}\tag{4.3}$$

In this representation the Dirac operator for Ψ is given by

$$D=i\rho_1\frac{\partial}{\partial t}+\rho_2\frac{\partial}{\partial x}+i\rho_3\otimes D_{S^2}^{(0)}(A^{\rm mon})\ ,$$

where $D_{S^2}^{(0)}(A^{\rm mon})$ is defined in (2.4). In the rotationally symmetric representation it becomes

$$D=i\rho_1\frac{\partial}{\partial t}+\rho_2\frac{\partial}{\partial x}+i\rho_3\otimes D_{S^2}(A^{\rm mon})\ ,\tag{4.4}$$

the chirality operator being given by

$$\Gamma_5=\rho_3\otimes\Omega\sigma_3\Omega^\dagger=\gamma_5\otimes\vec{e}_r\vec{\sigma}\ .\tag{4.5}$$

It is clear from (4.4) that an eigenstate of $D_{S^2}(A^{\rm mon})$ with an eigenvalue m corresponds to a particle with a mass $|m|$ in M^2. Only zero modes of $D_{S^2}(A^{\rm mon})$ are observable at low energies. From (2.6) we see that there are two zero modes:

$$\begin{aligned}&D_{S^2}(A^{\rm mon})u^{(\pm)}=0\ ,\\&u^{(\pm)}=\begin{bmatrix}u_{11}\\u_{12}\\u_{21}\\u_{22}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-\sin\theta\, e^{-i\phi}\\\cos\theta\mp 1\\\cos\theta\pm 1\\\sin\theta\, e^{i\phi}\end{bmatrix}\ .\end{aligned}\tag{4.6}$$

Here the first and second subscripts of u_{ij} refer to spin and isospin, respectively. $u^{(\pm)}$ satisfies

$$\begin{aligned}&\vec{e}_r\cdot\vec{\tau}u^{(\pm)}=-\vec{e}_r\cdot\vec{\sigma}u^{(\pm)}=\pm u^{(\pm)}\ ,\\&\int d\Omega\, u^{(\pm)\dagger}u^{(\pm)}=4\pi\ ,\\&\int d\Omega\, u^{(+)\dagger}u^{(-)}=0\ .\end{aligned}\tag{4.7}$$

Note an important relation

$$\vec{e}_r\cdot\vec{\tau}\vec{e}_r\cdot\vec{\sigma}=-1\ ,\tag{4.8}$$

where $\frac{1}{2}\vec{e}_r\cdot\vec{\tau}$ and $\vec{e}_r\cdot\vec{\sigma}$ represents the unbroken U(1) charge and chirality operator on S^2, respectively.

Ψ contains two massless fields χ and ξ in two dimensions:

$$\Psi(x,\theta,\phi)=\frac{1}{(4\pi r^2)^{1/2}}\begin{bmatrix}\chi_1(x)u^{(+)}(\theta,\phi)+\xi_1(x)u^{(-)}(\theta,\phi)\\\chi_2(x)u^{(+)}(\theta,\phi)+\xi_2(x)u^{(-)}(\theta,\phi)\end{bmatrix}\ .\tag{4.9}$$

χ_1 and ξ_2 have negative chirality ($\Gamma_5=-1$), whereas χ_2 and ξ_1 have positive chirality ($\Gamma_5=+1$). By using (4.7) we find

$$\int r^2 d\Omega\, d^2x\, \overline{\Psi}D\Psi=\int d^2x[\overline{\chi}i\gamma^m(\partial_m-i\overline{g}a_m)\chi+\overline{\xi}i\gamma^m(\partial_m+i\overline{g}a_m)\xi]\ .\tag{4.10}$$

Here the U(1) gauge coupling constant $\bar{g}$ is

$$\bar{g}=\frac{g}{4\sqrt{\pi}r} . \tag{4.11}$$

Note that χ and ξ have opposite charges.

The effective low-energy theory is described by (4.2) and (4.10). It is an Abelian gauge theory with two massless fermions.

V. ANOMALY

In this section we relate the four-dimensional anomaly to the two-dimensional anomaly. In $M^2\times S^2$ we have the axial-vector current anomaly[22]

$$(J_5^\mu)_{;\mu}=-\frac{g^2}{8\pi^2}\mathrm{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu} , \qquad J_5^\mu=e_a^\mu\bar{\Psi}\Gamma^a\Gamma_5\Psi , \tag{5.1}$$

where e_a^μ is a vierbein. Define a two-dimensional axial-vector current by

$$j_5^m=\int r^2 d\Omega\, J_5^m = -\bar{\chi}\gamma^m\gamma_5\chi+\bar{\xi}\gamma^m\gamma_5\xi . \tag{5.2}$$

Here we have used (4.3), (4.7), and (4.9). Then Eq. (5.1) implies, with (4.1),

$$\begin{aligned}\partial_m j_5^m &= \int r^2 d\Omega\, \partial_m J_5^m \\ &= \int r^2 d\Omega (J_5^\mu)_{;\mu} \\ &= -\frac{g^2}{8\pi^2}\int r^2 d\Omega\, \mathrm{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \frac{\bar{g}}{\pi}\epsilon^{mn}(\partial_m a_n-\partial_n a_m) .\end{aligned} \tag{5.3}$$

This is exactly what is expected, if one starts from the two-dimensional theory (4.2) and (4.11). There[23]

$$\begin{aligned}\partial^m(\bar{\chi}\gamma_m\gamma_5\chi)&=-\frac{\bar{g}}{2\pi}\epsilon^{mn}(\partial_m a_n-\partial_n a_m) ,\\ \partial^m(\bar{\xi}\gamma_m\gamma_5\xi)&=+\frac{\bar{g}}{2\pi}\epsilon^{mn}(\partial_m a_n-\partial_n a_m) .\end{aligned} \tag{5.4}$$

The sign is different for χ and ξ, because they have opposite charges in U(1). The result is nontrivial. The four-dimensional equation (5.1) contains all degrees of freedom associated with heavy particles in the two-dimensional language. We kept only massless modes a_m,χ, and ξ to obtain Eq. (5.3), i.e., the anomaly equation closes in the massless sector.

VI. LEFT-RIGHT ASYMMETRY AND ANOMALY CANCELLATION

The problem of left-right asymmetry in higher-dimensional theories lies in the fact that even if one starts from a left-right—asymmetric theory in higher dimensions, one usually ends up with a left-right—symmetric theory in lower dimensions. One way to get left-right asymmetry is to start from topologically nontrivial extra-dimensional manifolds like $K3$.[24] Instead, we show that left-right asymmetry naturally arises from a topologically trivial manifold like S^2 as a consequence of dynamical gauge-symmetry breaking discussed in the previous section.

Let us take SU(2)×U(1) gauge theory in $M^2\times S^2$ as an example. As has been shown earlier, a sufficiently large number of SU(2)-doublet massive fermions induce gauge-symmetry breaking SU(2)×U(1)→U(1)×U(1). In addition to them we introduce massless Weyl fermions such that all left-handed Weyl fermions are SU(2) doublets, while all right-handed Weyl fermions are SU(2) singlets. Since zero modes of $D_{S^2}(A^{\mathrm{mon}})$ exist only for SU(2) doublets, only left-handed Weyl fermions survive at low energies.

We denote SU(2) and U(1) gauge fields by A_μ and B_μ with coupling constants g and $g'Y$. (Y is a "hypercharge.") Corresponding two-dimensional U(1) gauge fields are denoted by a_m and b_m, respectively. As in Eq. (4.9) a left-handed massless fermion has decomposition given by

$$\Psi_L(x,\theta,\phi)=\frac{1}{(4\pi r^2)^{1/2}}\begin{bmatrix}\xi_1(x)u^{(-)}(\theta,\phi)\\ \chi_2(x)u^{(+)}(\theta,\phi)\end{bmatrix} . \tag{6.1}$$

The effective Lagrangian in M^2 is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}^{\mathrm{int}}&=-\bar{g}a_m(\bar{\chi}_R\gamma^m\chi_R-\bar{\xi}_L\gamma^m\xi_L)\\ &\quad -\bar{g}'Yb_m(\bar{\chi}_R\gamma^m\chi_R+\bar{\xi}_L\gamma^m\xi_L) ,\\ \bar{g}&=\frac{g}{4\sqrt{\pi}r},\quad \bar{g}'=\frac{g'}{2\sqrt{\pi}r} .\end{aligned} \tag{6.2}$$

As promised, the resultant U(1)×U(1) gauge theory is left-right asymmetric.

Since the original four-dimensional theory contains axial-vector gauge couplings, it must satisfy anomaly-free conditions. In particular, from the $\langle A_\mu A_\nu B_\lambda\rangle$ vertex we have

$$\mathrm{Tr}_L Y=0 . \tag{6.3}$$

Here the trace is over all left-handed Weyl fermions.

The effective two-dimensional theory also contains axial-vector gauge couplings. The question arises whether it is anomaly free or not. A dangerous vertex here is $\langle a_m b_n\rangle$. Noting that $\chi(\xi)$ is right (left) handed, we see

$$\begin{aligned}\langle a_m b_n\rangle &\to \bar{g}\bar{g}'\Sigma Y-(-\bar{g}\bar{g}')\Sigma Y\\ &\propto \Sigma Y=0 .\end{aligned} \tag{6.4}$$

That is, the condition (6.3) guarantees that the resultant two-dimensional theory is anomaly free.

VII. SUMMARY AND DISCUSSIONS

In this paper we investigated gauge theory in $M^2\times S^2$ as a toy model of higher-dimensional theories. Owing to the compactness of the extra-dimensional space S^2 fermion zero-point energies are shifted so that the monopole configuration A^{mon} in S^2 has lower energy density than the pure gauge configuration. Its implication is very large. First of all, SU(2) gauge symmetry breaks down to U(1). This is a new mechanism for dynamical gauge-symmetry breaking, and could replace the Higgs mechanism. Second, dynamically chosen $A^{\rm mon}$ admits two zero modes in a left-right-asymmetric way so that we have in the reduced two dimensions M^2 left-right-asymmetric massless fermions.

We restricted ourselves to $M^2\times S^2$. We can extend our analysis to any even dimensions $M^n\times S^2$, assuming that divergences can be consistently removed at least at the one-loop level. Then we would find that for $n=2,6,10,\ldots$ gauge symmetry is dynamically broken, provided that there exists a very heavy fermion. For $n\geq 6$, only one very heavy fermion ($mr>5$) is enough to induce gauge-symmetry breaking.

Similar phenomena are expected to happen for $M^n\times S^q$ ($q\geq 3$), though we have not examined it yet. The nonvanishing curvature of the extra-dimensional space seems crucial in our arguments. It must also be responsible for the fact that $A^{\rm mon}$, namely $l=1$ components, and heavy fermions are important to derive gauge-symmetry breaking,[25] though a consistent effective field theory for light particles exists.

We analyzed gauge theory on $M^n\times S^2$. The existence of compact extra dimensions must be justified by solving the Einstein equation simultaneously. This, with construction of more realistic theories, is left to be investigated.

Finally we note that our results indicate that strong gravity with quantum effects can lead to gauge-symmetry breaking in four dimensions.

Note added in proof. After submitting this paper, the author found E. Witten's paper which contains thorough discussions of the problem of left-right asymmetry in higher-dimensional theories [E. Witten, Princeton report, 1983 (unpublished)].

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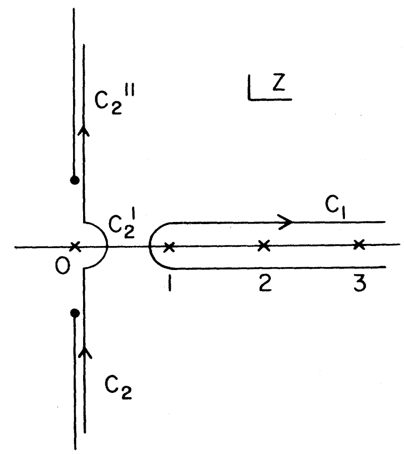

FIG. 1. Integration contours in Eqs. (A3) and (A5).

APPENDIX: DERIVATION OF (2.10)

The discrete sums in (2.7) and (2.8) are given by

$$f_2(n,b)=-\sum_{j=1}^{\infty} j(j^2+b^2)^{n/2}\,, \tag{A1}$$

$$f_3(n,b)=-\tfrac{1}{4}b^n-\sum_{j=1}^{\infty}(j+\tfrac{1}{2})[j(j+1)+b^2]^{n/2}\,. \tag{A2}$$

f_2 and f_3 are defined by (A1) and (A2) for $n<-2$, and are to be given by analytic continuation for positive n. Let us first consider f_2:

$$\begin{aligned} f_2(n,b)&=-\int_{C_1}dz\frac{1}{1-e^{2\pi iz}}z(z^2+b^2)^{n/2} =-\int_{C_2+C_2'}dz\frac{z(z^2+b^2)^{n/2}}{1-e^{2\pi iz}}-\int_{i\epsilon}^{\infty+i\epsilon}dz\left[1+\frac{1}{e^{-2\pi iz}-1}\right]z(z^2+b^2)^{n/2}\\ &=\frac{1}{n+2}b^{n+2}-\int_{C_2'}dz\cdots-\int_{C_2}dz\frac{z(z^2+b^2)^{n/2}}{1-e^{2\pi iz}}-\int_{C_2''}dz\frac{z(z^2+b^2)^{n/2}}{e^{-2\pi iz}-1}\,. \end{aligned} \tag{A3}$$

Contours are given in Fig. 1. The second term vanishes in the $\epsilon\to 0$ limit. The third and last terms are complex conjugates of each other. The expression (A3) applies to all n. Noting cuts extending from $z=\pm ib$, we have for $n>-2$

$$f_2(n,b)=\frac{b^{n+2}}{n+2}+2\int_0^b dx\frac{x(b^2-x^2)^{n/2}}{e^{2\pi x}-1}+2\cos\frac{n\pi}{2}\int_b^\infty dx\frac{x(x^2-b^2)^{n/2}}{e^{2\pi x}-1}\,. \tag{A4}$$

The formula for f_3 is obtained in a similar manner:

$$\begin{aligned} f_3(n,b)&=-\tfrac{1}{4}b^n-\int_{C_1}dz\frac{1}{1-e^{2\pi iz}}(z+\tfrac{1}{2})[z(z+1)+b^2]^{n/2}\\ &=-\tfrac{1}{4}b^n-\int_{C_2+C_2'}dz\frac{1}{1-e^{2\pi iz}}\cdots-\int_{i\epsilon}^{\infty+i\epsilon}dz\left[1+\frac{1}{e^{-2\pi iz}-1}\right]\cdots\\ &=\frac{1}{n+2}b^{n+2}-\tfrac{1}{4}b^n-\int_{C_2'}dz\frac{1}{1-e^{2\pi iz}}\cdots-\left[\int_{C_2}dz\frac{1}{1-e^{2\pi iz}}+\int_{C_2''}dz\frac{1}{e^{-2\pi iz}-1}\right]\cdots\,. \end{aligned} \tag{A5}$$

This time the third term gives a nonvanishing contribution in the $\epsilon\to 0$ limit to cancel the second term:

$$-\int_{C_2'} dz \cdots = -\int_{-\pi/2}^{\pi/2} i\epsilon e^{i\theta} d\theta \frac{\frac{1}{2}b^n}{-2\pi i\epsilon e^{i\theta}}$$

$$= +\tfrac{1}{4}b^n .$$

The expression (A5) is valid for all n. By introducing $z=+ix$ and

$$Re^{iw}=z(z+1)+b^2=-x^2+b^2+ix ,$$

we find for $n>-2$

$$f_3(n,b)=\frac{1}{n+2}b^{n+2} + \int_0^\infty dx \frac{R^{n/2}}{e^{2\pi x}-1}\left[\sin\frac{nw}{2}+2x\cos\frac{nw}{2}\right] . \tag{A6}$$

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