Two-body relativistic scattering of directly interacting particles

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Two-body relativistic scattering via direct interaction is discussed. For a certain class of interactions the problem is equivalent to a family of nonrelativistic scattering problems. The relativistic mass shell is shown to correspond with the kinetic-energy shell of the reduced problem. Under certain sufficient assumptions it is shown that known proofs apply, which ensure that the wave operators exist off-shell. Under more precise conditions, the scattering operator S can be defined also on the mass shell, which is stable by its action. Moreover, S preserves the positivity of energy and is unitary with respect to the inner product of on-shell states. In a more particular case, we obtain a natural normalization in the interacting mass shell.

I. INTRODUCTION

Relativistic dynamics of directly interacting particles has received growing attention during the last decade. The physical relevance of N-body relativistic quantum mechanics and whether it can be considered as a (generally nonlocal) simplification with respect to quantum field theory has been discussed widely in the literature.[1] Starting from a quantized version of predictive mechanics in its *a priori* Hamiltonian form,[2] we presented in a previous article a covariant framework for N-body scattering.[3] Quite naturally, the multitime character of predictive mechanics gave rise to a multitime formulation of relativistic scattering.

Formally, the transposition of the machinery of scattering theory into the language of relativity is rather easy. The Newtonian time is replaced by N parameters which generalize the proper times, and the role of the energy in nonrelativistic mechanics is now played by N half squared masses.[4] But, whereas in nonrelativistic dynamics most conventional results of scattering theory take place within a Hilbert space of states which do not lie on the *energy* shell, playing the same game in relativity, it seems at first sight that correct mathematical statements are applicable only to the states which lie off the *mass* shell. This looks rather frustrating. Of course, the Galilean physicist does not mind having to consider wave packets instead of plane waves and, thus accepts without difficulty a smearing of the *energy* shell around some average value.

In the relativistic situation we consider a more delicate smearing of the *mass* shell, which has been proposed by some authors.[5] Their point of view is supported by the fact that most real particles have a finite lifetime and, therefore, could be slightly off-shell. This line of thought leads to interesting developments, but requires a careful discussion, which is beyond the scope of this paper.

We consider that the notion of asymptotic states with infinitely *sharp* masses is at least a very convenient idealization. This concept offers the advantage of providing a straightforward connection with familiar techniques which utilize the conventional scalar product of the solutions of the Klein-Gordon equation. Thus, in our opinion, it is relevant to undertake a sharp-mass-shell theory in a rigorous framework. (From now on, a mass shell will be strictly understood as sharp.) This article is mainly devoted to this task, in the simple case of two particles without spin. Consequently, we are going to improve and sophisticate our previous formulation in order to make it applicable to the mass shell also.

Coming to the properties of the scattering operator, we have been led to distinguish between its *mathematical unitarity*—a property which occurs in the off-shell framework—and its *physical unitarity,* defined with respect to the scalar product we just mentioned. This point is connected with the emergence of two kinds of inner products ("mathematical" and "physical") that we have stressed previously.[3] Nevertheless, and in spite of our need for going finally on the mass shell, we consider off-shell investigations as a provisional but necessary stage of the theory.

Considerable progress has been made in this area by Horwitz and Rohrlich.[5] Whether we follow their interpretation or not does not affect its computational interest. Independently, substantial advances in stationary relativistic scattering have been made by use of the quasipotential approach.[6] Though quasipotential methods are practically sufficient in most cases, they do not provide a complete theory from a conceptual viewpoint since they quantize the relative degrees of freedom only. For instance, having in mind the eventual development of a second-quantized program, we feel that it is necessary to be able to superpose states with various linear momenta, which requires quantizing the full motion of the system.[7]

In this article we restrict our effort to the unipotential case, when the interaction is mediated through a single relativistic potential. Indeed, in this situation the manytime formalism undergoes a simplification which provides a single-time scattering problem. Then we select a large class of potentials which allows reduction of the initial problem to a family of (parameter-dependent) nonrelativistic scattering problems. The parameter involved here is the total linear momentum, and the contact with a time-dependent quasipotential approach is manifest.

Note that in Sec. II we remain more or less within a conventional Hilbert-space method and display an off-shell treatment, while in Sec. III we are led to a different point of view and systematically manipulate continuous matrix elements which have a rigorous setting only within rigged Hilbert spaces.

Unipotential two-body scattering can be presented as follows. We take $c=\hbar=1$, signature $+---$, and $P=p_1+p_2$. In the *a priori* Hamiltonian formalism,[2] the most general two-body system is defined by two *commuting* operators:

$$H_1=\bar{H}_1+V_1,\quad H_2=\bar{H}_2+V_2\ , \tag{1.1}$$

where $\bar{H}_1$ and $\bar{H}_2$ are the free Hamiltonians $\bar{H}_1=\frac{1}{2}{p_1}^2$, $\bar{H}_2=\frac{1}{2}{p_2}^2$ for particles without spin. The evolution operator is

$$U=\exp[i(\tau_1 H_1+\tau_2 H_2)]\ , \tag{1.2}$$

where τ_1,τ_2 are the evolution parameters (suitable generalizations of the proper times).

Let

$$\bar{U}=\exp[i(\tau_1\bar{H}_1+\tau_2\bar{H}_2)]$$

describe the free-particle evolution. The relativistic Møller operators are naturally defined as

$$\Omega^{\pm}=\lim W \quad \text{for } \tau_1 \text{ and } \tau_2\rightarrow \pm\infty\ , \tag{1.3}$$

where $W=U^{-1}\bar{U}$. In the general case, the problem of the order of the limits has been solved in Ref. 5. In this paper we only consider *unipotential* interactions (i.e., single-potential systems) characterized by $V_1=V_2=V$, with

$$[V,y\cdot P]=0,\quad y=\tfrac{1}{2}(p_1-p_2)\ . \tag{1.4}$$

Therefore, the problem of the order of the limit does not arise. Indeed, as we earlier pointed out,[3] drastic simplifications occur.

Let us set $\tau=\tau_1+\tau_2$. Then, recalling that $[H_1+H_2, H_1-H_2]=0$, we obtain

$$W=\exp\left[-\frac{i}{2}(H_1+H_2)\tau\right]\exp\left[\frac{i}{2}(\bar{H}_1+\bar{H}_2)\tau\right]\ ,$$

which reduces the problem to a single-time scattering problem. Now, from the identity

$${p_1}^2+{p_2}^2\equiv\tfrac{1}{2}P^2+2y^2\ ,$$

and assuming a *translation-invariant* potential, we have

$$W=e^{-i\tau H}e^{i\tau\bar{H}} \tag{1.5}$$

with

$$\bar{H}=\tfrac{1}{2}(\bar{H}_1+\bar{H}_2)-\frac{P^2}{8}=\tfrac{1}{2}y^2\ , \tag{1.6}$$

$$H=\tfrac{1}{2}y^2+V\ . \tag{1.7}$$

If we introduce the operators

$$E=\bar{E}-V\ , \tag{1.8}$$

$$\bar{E}=-\tfrac{1}{2}\tilde{y}^2 \tag{1.9}$$

with $\tilde{y}=y-(y\cdot P)P/P^2$, then H differs from $-E$ only by $(y\cdot P)^2/P^2$, which commutes with $\tilde{y}^2$ and also with V. Hence, we also have

$$W=e^{i\tau E}e^{-i\tau\bar{E}}\ . \tag{1.10}$$

At this stage, our problem already exhibits some analogy with a nonrelativistic problem.[8]

Notation. We set $r^\alpha=x_1^\alpha-x_2^\alpha$. When K^α is a timelike vector, we define $\forall\xi$

$$\hat{\xi}=\xi-(K\cdot\xi)K/K^2\ .$$

Throughout this paper, P is just multiplication by K, in a suitable representation. Thus, our distinction between $\tilde{\xi}$ and $\hat{\xi}$ is somehow academic, but it helps in reminding that P stands for an operator, whereas the components of K are numbers.

Situations arise when the "hat" construction must be made with several vectors K, L, etc. In this case, the hat is accompanied by a subscript.

II. OFF-SHELL THEORY OF $\Omega^{\pm}$ AND S

A. States and operators

Consider the off-shell wave function $\Psi(X,r)$. In view of a more practical representation, let us make a Fourier transformation with respect to the external coordinates only. Thus, we set

$$\Psi(X,r)=\frac{1}{(2\pi)^2}\int e^{iK\cdot X}\psi(K,r)d^4K\ , \tag{2.1}$$

and ψ is the wave function in the K, r representation.

If the development (2.1) is considered as a superposition of waves with various linear momenta, we are led to some restrictions on the support of ψ. First of all, spacelike contributions will be discarded as unphysical. Thus, we shall immediately assume that ψ vanishes when $K^2<0$. Then, null contributions are to be avoided for a technical reason: the most tractable relativistic two-body potentials contain generally K^2 in the denominator. Thus, we also require that ψ vanishes for $K^2=0$. Moreover, we shall retain only positive-energy amplitudes. Hence, we finally impose that, in its argument K, ψ has support only inside the forward light cone. This property will be called *retarded* (ret) by definition.[9] Beside this condition, we suppose in this section that ψ is an ordinary function of K (never a distribution with singularity). More precisely, ψ is supposed to be (in the variable K) locally integrable and of polynomial tempered growth, which ensures that Ψ exists according to Eq. (2.1). We shall devote a particular attention to the space $\mathscr{F}$ of functions $F(K,r)=f(K,\hat{r})$ which are retarded in K as above and depend on r^α only through the variables

$$\hat{r}_K^\alpha=r^\alpha-\frac{r\cdot K}{K^2}K^\alpha\ . \tag{2.2}$$

The label K will be dropped from $\hat{r}$ whenever no confusion is possible. We shall use a similar convention for

$$r_K^{(0)} = \frac{r \cdot K}{|K|} \ . \quad (2.3)$$

If we consider $f(K,\hat{r})$ as a function of r, which also depends on K as a parameter, let us choose, for each K, an adapted frame in which

$$r^{(0)} = r^0, \quad \hat{r}^i = r^i, \text{ etc.}$$

Then, $f(K,\hat{r})$ is explicitly given in terms of the variables r^i by a function $g_K(\vec{r})$.

The integral

$$I_K = \int g_K(\vec{r}) d^3\vec{r} \quad (2.4)$$

is manifestly invariant by any change of adapted frame (keeping K fixed). We shall note

$$I_K = \int f(K,\hat{r}) d^3\hat{r} \ . \quad (2.5)$$

Remark. We can write equivalently[10]

$$\int f(K,\hat{r}) d^3\hat{r} = \int F(K,r)\delta(r_K^{(0)}) d^4r \ . \quad (2.6)$$

Remark. When f depends on K only through $\hat{r}^2$, then I_K is independent of K. [*Proof.* $f(K,\hat{r}) = f(\hat{r}^2)$. Then, $g_K(\vec{r})$ is simply $f(-\vec{r}^2)$ for any K and an adapted frame. Hence, K disappears from the right-hand side of integral (2.4).]

Let us define, for $\varphi(K,\hat{r})$ and $\varphi'(K,\hat{r})$ in $\mathscr{F}$, the sesquilinear form

$$\langle \varphi, \varphi' \rangle_K = \int \varphi^*(K,\hat{r}) \varphi'(K,\hat{r}) d^3\hat{r} \ . \quad (2.7)$$

For each admissible K, the condition

$$\langle \varphi, \varphi \rangle_K < \infty$$

defines a K-dependent Hilbert space $L_K^2(d^3\hat{r})$, which is a copy of $L^2(\mathrm{R}^3, d^3\vec{r})$. We now turn to an application of this.

In the wave function $\psi(K,r)$, we can split the variable r into $r^{(0)}$ and $\hat{r}$ and Fourier transform with respect to $r^{(0)}$, which yields an expression of the form

$$\psi = \frac{1}{\sqrt{2\pi}} \int e^{iur^{(0)}} \varphi(K,u,\hat{r}) du \ , \quad (2.8)$$

which permits representation of the state of the system by a function φ, which belongs to $\mathscr{F}$ in the variables $K,\hat{r}$ and depends additionally on u. This method is of great convenience in view of reducing our problem to a three-dimensional one. For instance, let $\varphi'(K,u,\hat{r})$ correspond to another wave function ψ' by a development analogous to (2.8) and calculate the quantity $\langle \varphi, \varphi' \rangle_K$ according to (2.7). In this case $\langle \varphi, \varphi' \rangle_K$ still depends on u. For each admissible K and for each u, the condition $\langle \varphi, \varphi' \rangle_K < \infty$ defines a (K,u)-dependent Hilbert space $L_{Ku}^2(\mathrm{R}^3, d^3\hat{r})$, which is a copy of $L^2(\mathrm{R}^3, d^3\vec{r})$. Whenever the dependence on u can be ignored, we shall write L_K^2 instead of L_{Ku}^2.

Now, consider the scalar product

$$\langle \psi, \psi' \rangle = \int \psi^* \psi' d^4K \, d^4r \ , \quad (2.9)$$

which appears in the definition of $L^2(\mathrm{R}^8, d^4K \, d^4r)$. We can write

$$\langle \psi, \psi' \rangle = \int J(K) d^4K$$

with

$$J(K) = \int \psi^*(K,r) \psi'(K,r) d^4r \ .$$

Even if $J(K)$ exists, the integral (2.9) is not necessarily convergent. By a notation which refers to our restrictions about the dependence on K, we shall say that $\psi \in L^2(\mathrm{ret}K, \mathrm{R}^4, d^4r)$ when J exists.

Replacing ψ and ψ' by their development (2.8) yields

$$J(K) = \frac{1}{2\pi} \int e^{i(u'-u)r^{(0)}} \varphi^* \varphi' du \, du' d^4r \ .$$

Choosing an adapted frame, we get

$$r^{(0)} = r^0 \ ,$$

$$\hat{r} = (0, \vec{r}) \ ,$$

$$d^4r = dr^0 d^3\vec{r} \ .$$

Since φ^* and φ do not depend on r^0, we have

$$J = \frac{1}{2\pi} \int \varphi^* \varphi' \int e^{i(u'-u)r^0} dr^0 du \, du' d^3\vec{r} \ .$$

Integrating over r^0, $\delta(u-u')$ appears. Then, integrating over u' yields

$$J = \int \varphi^* \varphi' du \, d^3\vec{r} \ .$$

But, since $\hat{r} = (0, \vec{r})$,

$$\langle \varphi, \varphi' \rangle_K = \int \varphi^*(K,u,0,\vec{r}) \varphi'(K,u,0,\vec{r}) d^3\vec{r} \ .$$

Thus,

$$J = \int \langle \varphi, \varphi' \rangle_K du \ .$$

Hence, we obtain the important formula

$$\langle \psi, \psi' \rangle = \int \langle \varphi, \varphi' \rangle_K d^4K \, du \ . \quad (2.10)$$

Accordingly, when $\psi \in L^2(\mathrm{R}^8)$, then $\varphi(K,u,\hat{r})$ is certainly in $L_K^2(\mathrm{R}^3, d^3\hat{r})$ for any K. The converse is not always true, since it requires not only that $\langle \varphi, \varphi \rangle_K$ exists, but also that

$$\int \langle \varphi, \varphi \rangle_K d^4K \, du < \infty \ ,$$

which can be stated in terms of the variables $u, \hat{r}$, saying that

$$\varphi(K,u,\hat{r}) \in L^2(\mathrm{R}^8, d^4K, du, d^3\hat{r}) \ .$$

We see also that, as a Hilbert space of functions in the variable r^α, our space of wave functions admits the direct integral decomposition

$$L^2(\mathrm{ret}K, \mathrm{R}^4, d^4r) = \int^{\oplus} L_{Ku}^2 du \ .$$

1. Operators

In the K,r representation, all the translation-invariant operators have the form $A(K,r,y)$ with

$$y^\alpha = \frac{1}{i} \frac{\partial}{\partial r^\alpha} \ .$$

The case when additionally A commutes with $r^{(0)}$ and $y^{(0)}$

(then A is also, in particular, a relative observable in the sense of Todorov[10]) is of particular interest for the simplifications it implies.

Going to the $K,u,\hat{r}$ representation [where $r^{(0)}$ and $y^{(0)}$ are, respectively, represented by $(1/i)\partial/\partial u$ and u], the action of A ignores the variable u. In other words

$$A\psi(K,r)=\frac{1}{\sqrt{2\pi}}\int e^{iur^{(0)}}A_K\varphi(K,u,\hat{r})du\ , \qquad (2.11)$$

where A_K depends on K, $\hat{r}$, and $\hat{y}$ only. Considering φ as a function which depends on K and u as parameters, we see that A induces a K-dependent operator A_K in L_K^2. For example, $A=\hat{y}^2$,

$$A_K=\Delta_K=-\Box_r+(K\cdot\partial/\partial r)^2/K^2\ ,$$

which reduces to the ordinary Laplacian in adapted coordinates. *Conversely,* let us give in L_K^2 an operator A_K (which depends on K as a parameter). We define $A\varphi=A_K\varphi$ for each value of K in the support of the wave functions with respect to the retarded conditions previously stated. Then, $A\psi$ is defined through Eq. (2.11). Beware that even if φ belongs, $\forall K$, to the domain of A_K, assuming $\varphi\in L^2(\mathrm{R}^8,d^4K,du,d^3\hat{r})$ in general does not imply that $\varphi'=A_K\varphi$ would lie in $L^2(\mathrm{R}^8,d^4K,du,d^3\hat{r})$. Actually, in such a case, the only thing which is always true is that $A_K\varphi$ is in L_K^2 for each K.

A sequence of linear operators A_τ of the above type is said to converge strongly in L_K^2 when $\forall K$, $A_{K\tau}\rightarrow B_K$ strongly. Setting $B\varphi=B_K\varphi$ for each K, we say that $B=\lim A_\tau$ in L_K^2. If we define $A^*\varphi=A_K^*\varphi$, where A_K^* is the conjugate of A_K in L_K^2 [i.e., with respect to the K-dependent scalar product (2.7)], it is easy to check from (2.8) and (2.10) that

$$\langle A^*\psi,\psi'\rangle=\int\langle A_K^*\varphi,\varphi'\rangle_K d^4K\,du\ ,$$

$$\langle\psi,A\psi'\rangle=\int\langle\varphi,A_K\varphi'\rangle_K d^4K\,du\ .$$

Whenever these integrals exist, they are equal. Thus, our convention is compatible with the usual definition. It is noteworthy that all the definitions in which we crucially utilize Eq. (2.11) are based upon the direct-sum structure of our space of states—in fact, direct integral.[11]

B. Møller and scattering operators

Under the assumptions of Sec. I, E commutes with P, $r\cdot P$, and $y\cdot P$. More precisely, in the notations of the K,r representation,

$$E=-\tfrac{1}{2}\hat{y}^2-V\ ,$$

where V is an operator which depends on K^α, $\hat{r}^\beta$, and $\hat{y}^\gamma$ only. We do not intend here to discuss under which assumptions $\Omega^\pm$ might be a strong (or weak) limit in $L^2(\mathrm{R}^8,d^4K,d^4r)$.

It must be clearly understood that the result presented below does not imply the existence of Ω in $L^2(\mathrm{R}^8)$. In fact, we have replaced $L^2(\mathrm{R}^8)$ by $L^2(\mathrm{ret}K,\mathrm{R}^4,d^4r)$, which is a direct integral of the various L_{Ku}^2, and Ω is defined as a direct integral of operators. We are able to define the wave operators as limits in L_K^2. Indeed, solving the question of convergence in L_K^2 only requires solving a K-dependent family of three-dimensional problems. Fortunately, some criteria are well known, which guarantees the convergence of W in L_K^2.[12]

In the case of a central potential $V(K^2,\hat{r}^2)$, a sufficient condition is that *for each K in the support of the wave functions,* we have

$$\int|V(K^2,\hat{r}^2)|^2d^3\hat{r}<\infty\ .$$

In the energy-independent case (simple central potential), this condition reduces to

$$\int|V(-\vec{r}^2)|^2d^3\vec{r}<\infty\ .$$

This permits us to incorporate some nonrelativistic results into the present covariant framework. In practical calculations, one should be aware of the fact that E and V are homogeneous to a squared mass. Note also that H and E appear in W with opposite signs [Eqs. (1.5) and (1.10)]. Therefore, $\Omega^\pm$ corresponds to the $\mp$ wave operator of the nonrelativistic system.

Owing to the relation (2.10) and its previously mentioned consequence with respect to Hermitian conjugation, the scattering operator $S=\Omega^{-*}\Omega^+$ acts according to the formula

$$S\psi=\frac{1}{\sqrt{2\pi}}\int e^{iur^{(0)}}S_K\varphi(K,u,\hat{r})du\ , \qquad (2.12)$$

where S_K is nicely defined and unitary in L_K^2. Since we have departed from a strict $L^2(\mathrm{R}^8)$ framework, it remains to be checked whether the relativistic intertwining relations hold. (In our first presentation of relativistic scattering,[3] they were derived from more restrictive assumptions.) This can be actually carried out as follows. First, let us prove that $\Omega^\pm$ intertwines H_1-H_2 with $\overline{H}_1-\overline{H}_2$. In our unipotential case, since $\overline{H}_1-\overline{H}_2=H_1-H_2$, we only have to verify that it commutes with $\Omega^\pm$. But

$$(\overline{H}_1-\overline{H}_2)_K=-iK\cdot\frac{\partial}{\partial r}\ ,$$

hence,

$$(\overline{H}_1-\overline{H}_2)\psi=\frac{1}{\sqrt{2\pi}}\int e^{iur^{(0)}}|K|u\varphi\,du\ ,$$

while

$$\Omega^\pm\psi=\frac{1}{\sqrt{2\pi}}\int e^{iur^{(0)}}\Omega_K^\pm\varphi\,du\ .$$

We see that, in the $K,u,\hat{r}$ representation (i.e., using the direct-sum decomposition), $\overline{H}_1-\overline{H}_2$ just corresponds to the multiplicative operator iKu, while Ω corresponds to Ω_K, which acts only on $\hat{r}$, not on K,u. Obviously, Ω_K commutes with iKu, therefore, $[\Omega^\pm,\overline{H}_1-\overline{H}_2]$ does vanish. Then, we see that $\Omega^\pm$ intertwines E with $\overline{E}$ (consider $\Omega_K^\pm$, E_K, and $\overline{E}_K$ and apply the nonrelativistic theory in L_K^2). Now recall that

$$p_1{}^2+p_2{}^2=\tfrac{1}{2}P^2+2y^2\ ,$$

as noted in Sec. I. The intertwining of H_1+H_2 with $\overline{H}_1+\overline{H}_2$ follows immediately, provided W exists, in the direct-sum sense. Thus, we have

$$\Omega^{\pm}\overline{H}_1 = H_1\Omega^{\pm} ,$$
$$\Omega^{\pm}\overline{H}_2 = H_2\Omega^{\pm} .$$

[Let us again emphasize that all of the operators considered in this section are not operators in $L^2(\mathrm{R}^8)$, but are those only defined in the direct-sum sense.] Hence, after an elementary algebraic manipulation, we deduce

$$[S,\overline{H}_1] = [S,\overline{H}_2] = 0 , \tag{2.13}$$

which provides a heuristic argument for the claim that S leaves the free mass shells invariant.[13]

Unfortunately, the formalism displayed in this section will not encompass the case of on-shell states. In order to clarify this question and to settle a more meaningful description of scattering, we are going now to modify and to enlarge the formalism.

III. SCATTERING ON THE MASS SHELL

A. Mass shell and energy shell

In spite of its formal interest, the point of view developed in the previous section remains incomplete. Indeed, our main purpose was to obtain a description valid for the case of on-shell particles. It cannot be expected that such particles would match the L_K^2 scheme presented there. In fact, taking the sum and the difference of the wave equations in the K,r representation, namely

$$H_1\psi = \tfrac{1}{2}{m_1}^2\psi, \quad H_2\psi = \tfrac{1}{2}{m_2}^2\psi ,$$

and writing that P is just multiplication by K, we have explicitly

$$({m_1}^2 + {m_2}^2 - \tfrac{1}{2}K^2)\psi = 2y^2\psi + 4V\psi , \tag{3.1}$$

$$({m_1}^2 - {m_2}^2)\psi = 2K\cdot y\psi , \quad y = -i\partial/\partial r . \tag{3.2}$$

Equation (3.2) is immediately solved by taking ψ in the form

$$\psi = e^{i\mu r^{(0)}}\chi(K,\hat{r}) , \tag{3.3}$$

where

$$\mu = \frac{{m_1}^2 - {m_2}^2}{2|K|} = \frac{a}{|K|} ,$$

and $a = \frac{1}{2}({m_1}^2 - {m_2}^2)$. Inserting (3.3) into Eq. (3.1) yields

$$({m_1}^2 + {m_2}^2 - \tfrac{1}{2}K^2)\psi = \left[2\Delta + 2\frac{a^2}{K^2} + 4V \right]\psi$$

with $\Delta = \hat{y}^2$ (it reduces to the Laplacian in an adapted frame). Let us define

$$\epsilon(K) = \frac{a^2}{2K^2} - \frac{{m_1}^2 + {m_2}^2}{4} + \frac{K^2}{8} . \tag{3.4}$$

Then we have

$$(-\tfrac{1}{2}\Delta - V)\psi = \epsilon\psi .$$

Since Δ commutes with $r^{(0)}$, we can divide by $e^{i\mu r^{(0)}}$ and obtain

$$(-\tfrac{1}{2}\Delta - V)\chi = \epsilon\chi(K,\hat{r}) . \tag{3.5}$$

Note that from (3.3), ψ corresponds to $\varphi(K,u,\hat{r})$ according to (2.8) with

$$\varphi = \sqrt{2\pi}\,\delta(u-\mu)\chi . \tag{3.6}$$

On-shell states are normalizable neither in $L^2(\mathrm{R}^8)$ nor in L_K^2. Actually, we see that φ (respectively χ) is an eigenstate of $E = -\frac{1}{2}\Delta - V$ for the eigenvalue ϵ. In other words, ψ is on the mass shell if and only if φ (or χ) is on the energy shell of the reduced system (with unit mass and potential energy at $-V$). Under the assumptions which allow $\Omega^{\pm}$ to exist, states on the energy shell in the continuous part of the spectrum are not in L_K^2. By a similar argument (with a vanishing V), we can map the free mass shell onto the kinetic-energy shell of the reduced two-body system with ϵ as in (3.4).

B. Plane waves and positivity

On the free mass shell, the prototype is a plane wave. In the K,r representation, plane waves are of the form

$$\psi = (2\pi)^2\delta(K-L)e^{iY\cdot r} . \tag{3.7}$$

In this formula L and Y are the eigenvalues of P and y, respectively. Accordingly, L is a timelike, future-oriented, constant vector. Y is a constant vector, and they are submitted to the *mass-shell conditions:*

$$L\cdot Y = a , \tag{3.8}$$

$$L^2 + 4Y^2 = 2({m_1}^2 + {m_2}^2) . \tag{3.9}$$

Splitting Y into $Y_L^{(0)}$ and $\hat{Y}_L$ as in Eqs. (2.2) and (2.3), we see easily that

$$Y_L^{(0)} = \frac{a}{|L|} , \qquad \hat{Y}_L^2 = -2\epsilon(L) . \tag{3.10}$$

It reduces to $-\vec{Y}^2$ in the frames adapted to L. Clearly, $\epsilon(L)$ cannot be negative. Let us reintroduce individual variables, setting

$$l_1 = \frac{L}{2} + Y, \quad l_2 = \frac{L}{2} - Y . \tag{3.11}$$

Thus, (3.8) and (3.9) read ${l_1}^2 = {m_1}^2$, ${l_2}^2 = {m_2}^2$.

The plane wave (3.7) represents a system of two free particles with respective momenta l_1, l_2. Though we have assumed *total positivity* from the outset (i.e., L points forward in the future), we have not necessarily *individual positivity* (i.e., both l_1 and l_2 future-oriented timelike vectors). The formulas written so far in this subsection are quite general. They also apply in a mixed situation, which could be interpreted as featuring particle with antiparticle, provided the total momentum is future oriented and timelike. Such a case will not be discussed in this work. From now on, we shall be concerned with both l_1 and l_2 pointing in the future. In fact, a criterion of individual positivity is available. We can check that

$$L^2 > |{l_1}^2 - {l_2}^2| \tag{3.12}$$

is equivalent to

$$L\cdot l_2>0 \text{ and } L\cdot l_1>0\ . \tag{3.13}$$

Proof. Write (3.12) as

$$L^2>l_1{}^2-l_2{}^2\ ,$$

$$L^2>l_2{}^2-l_1{}^2\ ,$$

insert

$$L^2\equiv L\cdot l_1+L\cdot l_2\ ,$$

and develop and subtract $L\cdot l_1$ (respectively, $L\cdot l_2$) from both sides (this argument is general and does not require the timelikeness of l_1,l_2). In this situation, where L is timelike and future oriented, and while $m_1{}^2$ and $m_2{}^2$ are *a priori* assumed non-negative, Eq. (3.12) expresses that both particles have positive energy.

C. Extension of the scattering operator

The remark made in Sec. III A implies that the scattering operator can be extended to on-shell states *insofar as* (for each admissible K) S_K can be extended to the kinetic-energy shell. This is actually possible provided S_K maps into itself the space $\mathscr{S}_K$ made of the functions of fast decrease in $\hat r$. In this case, S_K is defined as a linear operator in the space $\mathscr{S}_K^*$ of tempered distributions in $\hat r$, and it is unitary in the rigged Hilbert space[14]:

$$\mathscr{S}_K\subset L_K^2\subset \mathscr{S}_K^*\ .$$

In particular, the action of S_K on $\exp(i\hat Y_K\hat r_K)$ yields a distribution. Let us develop it on the basis $\exp(i\hat Z_K\hat r_K)$. We have

$$S_K\exp[i(\hat Y\cdot\hat r)_K]=\int \sigma(K,\hat Y_K,\hat Z_K)\exp[i(\hat Z\cdot\hat r)_K]d^3\hat Z_K\ , \tag{3.14}$$

where the (continuous) matrix element σ is a distribution in $\hat Y$ and $\hat Z$, but depends on K as a parameter. By unitarity, σ satisfies

$$\int \sigma^*(K,\hat Y,\hat Z)\sigma(K,\hat Y',\hat Z)d^3\hat Z=\delta(\hat Y-\hat Y')\ , \tag{3.15}$$

$$\int \sigma(K,\hat Y,\hat Z)\sigma^*(K,\hat Y',\hat Z)d^3\hat Z=\delta(\hat Y-\hat Y') \tag{3.16}$$

(with three-dimensional δ functions).

Of course, if we consider two different wave functions (like in Sec. II), Eqs. (3.15) and (3.16) imply that

$$\langle S_K\varphi,S_K\varphi'\rangle_K=\langle\varphi,\varphi'\rangle_K\ .$$

Applying (2.11) with S as A and (2.8) in (2.10), we find

$$\langle S\psi,S\psi'\rangle=\langle\psi,\psi'\rangle\ .$$

The invertibility of S follows from that of S_K by the same way, and finally S is unitary in $L^2(\mathsf{R}^8,d^4Kd^4r)$.

Remark. When the potential depends upon K only through $\hat r$ and $\hat y$, the question of whether or not σ actually exists is a purely nonrelativistic problem [existence of the S matrix for the potential $V(\vec r,\vec y)$]. We assume hereafter that S_K does map $\mathscr{S}_K$ into itself ($\forall K$). Accordingly, (3.14)—(3.16) are valid. We do not examine here the detailed condition under which this property holds true. Indeed, we consider that this nonrelativistic problem has been previously solved (at least implicitly—see the specialized literature about scattering theory). We also remind the reader that (for each K and u) S_K is the scattering operator of a nonrelativistic system. In particular, S_K commutes with $\bar E$ and does not lead out of the kinetic-energy shell.

Now S can be extended, by (2.8) and (2.12), to a larger class of states characterized by the fact that $\varphi(K,u,\hat r)$ is a tempered distribution in $\hat r$ but remains an ordinary (retarded) function of K. Such states can possibly be on the mass shell. But still, the plane wave (3.7) is not attainable from this definition. In fact, for (3.7) we have

$$\varphi(K,u,\hat r)=(2\pi)^{5/2}\delta(K-L)\delta(u-Y^{(0)})e^{i\hat Y\cdot\hat r}\ . \tag{3.17}$$

Thus, we are obliged to go one step further, setting up a definition which applies to any plane wave. We proceed as follows. Let $A(K,\hat r,\hat y)$ be an operator of the type considered in Sec. II A, with this additional property that A_K carries $\mathscr{S}_K$ into itself, $\forall K$. Then A_K is extended to the whole $\mathscr{S}_K^*$ and satisfies an equation similar to (3.14). Let α be its matrix element in $\mathscr{S}_K^*$ with respect to the extended basis $e^{i(\hat Y\hat r)_K}$. Now, when ψ is of the form (3.7), we *define*

$$A\psi=(2\pi)^2\delta(K-L)e^{iY^{(0)}r^{(0)}}A_Le^{i(\hat Y\hat r)_L}\ , \tag{3.18}$$

assuming (3.7) means that ψ is an eigenvector of K and y with the respective eigenvalues L,Y. This suggests we call $|L,Y\rangle$ the corresponding state vector in the *bracket notation.*

Expression (3.18) is manifestly a tempered distribution in the variables K,r. Take a frame adapted to L, write $e^{iY^0r^0}$ as

$$\int e^{iZ^0r^0}\delta(Z^0-Y^0)dZ^0\ ,$$

and develop the last term in (3.18) with the help of α, the R_L^3 kernel of A. This yields,[15] in bracket notation,

$$A\,|\,L,Y\rangle=\int |\,L,Z\rangle\delta(Y^0-Z^0)\alpha_L(\vec{\mathrm{Y}},\vec{\mathrm{Z}})d^4Z\ . \tag{3.19}$$

Hence, we can compute, in the plane-wave basis, the matrix element of A as a linear operator in $\mathscr{S}^*(\mathsf{R}^8,d^4K,d^4r)$:

$$\langle L'Z\,|\,A\,|\,LY\rangle=\delta(L-L')\delta(Y_L^{(0)}-Z_L^{(0)})\alpha(L,\hat Y,\hat Z)\ . \tag{3.20}$$

There is no difficulty in checking that this generalization of A is compatible with the point of view developed in Sec. II; we leave it to the reader.

When A is the scattering operator S, Eq. (3.20) gives the S matrix by substitution of σ in place of α. Equation (3.18) becomes explicitly

$$S\psi=(2\pi)^2\delta(K-L)e^{iY^{(0)}r^{(0)}}S_Ke^{i\hat Y\cdot\hat r}\ . \tag{3.21}$$

It is obvious that $[S,K]\psi=0$. Moreover, $[y\cdot K,S]\psi=0$ for the following reason. On the one hand, $y\cdot K\psi=Y\cdot L\psi$, hence $Sy\cdot K\psi=Y\cdot LS\psi$. On the other hand, we can compute $y\cdot KS\psi$ from (3.21). Since $y\cdot K$ does not act on $\hat r$, one finds $y\cdot KS\psi = Y\cdot LS\psi$. Finally, we also have

$$[S,\hat{y}^2]=0 .$$

Indeed, our assumptions mean that S_K commutes with $\bar{E}$, which is proportional to $\hat{y}^2$ (in other words, S_K does not lead out of the free kinetic-energy shell), and noticing that $\hat{y}^2$ acts neither on K nor on $r^{(0)}$, we deduce easily by (3.21) that S commutes with $\hat{y}^2$.

Now we can use the identity expressing y^2 in terms of $\hat{y}^2$, $y\cdot K$, and K^2. From commutation with $\hat{y}^2$, $y\cdot K$, and K^2, it follows that S commutes with y^2 also. Recalling that the plane waves $|LY\rangle$ form a basis, we summarize these results and write

$$[S,K^\alpha]=[S,y\cdot K]=[S,y^2]=0 \tag{3.22}$$

in the whole space $\mathscr{S}^*_+$ of distributions in K,r retarded in K. A look at the wave equations (3.1) and (3.2) with vanishing V shows that the free mass shell is made of the eigenstates common to $y\cdot K$ and $2y^2+\frac{1}{2}K^2$. S commutes with them, and thus *the mass shell for each particle is stable by the action of S* as indicated in Sec. II. Still, we face the question of making sure that the S operator transforms correctly the states of positive energy.

Pure positive-energy states are superpositions of plane waves with individual positivity (both l_1 and l_2 point toward the future). (The mixed case of $L^2>0$ with l_1 pointing towards the future and l_2 towards the past—or the reverse—will not be considered here.) Let us prove that actually *pure positivity is preserved by the action of S*. By superposition, it is sufficient to prove it when the incident wave is plane. Supposing that $|L,Y\rangle$ enjoys individual positivity in the sense of Sec. III B, consider $S|LY\rangle$, which is given by (3.21). From (3.14) we have, in fact,

$$S|LY\rangle=\int |LZ\rangle\sigma(L,\hat{Y},\hat{Z})d^3\hat{Z}_L , \tag{3.23}$$

where $Z_L^{(0)}=Y_L^{(0)}$ [utilize (3.21) with S_L and $(\hat{Y}\cdot\hat{r})_L$]. Let us see that all the waves $|LZ\rangle$ in the above development exhibit individual positivity.

Like $|L,Y\rangle$, each wave $|L,Z\rangle$ is on a mass shell. It is an eigenstate of P with the same eigenvalue L. But, L and Z satisfy a couple of equations similar to (3.8) and (3.9) with, perhaps, other values of a, m_1, and m_2. Since $Z_L^{(0)}=Y_L^{(0)}$, Z is not completely arbitrary. We have, in fact,

$$L\cdot Z=Y\cdot L .$$

By (3.10), we see that $|LY\rangle$ and $|LZ\rangle$ correspond to the same value of a, where a is defined (Sec. III A) as $\frac{1}{2}({m_1}^2-{m_2}^2)$. So, $|L,Z\rangle$ and $|LY\rangle$ correspond to the same L and the same ${m_1}^2-{m_2}^2$. But the positivity criterion (3.12) depends only upon L and ${m_1}^2-{m_2}^2$. Being satisfied by $|L,Y\rangle$ it is also satisfied by $|L,Z\rangle$, which proves our statement.

IV. PHYSICAL UNITARITY

A. A useful property

As mentioned previously, from (3.20) we have explicitly the S matrix

$$\langle L'Z|S|LY\rangle=\delta(L-L')\delta(Y_L^{(0)}-Z_L^{(0)})\sigma(L,\hat{Y},\hat{Z}) . \tag{4.1}$$

From (3.15) and (3.16) it is easy to check that

$$S^*S=SS^*=1 , \tag{4.2}$$

where S^* is conjugate to S with respect to the scalar product (2.9), or equivalently, defining S^* by its matrix elements from (4.1). The physical interest of this property already emerges if one is aware of asymptotic completeness. Its very meaning could be clarified if a probabilistic interpretation of the scalar product (2.9) were to be found. This interesting question opens a number of topics which deserve investigation. But for the moment, we refer to (4.2) as *mathematical unitarity* because we like to stress its importance with respect to the formalism.

B. The physical inner product

Let $|L,Y\rangle$ and $|L',Y'\rangle$ be two distinct plane waves *on the same* (free) mass shell. That is to say L' and Y' as well as L and Y satisfy Eqs. (3.8) and (3.9) with the same *fixed* values of m_1,m_2. In terms of l_1,l_2 and l'_1,l'_2, defined as in (3.11), Eqs. (3.8) and (3.9) represent the double hyperboloid $\eta_{m_1}\times\eta_{m_2}$ associated with the masses m_1,m_2. Let us assume that both $|LY\rangle$ and $|L'Y'\rangle$ are of positive energy.

For the sake of convenience in the forthcoming calculations, let us revert to the initial X,r representation. [We may notice that a lot of useful formulas—for instance Eq. (3.23)—are not representation dependent.] Now, the plane waves are represented by

$$\Psi\equiv\Psi_{LY}=e^{iL\cdot X}e^{iY\cdot r} ,$$

$$\Psi'\equiv\Psi_{L'Y'}=e^{iL'\cdot X}e^{iY'\cdot r} .$$

The physical inner product $(\Psi_{LY},\Psi_{L'Y'})$ can be defined as follows:

$$D_{m_1m_2}(L,L',Y,Y')=\left[\frac{L^0}{2}+Y^0\right]\left[\frac{L^0}{2}-Y^0\right]\times\delta(\vec{L}-\vec{L}')\delta(\vec{Y}-\vec{Y}') , \tag{4.3}$$

in which L^0 and Y^0 must be understood as related to $\vec{L}$ and $\vec{Y}$ through (3.8) and (3.9) (hence, the dependence upon m_1,m_2).

From (3.11) we easily recognize the product of Jordan Pauli distributions in

$$l_1-l_2,l'_1-l'_2 . \tag{4.4}$$

This point of view is automatically consistent with the usual definition of the scalar product for any pair of wave functions which are on the m_1,m_2 mass shell with positive energy (sixfold integration over the positive sheet of $\eta_{m_1}\times\eta_{m_2}$).

Then the question arises whether this physical scalar product is invariant by S (more precisely, by the on-shell m_1,m_2 restriction of S). In fact, the answer is yes. Since the plane waves $|LY\rangle$ satisfying (3.8) and (3.9) for a basis of the on-shell space, it is enough to prove that S leaves invariant their physical scalar product.

We shall compute $(S\Psi,S\Psi')$ and show that it is just equal to $(\Psi,\Psi')=D$. It is useful to notice that when $f(L,Y)$ is a distribution defined on the positive sheet of $\eta_{m_1}\times\eta_{m_2}$ [the six-dimensional surface defined by (3.8), (3.9), and (3.12)], then

$$f(L,Y)D=f(L',Y')D \ . \tag{4.5}$$

Note that the products in (4.5) are well defined. Indeed, D is obtained [by Eqs. (4.3)] from a function of polynomial growth multiplied by $\delta(\vec{L}-\vec{L}')\delta(\vec{Y}-\vec{Y}')$, which is a distribution but in the variables $\vec{L}-\vec{L}'$, $\vec{Y}-\vec{Y}'$, which are independent from the arguments of f. As seen in (3.23),

$$S\Psi_{LY}=\int \Psi_{LZ}\sigma(L,\hat{Y}\hat{Z})d^3\hat{Z}_L \ , \tag{4.6}$$

where Z is submitted to the condition $Z_L^{(0)}=Y_L^{(0)}$. Similarly,

$$S\Psi_{L'Y'}=\int \Psi_{L'Z'}\sigma(L',\hat{Y}'\hat{Z}')d^3\hat{Z}'_{L'} \ , \tag{4.7}$$

where Z' is submitted to the condition $Z_L'^{(0)}=Y_L'^{(0)}$. Though the subscripts are omitted in the variables with carets for typographic simplicity, L is understood in (4.6) while L' is understood in (4.7).

The physical inner product $(\Psi_{LZ},\Psi_{L'Z'})$ is given by (4.3), that is,

$$(LZ,L'Z')=D(L,L',Z,Z') \ .$$

By sesquilinearity we have, thus,

$$(S\Psi,S\Psi')=\int \sigma^*(L,\hat{Y}_L,\hat{Z}_L)\sigma(L',\hat{Y}'_{L'},\hat{Z}'_{L'}) \times D(L,L',Z,Z')d^3\hat{Z}_L d^3\hat{Z}'_{L'} \ .$$

Taking (4.5) into account, we may simply replace L' by L. Hence,

$$(S\Psi,S\Psi')=\int \sigma^*(L,\hat{Y},\hat{Z})\sigma(L,\hat{Y}',\hat{Z}') \times D(L,L',Z,Z')d^3\hat{Z}d^3\hat{Z}' \ , \tag{4.8}$$

where all the quantities with carets are orthogonal to the same L.

In an L-adapted frame we have, more explicitly,

$$(S\Psi,S\Psi')=\int \sigma_L^*(\vec{Y},\vec{Z})\sigma_L(\vec{Y}',\vec{Z}') \times D(L,L',Z,Z')d^3\vec{Z}d^3\vec{Z}' \tag{4.9}$$

with $\sigma_L(\vec{Y},\vec{Z})=\sigma(L,\hat{Y},\hat{Z})$, etc. From (4.3) and the conditions $Z^0=Y^0$, $Z'^0=Y'^0$ we have

$$D(L,L',Z,Z')=\left[\frac{L^0}{2}+Y^0\right]\left[\frac{L^0}{2}-Y^0\right] \times\delta(\vec{L}-\vec{L}')\delta(\vec{Z}-\vec{Z}') \ . \tag{4.10}$$

Insert (4.10) into (4.9) and integrate over $\vec{Z}'$. This provides

$$(S\Psi,S\Psi')=\int \sigma_L^*(\vec{Y},\vec{Z})\sigma_L(\vec{Y}',\vec{Z})\left[\frac{L^0}{2}+Y^0\right] \times\left[\frac{L^0}{2}-Y^0\right]\delta(\vec{L}-\vec{L}')d^3\vec{Z}$$

$$=\left[\frac{L^0}{2}+Y^0\right]\left[\frac{L^0}{2}-Y^0\right]\delta(\vec{L}-\vec{L}') \times\int \sigma_L^*(\vec{Y},\vec{Z})\sigma_L(\vec{Y}'\vec{Z})d^3\vec{Z} \ .$$

But (3.15) reads simply here

$$\int \sigma_L^*(\vec{Y},\vec{Z})\sigma_L(\vec{Y}'\vec{Z})d^3\vec{Z}=\delta(\vec{Y}-\vec{Y}') \ .$$

Therefore,

$$(S\Psi,S\Psi')=\left[\frac{L^0}{2}+Y^0\right]\left[\frac{L^0}{2}-Y^0\right] \times\delta(\vec{L}-\vec{L}')\delta(\vec{Y}-\vec{Y}') =D(L,L',Y,Y') \ ,$$

which proves that S is unitary with respect to the on-shell product (Ψ,Ψ').

1. *Application*

We consider here the simplest case of scattering, assuming *a purely continuous spectrum* (no bound state).[16]

Let ϕ be on the interacting mass shell [i.e., ϕ satisfies a system equivalent to (3.1) and (3.2)] with the interaction term. For each ϕ of this kind, one can associate the asymptotic states $\phi_{\text{in}},\phi_{\text{out}}$, which are *free*, through the formulas

$$\phi=\Omega^-\phi_{\text{out}}=\Omega^+\phi_{\text{in}} \ . \tag{4.11}$$

Let ϕ' be another state on the interacting mass shell. By (4.11), we associate to it the free states $\phi'_{\text{in}},\phi'_{\text{out}}$. Multipling the first equation (4.11) by Ω^{-*} and using the isometry of the Møller operators,[17] we obtain

$$\phi_{\text{out}}=\Omega^{-*}\phi \ .$$

Using the second equation (4.11), we replace ϕ by $\Omega^+\phi_{\text{in}}$, which finally yields

$$\phi_{\text{out}}=S\phi_{\text{in}} \ , \tag{4.12}$$

as usual. Then, the physical unitarity of S provides a solution to the problem of normalizing the interacting mass shell.

Indeed, it is natural to define

$$(\phi,\phi')_{\text{in}}=(\phi_{\text{in}},\phi'_{\text{in}}) \ ,$$

$$(\phi,\phi')_{\text{out}}=(\phi_{\text{out}},\phi'_{\text{out}}) \ .$$

It is obvious from (4.12) that these two definitions coincide because S is unitary with respect to the physical inner product of the free states. The above formula defines in the present case a unique (ϕ,ϕ'), which reduces to the scalar product of free states when no interaction is present.

V. CONCLUSION

We have shown that a wide class of potentials permits us to set the problem of relativistic scattering as a parameter-dependent family of problems tractable by well-known methods of nonrelativistic scattering. Note that, even in the off-shell case, we have used a framework which is more flexible than an $L^2(\mathrm{R}^8)$ axiomatic that we had tentatively proposed in our first article on this subject.[3]

A more accurate mathematical treatment could be of interest. For instance, one can ask whether, in the simplest cases (simple central potential), a convergence stronger than that in L_K^2 might be obtained for the wave operators, and compare this with some of the results in Ref. 5.

The correspondence between the mas shell and the energy shell of the reduced problem has been emphasized. Intuitively, this correspondence is the key idea of our method. Under certain sufficient assumptions, it has been proven that the scattering operator does not lead out of the mass shell, preserves the positivity of the energy, and is unitary in two ways: not only in the off-shell framework, but also with respect to the usual inner product of the solutions of the Klein-Gordon equation. These properties are essential in view of the probabilistic interpretation. That is why it would be worthwhile to try to extend these results for more general interactions.

The question of N-body scattering is also very exciting, but it is already rather complicated to construct N-body quantum systems. Thus, we guess it will remain an open question for awhile.

Let us conclude with a remark. The results we have obtained are not at all surprising. They hold in quantum field theory insofar as S-matrix theory can be rigorously derived from quantum field theory. But in our approach it is ensured step by step that these expected scattering properties are consistent with a certain explicit model of interaction.[18]

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[1]For a recent survey of these matters, see *Relativistic Action at a Distance: Classical and Quantum Aspects (Lecture Notes in Physics),* proceedings of the Workshop, Barcelona, Spain, 1981, edited by J. Llosa (Springer, Berlin, 1982), Vol. 162.

[2]Ph. Droz-Vincent, Rep. Math. Phys. 8, 79 (1975).

[3]Ph. Droz-Vincent, Nuovo Cimento 58A, 355 (1980).

[4]Alternatively, the N masses for particles with spin.

[5]L. P. Horwitz and F. Rohrlich, Phys. Rev. D 24, 1528 (1981); 26, 3452 (1982).

[6]I. T. Todorov, Phys. Rev. D 3, 2351 (1971); Dubna Report No. JINR E2-10125, 1976 (unpublished), and contribution in Ref. 1.

[7]For a possible second quantization, see Ph. Droz-Vincent, contribution in Ref. 1.

[8]Similar simplifications have been recognized also at the classical level: Ph. Droz-Vincent, Ann. Inst. Henri Poincaré XXVII, 407 (1977); C. R. Acad. Sci. B290, 115 (1980).

[9]If it were technically necessary we could impose a stronger retarded condition $K^2>\zeta^2>0$ for some fixed but arbitrary ζ. When we say $\forall K$, we of course mean for each K satisfying the retarded conditions.

[10]I. T. Todorov, contribution in Ref. 1, Sec. 8.

[11]For the rigorous theory of direct-*sum* spaces and operators defined thereon, see F. Riesz and B. Sz. Nagy, *Lecons d'Analyse Fonctionnelle* (Gauthiers-Villars, Paris, 1968), Chap. VIII, Sec. 120, p. 312; N. Dunford and J. Schwartz, *Linear Operators* (Interscience, New York, 1958); for direct *integration*, see J. Dixmier, *Les Algèbres d'Opérateurs dans L'espace Hilbertien* (Gauthier-Villars, Paris, 1969), Chap. II, Sec. 1.5; Chap. II. Sec. 2.3, pp. 158—160.

[12]J. M. Cook, J. Math. Phys. (Cambridge, Mass.) 36, 82 (1957); M. N. Hack, Nuovo Cimento 9, 731 (1958); S. T. Kuroda, *ibid.* 12, 431 (1959).

[13]Under different assumptions, Eq. (2.13) was derived as Eq. (20) in Ref. 3. The conservation of individual masses has been discussed recently in the framework of constraint dynamics (see the last article quoted in Ref. 5).

[14]For an introduction to rigged Hilbert-space methods in physics, see N. N. Bogoliubov, A. A. Logunov, and I. T. Todorov, *Introduction to Quantum Field Theory* (Benjamin, Reading, Mass., 1975), Chap. I, and Secs. 4—6, p. 32 *et seq*; A. Böhm, in *The Rigged Hilbert Space and Quantum Mechanics (Lecture Notes in Physics),* edited by J. Ehlers *et al.* (Springer, Berlin, 1978); A. Böhm, in *Quantum Mechanics* (Springer, Berlin, 1979), Chap. II, Sec. 10 and Chap. XV; such methods have been recently applied in V. Gorini, G. Parravicini, and E. C. G. Sudarshan, J. Math. Phys. 21, 2208 (1980).

[15]We have explicitly

$$\alpha(L\hat{Y}\hat{Z})=\alpha_L(\vec{Y}\vec{Z})$$

in any frame adapted to L.

[16]We mean the case where E_K has a purely continuous spectrum for each K.

[17]As we depart from the $L^2(\mathrm{R}^8)$ scheme, this property has to be checked. In fact, owing to the direct integral structure present in our formulation, and to the isometry of $\Omega_K^{\pm}$ in L_K^2, it is easy to verify that

$$\Omega^{-*}\Omega^{-}=\Omega^{+*}\Omega^{+}=I\ .$$

[18]Some of the results we have presented here have been sketched in compact form (beware of the misprints): Ph. Droz-Vincent, Lett. Nuovo Cimento 33, 383 (1982); 35, 83 (1982).