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Theory of motion for monopole-dipole singularities of classical Yang-Mills-Higgs fields. II. Approximation scheme and equations of motion

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In the preceding paper, the laws of motion were established for classical particles with spin which are monopole-dipole singularities of Yang-Mills-Higgs fields. In this paper, a systematic approximation scheme is developed for solving the coupled nonlinear field equations in any order and for determining the corresponding equations of motion. In zeroth order the potentials are taken as the usual Liénard-Wiechert and Bhabha—Harish-Chandra potentials (generalized to isospace); in this order the solutions are necessarily Abelian, since the isovector describing the charge is constant. The regularization necessary to obtain expressions finite on the world lines of the particles is achieved by the method of Riesz potentials. All fields are taken as retarded and are expressed in integral form. Omitting dipole interactions, the integrals for the various terms are carried out as far as possible for general motions, including radiation-reaction terms. In first order, the charge isovectors are no longer necessarily constant; thus the solutions are not necessarily Abelian, and it is possible for charge to be radiated away. The cases of time-symmetric field theory and of an action-at-a-distance formulation of the theory are discussed in an appendix.

I. INTRODUCTION

In the preceding paper[1] the field equations and the laws of motion for a point particle moving in an (unspecified) classical Yang-Mills field $\vec{A}^{\mu}(x)$ and a classical scalar field (Higgs field) $\vec{\phi}(x)$ were established, using energy-momentum conservation for the coupled system of matter and fields as well as covariant charge conservation. The source terms in the field equation (I2.9) and (I2.12) were given by the expressions (I2.23) and (I2.24) characterizing the densities $\vec{j}^{\mu}(x)$ and $\vec{\rho}(x)$ as being due to the motion of N classical point particles each carrying a monopole and a dipole moment. In this paper we establish the equations of motion, i.e., the laws of motion of the particles in the explicitly determined fields, in various orders of an approximation procedure described in Sec. II, including the radiation-reaction terms of the particles in the order of iteration considered. We first describe the regularization procedure adopted to yield expressions for the fields $\vec{A}^{\mu}(x)$ and $\vec{\phi}(x)$ due to each particle which are finite on its world line (see the first parts of Secs. III and IV, respectively). Use is made of the Riesz method[2–7] which has been applied to similar problems before, e.g, yielding directly the Lorentz-Dirac equation for the motion of a classical relativistic point charge. Some technical details needed in Sec. IV are presented in Appendices A and B.

In the zeroth order of approximation the equations for the $\vec{\phi}$ and $\vec{A}^{\mu}$ fields decouple, while this is no longer the case in the first and higher orders.[8] The fields used in all these calculations are taken to be purely retarded. Evaluating them on the world line then leads to the appearance of radiation-reaction terms in the equations of motion in all orders. No such terms appear if time-symmetric fields are assumed; such a theory is briefly discussed in Appendix C. It is shown there that radiation effects can be obtained by Wheeler-Feynman-type considerations familiar from electro- and mesodynamics. Appendix C also contains a discussion of the possibility of an action-at-a-distance formulation of the theory, and of the difference between this formulation and that of the time-symmetric field theory.

One of our main aims is to study the dynamical effects of the scalar or Higgs field $\vec{\phi}(x)$, which is a classical isovector field associated with a range parameter [cf. Eq. (I2.12)]. It appears that this classical field is a means to simulate, within the framework of a non-Abelian gauge theory with point sources, the effects of an extended source distribution; it allows us to avoid the difficulty of defining in a gauge-independent manner what is actually meant by an extended source distribution $\vec{j}^{\mu}(x)$ in such a theory. We use the Yang-Mills current (I2.24) (with the dipole terms being dropped later) for non-Abelian point particles and the gauge-covariant current density associated with the scalar field $\vec{\phi}(x)$ [Eq. (I2.14)]. This will induce a spread-out non-Abelian charge distribution and thus introduce the effects of an extended source into the classical theory in a gauge covariant manner.[9] The range parameter (with dimension of an inverse length) determines the domain in space-time over which $\vec{\phi}(x)$ varies appreciably.

Furthermore, the classical Higgs field represents an isovector field which is determined as part of the nonlinear dynamics, and simultaneously provides a direction field which can be used to form appropriate gauge-invariant quantities, i.e., one can project a gauge-dependent quantity appearing in the equations onto the Higgs direction and thereby obtain a gauge-invariant expression. This can be done order by order in the approximation scheme described in Sec. II. The charge associated with a particle interacting with the Yang-Mills field is proportional to a classical isospin vector $\vec{\tau}$. The equation determining the time variation of this charge is determined in each order in the process of solving the field equations. In the absence of dipole Yang-Mills interactions, it is the interaction of the particle with the Yang-Mills and Higgs fields which provides the possibility of such a time variation in first and higher orders; in zeroth order $\vec{\tau}$ is necessarily constant and thus the solutions are Abelian. The possibility of nonvanishing time variation of $\vec{\tau}$ in first order provides a mechanism for radiation of charge from a system of interacting particles. The question whether such a classical radiation is possible was one of the main motivations for the present investigation, as discussed in the preceding paper, and it appears that an important dynamical effect of the Higgs field is that it contributes to such a process.

II. THE APPROXIMATION METHOD

The laws of motion found in the preceding paper are exact. The fields entering these laws must be determined from the nonlinear set of coupled partial differential equations (I2.11) and (I2.12).

To avoid unnecessary notational complications, we shall in the following concentrate our attention on a single particle and drop all subscripts i. The results obtained can be generalized immediately to the case of N particles, however, and we shall restore the subscripts when needed.

In Eqs. (I2.11) and (I2.12) it is clear that the nonlinear terms due originally to covariant derivatives are characterized by appropriate powers of the dimensionless constant g, and similarly for the nonlinear terms due to $F(g^2\vec{\phi}^{\,2}/\chi^2)$ [defined in (I2.10), where g^2 had been explicitly introduced for that purpose]. We now consider the formal expansions

$$\vec{A}^{\mu}=\sum_{n=0}^{\infty} g^n{}_n\vec{A}^{\mu}\ ;\ \ \vec{\phi}=\sum_{n=0}^{\infty} g^n{}_n\vec{\phi}\ . \tag{2.1}$$

Then we can write Eqs. (I2.11) and (I2.12) as

$$\sum_{n=0}^{\infty} g^n(\Box_n\vec{A}^{\nu}-\partial^{\nu}{}_{\mu n}\vec{A}^{\mu})=4\pi\sum_{n=0}^{\infty} g^n{}_n\vec{j}^{\,\nu}+\sum_{n=0}^{\infty} g^n{}_n\vec{K}^{\nu}\ , \tag{2.2}$$

$$\sum_{n=0}^{\infty} g^n(\Box+\chi^2)_n\vec{\phi}=4\pi\sum_{n=0}^{\infty} g^n{}_n\vec{\rho}+\sum_{n=0}^{\infty} g^n{}_n\vec{\Lambda}\ , \tag{2.3}$$

where the ${}_n\vec{K}$'s and ${}_n\vec{\Lambda}$'s stand for the nonlinear terms of the appropriate orders, while the first sums on the right-hand sides of Eqs. (2.2) and (2.3) represent the expansions of the sources [compare Eqs. (I2.23) and (I2.24)].

In an earlier paper[6] (in which we did not introduce Higgs fields) as well as in the corresponding treatment of the nonlinear meson theory described in Ref. 8 we did not have to expand the source terms, because we restricted ourselves to monopole moments. Here, however, we have included dipole moments, and by Eqs. (I2.23) and (I2.24) the corresponding source terms involve the fields through the covariant derivatives.

The matter tensor also involves fields, as can be seen from Eqs. (I2.93) and (I2.99), because several of the terms introduced in the break-up (I2.48) must include contributions due to the fields for all conditions required to be satisfied.[10] Thus we must expand $T_m^{\mu\nu}$ as

$$T_m^{\mu\nu}=\sum_{n=0}^{\infty} g^n\int_{-\infty}^{\infty}\{{}_np^{\mu\nu}(\tau)\delta^4(s^\sigma)+\partial_\rho[{}_np^{\rho\mu\nu}(\tau)\delta^4(s^\sigma)]\}d\tau\ . \tag{2.4}$$

In the lowest order in g, Eqs. (2.2) and (2.3) reduce to

$$\Box_0\vec{A}^{\nu}-\partial^{\nu}{}_{\mu\,0}\vec{A}^{\mu}=4\pi_0\vec{j}^{\,\nu}\ , \tag{2.5}$$

$$(\Box+\chi^2)_0\vec{\phi}=4\pi_0\vec{\rho}\ , \tag{2.6}$$

with

$${}_0\vec{j}^{\,\nu}=\int_{-\infty}^{\infty}\{\vec{Q}^{\nu}(\tau)\delta^4(s^\sigma)+\partial_\rho[\vec{S}^{\rho,\nu}(\tau)\delta^4(s^\sigma)]\}d\tau\ , \tag{2.7}$$

$${}_0\vec{\rho}=\int_{-\infty}^{\infty}\{\vec{S}(\tau)\delta^4(s^\sigma)+\partial_\rho[\vec{S}^{\rho}(\tau)\delta^4(s^\sigma)]\}d\tau\ . \tag{2.8}$$

Therefore, in zeroth order, the Yang-Mills and Higgs fields are decoupled. Equations (2.6) and (2.8) are identical (except for notation and inclusion of a dipole term) with the equations of lowest order of the scalar meson theory treated in Ref. 8. Equations (2.5) and (2.7) differ from those of the electromagnetic field only in referring to vectors in charge space. The energy-momentum tensor (I2.16) of the fields reduces to

$$4\pi_0T_f{}^{\mu\nu}={}_0\vec{F}^{\mu}\cdot{}_0\vec{F}^{\nu}-\tfrac{1}{2}\eta^{\mu\nu}{}_0\vec{F}^{\rho}\cdot{}_0\vec{F}_{\rho}+\tfrac{1}{2}\chi^2\eta^{\mu\nu}{}_0\vec{\phi}\cdot{}_0\vec{\phi}-{}_0\vec{F}^{\mu\rho}\cdot{}_0\vec{F}^{\nu}{}_{\rho}+\tfrac{1}{4}\eta^{\mu\nu}{}_0\vec{F}_{\rho\sigma}\cdot{}_0\vec{F}^{\rho\sigma}\ , \tag{2.9}$$

where

$${}_0\vec{F}^{\mu}=\partial^{\mu}{}_0\vec{\phi};\ \ {}_0\vec{F}^{\mu\nu}=\partial^{\mu}{}_0\vec{A}^{\nu}-\partial^{\nu}{}_0\vec{A}^{\mu}\ , \tag{2.10}$$

and similarly

$${}_0T_m^{\mu\nu}=\int_{-\infty}^{\infty}\{{}_0p^{\mu\nu}(\tau)\delta^4(s^\sigma)+\partial_\rho[{}_0p^{\rho\mu\nu}(\tau)\delta^4(s^\sigma)]\}d\tau\ . \tag{2.11}$$

We have

$$\partial_\mu({}_0T_m^{\mu\nu}+{}_0T_f^{\mu\nu})=0=\int_{-\infty}^{\infty}\{{}_0p^{\mu\nu}(\tau)\partial_\mu\delta^4(s^\sigma)+\partial_{\rho\mu}[{}_0p^{\rho\mu\nu}(\tau)\delta^4(s^\sigma)]\}d\tau+{}_0\vec{\rho}\cdot\partial^{\nu}{}_0\vec{\phi}+{}_0\vec{j}_{\mu}\cdot{}_0\vec{F}^{\mu\nu}\ , \tag{2.12}$$

and from Eq. (2.5)

$$\partial_{\nu\,0}\vec{\mathrm{j}}^{\,\nu}=0\;. \tag{2.13}$$

We can proceed from here as we did from Eq. (I2.13) previously and obtain instead of Eq. (I2.43) in zeroth order

$$\frac{d\vec{\mathrm{Q}}}{d\tau}=0\;, \tag{2.14}$$

and thus, using Eq. (I2.77),

$$\frac{d\vec{\tau}}{d\tau}=0\;. \tag{2.15}$$

To obtain the translational and rotational laws of motion, we can proceed from Eq. (2.12) as we did from Eq. (I2.26) before; we note that in the equation for M corresponding to Eq. (I2.98) the second term is absent because of Eq. (2.15), so that the problem of integrability is simplified. We finally obtain instead of Eqs. (I2.100) and (I2.101)

$$\frac{d}{d\tau}\left[[m-\vec{\tau}\cdot(h_1{}_0\vec{\phi}-h_2S_\rho\partial^\rho{}_0\vec{\phi}+\tfrac{1}{2}fB^{\mu\rho}{}_0\vec{\mathrm{F}}_{\mu\rho})]v^\nu+\vec{\tau}\cdot\left[h_2S^\nu\frac{d_0\vec{\phi}}{d\tau}-fB^{\nu\rho}v^\sigma{}_0\vec{\mathrm{F}}_{\sigma\rho}\right]+\dot{B}^{\nu\rho}v_\rho\right]$$
$$=\vec{\tau}\cdot[-h_1\partial^\nu{}_0\vec{\phi}+h_2S_\rho\partial^{\rho\nu}{}_0\vec{\phi}+I_0\vec{\mathrm{F}}^{\nu\rho}v_\rho+fB_{\rho\sigma}\partial^\rho{}_0\vec{\mathrm{F}}^{\nu\sigma}]\;, \tag{2.16}$$

$$\dot{B}^{\mu\nu}-\dot{B}^{\mu\rho}v_\rho v^\nu+\dot{B}^{\nu\rho}v_\rho v^\mu=\vec{\tau}\cdot\left\{h_2\left[S^\mu\left[\partial^\nu{}_0\vec{\phi}-v^\nu\frac{d_0\vec{\phi}}{d\tau}\right]-S^\nu\left[\partial^\mu{}_0\vec{\phi}-v^\mu\frac{d_0\vec{\phi}}{d\tau}\right]\right]\right.$$
$$\left.-f[B^{\mu\rho}({}_0\vec{\mathrm{F}}^\nu{}_\rho+v^\nu{}_0\vec{\mathrm{F}}_{\rho\sigma}v^\sigma)-B^{\nu\rho}({}_0\vec{\mathrm{F}}^\mu{}_\rho+v^\mu{}_0\vec{\mathrm{F}}_{\rho\sigma}v^\sigma)]\right\}\;. \tag{2.17}$$

We now have to determine the form of the zero-order fields entering these equations. The solutions of Eq. (2.6) are well known from classical meson theory.[8] Those of Eq. (2.5), because of the forms (2.7) and (2.8) of the multipole moments, differ from those of classical electrodynamics[12] only by multiplication by $\vec{\tau}$, which is constant by Eq. (2.15).

Equation (2.5) is solved most simply by introducing the Lorentz condition

$$\partial_{\nu 0}\vec{\mathrm{A}}^{\,\nu}=0\;, \tag{2.18}$$

which implies

$$\Box_0\vec{\mathrm{A}}^{\,\nu}=4\pi\,{}_0\vec{\mathrm{j}}^{\,\nu}\;. \tag{2.19}$$

In the following we shall be mainly interested in the retarded solutions of the wave equations (2.6) and (2.19). Although the fields become infinite at the position of the particle, we can obtain finite equations of motion by a variety of methods.[2–8,11–14] All these methods give the same result. In Sec. III below we shall use the Riesz method[2–9] to regularize the fields $\vec{\mathrm{A}}^{\,\nu}$ and $\vec{\phi}$ in various orders of approximation.

It should be noted that, in any classical theory, none of the methods mentioned imply a "renormalization," since one never has to consider a theory without interactions (which would have to be renormalized after interactions are introduced). In the fields calculated by integrating the field equations under consideration some infinities will appear. However, they never enter the equations of motions, since they can be shown to be compensated by appropriate terms in the four-momenta A_i^μ, the angular momentum $B_i^{\mu\nu}$,[15] and (in our case) the Yang-Mills charges.[16] Most methods developed prove this only for particular linear field theories, but a method due to Mathisson[14] proves it for any special-relativistic field theory independent of the particular form of the field equations.

For any particular linear field equation, all the methods developed yield the same results for the fields to be inserted into the laws of motion, i.e., for the particular finite part of the field of the particle evaluated on its own world line; the fields of the other particles, of course, are finite and can be inserted without any difficulty. We have found it most convenient to use the Riesz method for the evaluation of the finite "self-action" part of the field (see Sec. III).

Of course, for any equation of the types (2.6) or (2.19) the solution is not unique, but depends on the Green's functions chosen. In this and the next section, we shall always take the retarded Green's functions. Evaluating the field of the particle on its own world line then yields finite self-action ("radiation reaction") terms. Because of Eq. (2.15) the self-action terms due to the Higgs field reduce to those of the neutral scalar meson field in the order considered now; however, they will have to be reevaluated in the next order, when Eq. (2.15) no longer holds.

However, even with the simplification due to Eq. (2.15) the radiation reaction terms originating from the dipoles (though known, as noted above), are very lengthy, and in higher orders the calculations would become prohibitively complicated. Therefore, we shall *drop all dipole terms from here on* and deal only with the case in which the exact equations are given by (I3.1)—(I3.3). Specializing to spinless particles one would have to set in addition $B^{\mu\nu}$ equal to zero in these equations. However, we shall keep the $B^{\mu\nu}$ terms and the rotational equations of motion

(I3.2) which are identical to (2.17) with the right-hand side (RHS) put equal to zero. If desired, one can easily specialize the subsequent formulas to the case of spinless Yang-Mills-Higgs particles by setting $B^{\mu\nu}=0$, satisfying the rotational equations of motion in a trivial way. Equations (2.7) and (2.8) are thus replaced by (now dropping the superfluous subscript 0, but remembering that we are dealing with a many-particle problem)

$$\vec{j}^{\,\mu}=\int_{-\infty}^{\infty}\vec{Q}^{\mu}(\tau)\delta^4(x-z(\tau))d\tau\ , \tag{2.20}$$

$$\vec{\rho}=\int_{-\infty}^{\infty}\vec{S}(\tau)\delta^4(x-z(\tau))d\tau\ . \tag{2.21}$$

In Eqs. (2.20) and (2.21) we considered a single particle; for many particles, each would be represented by a similar term (with label i) on the RHS of (2.20) and (2.21) and would thus contribute an expression of the form (2.22) or (2.25) below to the total ${}_0\vec{A}^{\nu}$ or ${}_0\vec{\phi}$ field, respectively. Also, as long as we are dealing with a single particle it is notationally shorter to continue using $\vec{Q}$, $\vec{Q}^{\mu}$, and $\vec{S}$ in the equations without relating them to $\vec{\tau}$ through Eqs. (I2.77), (I2.78), and (I2.80).

For Eq. (2.19) the retarded solution is the familiar Liénard-Wiechert (LW) potential generalized to isospace

$$ {}_0\vec{A}^{\nu}(x)=\left.\frac{\vec{Q}v^{\nu}}{R^{\mu}v_{\mu}}\right|_{\tau=\tau_{\rm ret}}\ , \tag{2.22}$$

where we have used Eq. (I2.78) to replace $\vec{Q}^{\nu}$ by $\vec{Q}v^{\nu}$, and where

$$\begin{aligned} &R^{\mu}(\tau_{\rm ret})=x^{\mu}-z^{\mu}(\tau_{\rm ret})\ ,\\ &R^{0}(\tau_{\rm ret})>0 \end{aligned} \tag{2.23}$$

is the retarded distance of field and source point obeying

$$R^{\mu}(\tau_{\rm ret})R_{\mu}(\tau_{\rm ret})=0\ . \tag{2.24}$$

Retarded differentiation of (2.22) together with Eq. (2.14) shows that the Lorentz condition (2.18) is fulfilled. The zeroth-order field strength ${}_0\vec{F}^{\mu\nu}$ is given by Eq. (2.10).

Similarly, for Eq. (2.6) with the source term (2.21) one has the Bhabha—Harish-Chandra (BH) solution[13]

$$ {}_0\vec{\phi}(x)=\left.\frac{\vec{S}}{R^{\mu}v_{\mu}}\right|_{\tau=\tau_{\rm ret}}-\chi\int_{-\infty}^{\tau_{\rm ret}}\frac{\vec{S}(\tau)}{R}J_1(\chi R)d\tau\ , \tag{2.25}$$

where J_n is the Bessel function of order n, with

$$\begin{aligned} &R=[R^{\mu}(\tau)R_{\mu}(\tau)]^{1/2}\ ,\\ &R^{\mu}(\tau)=x^{\mu}-z^{\mu}(\tau)\ . \end{aligned} \tag{2.26}$$

The first term on the RHS of Eq. (2.25) is analogous to the retarded LW expression (2.22) for the zeroth-order Yang-Mills potential. The second term is an integral over the history of the motion of the particle which is proportional to the inverse length parameter χ appearing in Eq. (2.6). For a "massless" Higgs field this second part in Eq. (2.25) would be absent. The expressions for the regularized fields ${}_0\vec{A}^{\nu}$ and ${}_0\vec{\phi}$, i.e., for the "Riesz potentials," will be given in the next section. These fields assume finite values on the world line of the particle and allow the computation of finite self-action terms proceeding order by order in the iteration scheme; for fields which are finite *without* regularization one obtains the usual expressions. Of course, it has not been established (here or in any other iteration scheme for a nonlinear theory) that any such method of obtaining a series of finite terms approaches the regularized solution of the exact nonlinear equations.

III. RIESZ REGULARIZATION AND THE ZEROTH-ORDER FIELD EQUATIONS

The expressions (2.22) and (2.25) for the zeroth order retarded LW field ${}_0\vec{A}^{\nu}(x)$ and the retarded BH field ${}_0\vec{\phi}(x)$ are singular when considered for points x on the world line of the particle. Following Refs. 2—4 we regularize these expressions according to the Riesz method and define the Riesz potentials[17]

$$ {}_0\vec{A}^{(\alpha)}_{\nu}(x)=\alpha\int_0^{\infty}\frac{\vec{Q}v_{\nu}}{R^{\mu}v_{\mu}}R^{\alpha-1}dR \tag{3.1}$$

and

$$ {}_0\vec{\phi}^{(\alpha)}(x)=\frac{\chi}{2^{\alpha/2}\Gamma(\frac{1}{2}[\alpha+2])}\times\int_0^{\infty}\frac{\vec{S}}{R^{\mu}v_{\mu}}\left[\frac{R}{\chi}\right]^{\alpha/2}J_{(\alpha-2)/2}(\chi R)dR\ . \tag{3.2}$$

Here R is defined by Eq. (2.26), and the limit $\alpha\to 0$ is understood, yielding the physical fields both possessing the dimension $(\text{length})^{-1}$. The integrals over R from zero to infinity appearing in Eqs. (3.1) and (3.2) could easily be transformed into integrals over proper time running from $\tau=-\infty$ to $\tau=\tau_{\rm ret}$ by using

$$dR=-\frac{\kappa}{R}d\tau\ , \tag{3.3}$$

where

$$\kappa=R^{\mu}(\tau)v_{\mu}(\tau)\ . \tag{3.4}$$

In order to make contact with the solutions (2.22) and (2.25) of the zeroth-order field equations discussed above we use the following well-known formula for the Bessel functions in Eq. (3.2):

$$J_{(\alpha-2)/2}(x)=\frac{\alpha}{x}J_{\alpha/2}(x)-J_{(\alpha+2)/2}(x)\ . \tag{3.5}$$

Taking the limit $\alpha\to 0$ and switching over to τ as the integration variable in Eq. (3.2), it is an easy matter to see that for x *away from the world line* the results obtained for ${}_0\vec{A}^{(\alpha=0)}_{\nu}(x)$ and ${}_0\vec{\phi}^{(\alpha=0)}(x)$ are identical with the LW expression (2.22) and the BH expression (2.25), respectively. For values of x *on the world line,* i.e., for $x=z(\tau_0)$, the Riesz forms of the fields, (3.1) and (3.2), yield the finite results

$$ {}_0\vec{A}^{(\alpha=0)}_{\nu}(x=z(\tau_0))=-\left[\frac{d}{d\tau}(\vec{Q}v_{\nu})\right]_{\tau=\tau_0} \tag{3.6}$$

and

$$ {}_0\vec{\phi}^{(\alpha=0)}(x=z(\tau_0)) = -\left[\frac{d\vec{S}}{d\tau}\right]_{\tau=\tau_0} - \chi\int_{-\infty}^{\tau_0}\vec{S}(\tau)\frac{J_1(\chi R)}{R}d\tau\ , \quad (3.7) $$

where in the last integral

$$ R=[R^{\mu}(\tau)R_{\mu}(\tau)]^{1/2}\ ,\qquad R^{\mu}(\tau)=z^{\mu}(\tau_0)-z^{\mu}(\tau)\ . \quad (3.8) $$

Noting that

$$ \lim_{x\to 0}\frac{1}{x}J_1(x)=\tfrac{1}{2}\ , \quad (3.9) $$

one sees also that the integral on the RHS of Eq. (3.7) is finite. Equations (3.6) and (3.7) will be taken as defining the regularized zeroth order $\vec{A}_\nu$ and $\vec{\phi}$ fields for points on the world line of the particle. To obtain the Yang-Mills field strengths ${}_0\vec{F}_{\mu\nu}$ and the fields ${}_0\vec{F}_\mu=\partial_\mu\, {}_0\vec{\phi}$ appearing in the zeroth-order equations for the translational motion we define the quantities ${}_0\vec{F}_\mu^{(\alpha)}(x)$ and ${}_0\vec{F}_{\mu\nu}^{(\alpha)}(x)$ using Eqs. (2.10) and the Riesz expressions (3.1) and (3.2) and obtain for $\alpha\to 0$ (Ref. 18)

$$ {}_0\vec{F}_{\mu\nu}^{(\alpha=0)}(x)=\begin{cases} -\left[\frac{2\vec{Q}}{\kappa}[a_{[\mu}k_{\nu]}-(a\cdot k)v_{[\mu}k_{\nu]}]\right]_{\tau=\tau_{\text{ret}}}-\left[\frac{2\vec{Q}}{\kappa^2}v_{[\mu}k_{\nu]}\right]_{\tau=\tau_{\text{ret}}} \text{ for } x\neq z(\tau_0)\ , & (3.10\text{a})\\ [\tfrac{4}{3}\vec{Q}\dot{a}_{[\mu}v_{\nu]}]_{\tau=\tau_0} \text{ for } x=z(\tau_0)\ , & (3.10\text{b}) \end{cases} $$

and

$$ {}_0\vec{F}_{\mu}^{(\alpha=0)}(x)=\begin{cases} \left[\frac{1}{\kappa}\frac{d}{d\tau}\left[\frac{\vec{S}R_\mu}{\kappa}\right]\right]_{\tau=\tau_{\text{ret}}}-\frac{\chi^2}{2}\left[\frac{\vec{S}R_\mu}{\kappa}\right]_{\tau=\tau_{\text{ret}}}+\chi^2\int_{-\infty}^{\tau_{\text{ret}}}\vec{S}R_\mu\frac{J_2(\chi R)}{R^2}d\tau \text{ for } x\neq z(\tau_0)\ , & (3.11\text{a})\\ -\tfrac{1}{2}[\tfrac{2}{3}\vec{S}(a^2v_\mu+\dot{a}_\mu)+2\dot{\vec{S}}a_\mu+2\ddot{\vec{S}}v_\mu]_{\tau=\tau_0}-\frac{\chi^2}{2}[\vec{S}v_\mu]_{\tau=\tau_0}+\chi^2\int_{-\infty}^{\tau_0}\vec{S}R_\mu\frac{J_2(\chi R)}{R^2}d\tau \text{ for } x=z(\tau_0)\ . & (3.11\text{b}) \end{cases} $$

To obtain the form of the integrals in Eqs. (3.11) and of the quantity in the second square brackets (appearing as a boundary term), an integration by parts was performed in the original expression for ${}_0\vec{F}_\mu^{(\alpha)}$, using

$$ \frac{d}{dx}(x^{-n}J_n(x))=-x^{-n}J_{n+1}(x) \quad (3.12) $$

as well as Eq. (3.9).

We note that for values x on the world line the Lorentz force term for the regularized self-field of Eq. (3.10) produces the Abraham four-vector (except for the different coupling constants), i.e., the familiar radiation reaction term

$$ \vec{Q}\cdot{}_0\vec{F}_{\mu\nu}^{(\alpha=0)}(x=z(\tau))v^\nu=\vec{Q}^2\tfrac{2}{3}(\dot{a}_\mu+a^2v_\mu)\ , \quad (3.13) $$

as required. Here and in Eq. (3.14) below we have considered an arbitrary point on the world line and replaced τ_0 by τ.

In order to make use of Eqs. (3.11b) in determining that part of the force on the Yang-Mills particle associated with the Higgs field, we write the zeroth-order equations for the translational motion of a single monopole particle in the form

$$ [m-\vec{S}\cdot{}_0\vec{\phi}(x=z(\tau))]a^\nu+\frac{d}{d\tau}(\dot{B}^{\nu\rho}v_\rho) = -\vec{S}\cdot I^{\nu\rho}\,{}_0\vec{F}_\rho(x=z(\tau))+\vec{Q}\cdot{}_0\vec{F}^{\nu\rho}(x=z(\tau))v_\rho \quad (3.14) $$

and

$$ \frac{d\vec{Q}}{d\tau}=0\ . \quad (3.15) $$

Furthermore, (I3.2) is to be satisfied for the $B^{\mu\nu}$. In Eqs. (3.14) $I^{\nu\rho}=I^{\nu\rho}(\tau)$ is the projection operator

$$ I^{\nu\rho}=\eta^{\nu\rho}-v^\nu v^\rho \quad (3.16) $$

obeying[19]

$$ I^{\nu\rho}v_\rho=0\ ,\quad I^{\nu\rho}n_\rho=n^\nu\ ,\qquad n^\mu=k^\mu-v^\mu\ ,\quad n^\mu n_\mu=0\ ,\quad n^\mu k_\mu=-1\ . \quad (3.17) $$

Here $k^\mu=k^\mu(\tau,\theta,\rho)$ is a lightlike vector and n^μ is a spacelike unit vector (compare Appendix B). Equation (3.14) makes explicit that only the component of the fields ${}_0\vec{F}_\rho(x=z(\tau))$ perpendicular to the four-velocity at the retarded time τ enters the equations for the translational motion which is, of course, required by the form of the left-hand side (LHS) of that equation. Moreover, since both $\vec{Q}$ and $\vec{S}$ are proportional to $\vec{\tau}$ with $d\vec{\tau}/d\tau=0$ in zeroth order, the terms proportional to $\dot{\vec{S}}$ and $\ddot{\vec{S}}$ in Eqs. (3.11) disappear, while the terms proportional to v_ρ are annihilated by the projection operator $I^{\nu\rho}$. We thus find for the first term on the RHS of (3.14)

$$-\vec{S}\cdot I^{\nu\rho}{}_0\vec{F}_\rho(x=z(\tau))=\vec{S}^2\tfrac{1}{3}(\dot a^{\,\nu}+a^2v^\nu)-\vec{S}^2 I^{\nu\rho}\chi^2\int_{-\infty}^{\tau}\bar R_\rho\frac{J_2(\chi\bar R)}{\bar R^2}d\bar\tau-\vec{S}\cdot I^{\nu\rho}{}_0\vec{F}_{\mathrm{ex}\rho}(x=z(\tau)) \ . \tag{3.18}$$

Here we have called the integration variable $\bar\tau$, with $\bar R_\rho=\bar R_\rho(\bar\tau)=z_\rho(\tau)-z_\rho(\bar\tau)$, and $I^{\nu\rho}$ is associated with the value τ denoting the upper limit of the integration; the last term is of course due to the external field due to all other particles.

Furthermore, the effective mass appearing on the LHS of Eqs. (3.14) is given by

$$m-\vec{S}\cdot{}_0\vec\phi(x=z(\tau))=m+\vec{S}^2\chi\int_{-\infty}^{\tau}\frac{J_1(\chi\bar R)}{\bar R}d\bar\tau \ , \tag{3.19}$$

where we have used Eq. (3.7), remembering that $\dot{\vec{S}}=0$ in zeroth order. Thus, physically, a shift appears in the mass value for the non-Abelian point particle coupled to a classical Higgs field ${}_0\vec\phi$ which solves Eq. (2.6). This shift is proportional to $\vec{S}^2$ and to the inverse length parameter χ associated with the Higgs field.[20] Moreover, there is a force term (3.18) due to the $\vec\phi$ field appearing in the zeroth-order equations for the translational motion. It is at once apparent that this force term is zero for straight-line motion, i.e., for $\bar R_\rho(\bar\tau)=v_\rho(\tau-\bar\tau)$, where a_ρ vanishes, because of the property (3.17).

In the presence of an external Yang-Mills field ${}_0\vec{F}^{\nu\rho}_{\mathrm{ex}}$ and an external Higgs field ${}_0\vec\phi_{\mathrm{ex}}$ (with ${}_0\vec{F}^{\nu}_{\mathrm{ex}}=\partial^\nu{}_0\vec\phi_{\mathrm{ex}}$) which are the sum of retarded fields of all particles other than the one under consideration, the equations for the translational motion together with the equations for the rotational motion read in zeroth-order approximation

$$\frac{d}{d\tau}\left\{\left[m-h_1\vec\tau\cdot{}_0\vec\phi_{\mathrm{ex}}+h_1{}^2\chi\int_{-\infty}^{\tau}\frac{J_1(\chi\bar R)}{\bar R}d\bar\tau\right]v^\nu+\dot B^{\,\nu\rho}v_\rho\right\}=I\vec\tau\cdot{}_0\vec{F}^{\nu\rho}_{\mathrm{ex}}v_\rho-h_1\vec\tau\cdot{}_0\vec{F}^{\nu}_{\mathrm{ex}}+\tfrac{1}{3}(2I^2+h_1{}^2)(\dot a^{\,\nu}+a^2v^\nu)$$
$$+h_1{}^2\chi^2\left[\tfrac{1}{2}v^\nu-\int_{-\infty}^{\tau}\bar R^{\,\nu}\frac{J_2(\chi\bar R)}{\bar R^2}d\bar\tau\right] \ , \tag{3.20}$$

$$\dot B^{\,\mu\nu}-\dot B^{\,\mu\rho}v_\rho v^\nu+\dot B^{\,\nu\rho}v_\rho v^\mu=0 \ . \tag{3.21}$$

In Eq. (3.20), we have used Eqs. (3.7), (3.10), (3.11) and (2.15) together with $\vec\tau^{\,2}=1$, $\vec{Q}=I\vec\tau$, and $\vec{S}=h_1\vec\tau$ [compare Eqs. (I2.75), (I2.77), and (I2.80)]. For $h_1=0$ (i.e., in the absence of Higgs fields) and $\vec{Q}^2$ replaced by e^2 and, correspondingly, $\vec{Q}\cdot{}_0\vec{F}^{\mu\nu}_{\mathrm{ex}}$ replaced by $eF^{\mu\nu}_{\mathrm{ex}}$, where $F^{\mu\nu}_{\mathrm{ex}}$ is the tensor of the external electromagnetic field, Eq. (3.20) reduces to the Lorentz-Dirac equation of classical electrodynamics.

IV. THE FIRST-ORDER EQUATIONS

A. The first-order laws of motion

To obtain the equations to the next order we note first that since we have dropped the dipole terms, which through the covariant derivatives are field dependent, the source terms will give no *direct* contributions beyond the zero-order terms (2.20) and (2.21); however, they will contribute *indirectly* through various zero-order fields appearing on the RHS of the wave equations [for their final form see Eqs. (4.14) and (4.15) below]. Collecting the terms of orders g^0 and g^1 in Eqs. (2.2) and (2.3) we obtain

$$\Box({}_0\vec{A}^{\,\nu}+g\,{}_1\vec{A}^{\,\nu})-\partial^\nu{}_\mu({}_0\vec{A}^{\,\mu}+g\,{}_1\vec{A}^{\,\mu})=4\pi\vec{j}^{\,\nu}+g\partial_\mu({}_0\vec{A}^{\,\mu}\wedge{}_0\vec{A}^{\,\nu})+g\,{}_0\vec{A}_\mu\wedge{}_0\vec{F}^{\mu\nu}+g\,{}_0\vec\phi\wedge\partial^\nu{}_0\vec\phi \ , \tag{4.1}$$

$$(\Box+\chi^2)({}_0\vec\phi+g\,{}_1\vec\phi)=4\pi\vec\rho+g\,{}_0\vec{A}^{\,\mu}\wedge\partial_{\mu0}\vec\phi+g\partial^\mu({}_0\vec{A}_\mu\wedge{}_0\vec\phi) \ . \tag{4.2}$$

In passing from the zeroth to the first order we are confronted with a difficulty of the same kind as was encountered in the corresponding problem in general relativity (for the slow as well as for the fast approximation method), as well as in nonlinear meson theory, and which was discussed in detail in Refs. 11 and 8. The problem is the following: Eqs. (3.20) and (3.21) with (2.15) imply a certain motion, as well as corresponding fields which are finite along the world line corresponding to this zero-order motion. But these zero-order fields enter the terms of order g^1 in Eqs. (4.1) and (4.2) and through them the first-order equations of motion: they are regularized for the zero-order world line rather than the world line corresponding to the actual motion in the first order, thereby destroying the consistency of the first-order approximation. Similar difficulties occur in the transition from any higher order to the next one.

To avoid these difficulties, we proceed as in Refs. 11 and 8: We assume that as we go from one approximation to the next one, the explicit forms of the lower-order solutions for the fields and of the lower-order matter and field source densities are unchanged, but the restrictions imposed by the lower-order equations of motion as well as by the lower-order Lorentz condition are dropped. Thus, in the mth approximation the unknowns are ${}_m\vec{A}^{\,\mu}$, ${}_m\vec\phi$, and ${}_mp^{\mu\nu}$, and these quantities enter the mth order equations only linearly. Of course, the particle variables z_μ, $B^{\mu\nu}$, and $\vec{\tau}$ on which these quantities depend are expected to be *different* functions of τ in each order m. However, for notational convenience we do not indicate this explicitly.

Equation (4.1) implies

$$\partial_\nu(\vec{j}^{\,\nu}+g_1\vec{J}^{\,\nu})-g_0\vec{A}_\nu\wedge\vec{j}^{\,\nu}=0 \tag{4.3}$$

with

$$4\pi g_1\vec{J}^{\,\nu}=g_0\vec{\phi}\wedge\partial^\nu{}_0\vec{\phi} \tag{4.4}$$

and

$$g\partial_{\nu 1}\vec{J}^{\,\nu}=g_0\vec{\phi}\wedge\vec{\rho}\ . \tag{4.5}$$

We can proceed from here as before and obtain instead of Eq. (2.14)

$$\frac{d\vec{Q}}{d\tau}=g(\vec{S}\wedge{}_0\vec{\phi}+{}_0\vec{A}^{\,\nu}v_\nu\wedge\vec{Q}) \tag{4.6}$$

with ${}_0\vec{A}_\nu(x=z(\tau))$ and ${}_0\vec{\phi}(x=z(\tau))$ given by Eqs. (3.6) and (3.7), respectively. Using Eqs. (I2.77) and (I2.80) one thus has

$$I\frac{d\vec{\tau}}{d\tau}=g\vec{\tau}\wedge(h_1{}_0\vec{\phi}-I_0\vec{A}_\nu v^\nu)\ . \tag{4.7}$$

describing a precession of the charge vector $\vec{Q}=I\vec{\tau}$ during the motion in first order due to the Yang-Mills and Higgs fields.

The energy-momentum tensor ${}_0T^{\mu\nu}+g_1T^{\mu\nu}$ appropriate for Eqs. (4.1) and (4.2) can be obtained easily from Eq. (I2.16). The divergence of the total energy-momentum tensor becomes

$$\partial_\mu({}_0T_m^{\mu\nu}+{}_0T_f^{\mu\nu}+g_1T_m^{\mu\nu}+g_1T_f^{\mu\nu})$$
$$=0=\int_{-\infty}^{\infty}\{({}_0p^{\mu\nu}+g_1p^{\mu\nu})\partial_\mu\delta^4(s^\sigma)+\partial_{\rho\mu}[{}_0p^{\rho\mu\nu}\delta^4(s^\sigma)+g_1p^{\rho\mu\nu}\delta^4(s^\sigma)]\}d\tau+\vec{\rho}\cdot[{}_0\vec{F}^{\,\nu}+g_1\vec{F}^{\,\nu}]$$
$$+\vec{j}_\mu\cdot[{}_0\vec{F}^{\mu\nu}+g_1\vec{F}^{\mu\nu}]\ , \tag{4.8}$$

$$\begin{aligned}{}_1\vec{F}^{\,\nu}&=\partial^\nu{}_1\vec{\phi}-{}_0\vec{A}^{\,\nu}\wedge{}_0\vec{\phi}\ ,\\ {}_1\vec{F}^{\mu\nu}&=\partial^\mu{}_1\vec{A}^{\,\nu}-\partial^\nu{}_1\vec{A}^{\,\mu}-{}_0\vec{A}^{\,\mu}\wedge{}_0\vec{A}^{\,\nu}\ .\end{aligned} \tag{4.9}$$

We can now proceed as in Sec. II of the preceding paper, but in line with our previous arguments on passing from one order of approximation to the next, we break up the *sums* ${}_0p^{\mu\nu}+g_1p^{\mu\nu}$ and ${}_0p^{\rho\mu\nu}+g_1p^{\rho\mu\nu}$, rather than each term separately, into components as in Eqs. (I2.47) and (I2.48). In the equation analogous to Eq. (I2.98) now the second term vanishes to order g^1 because of Eq. (4.7), and the third term is absent, since we no longer include dipole moments. Thus we obtain for the law of translational motion

$$\frac{d}{d\tau}\{[m-h_1\vec{\tau}\cdot({}_0\vec{\phi}+g_1\vec{\phi})]v^\nu+\dot{B}^{\nu\rho}v_\rho\}$$
$$=-\vec{\tau}\cdot\{h_1({}_0\vec{F}^{\,\nu}+g_1\vec{F}^{\,\nu})-I({}_0\vec{F}^{\,\nu\rho}+g_1\vec{F}^{\,\nu\rho})v_\rho\}\ . \tag{4.10}$$

Equation (3.21) for the law of rotational motion remains unchanged.

B. The first-order fields

1. The structure of the fields

We now have to determine the fields which enter the law of translational motion (4.10) and the law of variation of charge (4.7) in first order to obtain the corresponding equations of motion; the law of rotational motion (3.21), of course, does not contain any fields and thus already is the first-order equation of motion.

We first note that Eq. (2.18) to first order is replaced by

$$\partial_\nu({}_0\vec{A}^{\,\nu}+g_1\vec{A}^{\,\nu})=0\ , \tag{4.11}$$

which to order g^1 reduces Eqs. (4.1) and (4.2) to

$$\Box({}_0\vec{A}^{\,\nu}+g_1\vec{A}^{\,\nu})=4\pi\vec{j}^{\,\nu}+g[\partial_\mu({}_0\vec{A}^{\,\mu}\wedge{}_0\vec{A}^{\,\nu})+{}_0\vec{A}_\mu\wedge{}_0\vec{F}^{\mu\nu}$$
$$+{}_0\vec{\phi}\wedge\partial^\nu{}_0\vec{\phi}]\ , \tag{4.12}$$

$$(\Box+\chi^2)({}_0\vec{\phi}+g_1\vec{\phi})=4\pi\vec{\rho}+2g_0\vec{A}^{\,\mu}\wedge\partial_{\mu 0}\vec{\phi}\ . \tag{4.13}$$

From our previous discussion of how to proceed from one order to the next it follows that we can still make use of the zero-order Eqs. (2.14) and (2.6) in these equations, provided that they are understood to contain the source densities corresponding to the motions determined by Eqs. (4.10) and (4.6). We thus get

$$\begin{aligned}\Box_1\vec{A}^{\,\nu}=&\ \partial^\mu{}_0\vec{A}_\mu\wedge{}_0\vec{A}^{\,\nu}+2{}_0\vec{A}_\mu\wedge\partial^\mu{}_0\vec{A}^{\,\nu}\\ &-{}_0\vec{A}_\mu\wedge\partial^\nu{}_0\vec{A}^{\,\mu}+{}_0\vec{\phi}\wedge\partial^\nu{}_0\vec{\phi}\ ,\end{aligned} \tag{4.14}$$

$$(\Box+\chi^2)_1\vec{\phi}=2_0\vec{A}^{\,\mu}\wedge\partial_{\mu 0}\vec{\phi}\ . \tag{4.15}$$

Thus the dynamics of the $\vec{A}^{\,\nu}$ and $\vec{\phi}$ fields is now coupled, and the field combinations appearing as effective source terms in these equations are functionals of the first-order world lines. The solutions of the equations for ${}_1\vec{A}^{\,\nu}$ and ${}_1\vec{\phi}$ have to be inserted in Eqs. (4.10) and (4.6) to give us the final form of the equations of motion in first order.

In accordance with Eqs. (2.2) and (2.3), we denote the effective source densities appearing in the first-order equations by ${}_1\vec{K}^{\,\nu}$ and ${}_1\vec{\Lambda}$:

$$\begin{aligned}{}_1\vec{K}^{\,\nu}=&\ \partial^\mu{}_0\vec{A}_\mu\wedge{}_0\vec{A}^{\,\nu}+2{}_0\vec{A}_\mu\wedge\partial^\mu{}_0\vec{A}^{\,\nu}\\ &-{}_0\vec{A}_\mu\wedge\partial^\nu{}_0\vec{A}^{\,\mu}+{}_0\vec{\phi}\wedge\partial^\nu{}_0\vec{\phi}\ ,\end{aligned} \tag{4.16}$$

$${}_1\vec{\Lambda}=2_0\vec{A}^{\,\mu}\wedge\partial_{\mu 0}\vec{\phi}\ . \tag{4.17}$$

Before solving Eqs. (4.14) and (4.15) we have to discuss the question of gauge fixing implicit in our method of constructing the solution of the nonlinear equations in terms of an iteration of regularized zero-order *Abelian* solutions. We first note that it is always possible to consider an Abelian subclass of solutions of a non-Abelian gauge theory. In our case this amounts to the embedding of the solutions of an Abelian theory into a nonlinear Yang-Mills framework by adopting a particular direction in isospace associated with the Abelian subclass. Usually such a choice of direction is associated with the electromagnetic interaction, which is described by an Abelian gauge group (see Sec. III of the preceding paper). Here we start our iteration scheme with an Abelian solution as an input, implying that we pick a particular direction in isospace (which is, in fact, the same everywhere in space-time). The particular direction we choose is immaterial, since we are free to perform a global isorotation. However, this choice of direction for the zero-order Abelian field must be made known to every observer. Physically one could think of using particles carrying an electric charge e in addition to the non-Abelian charge $\vec{Q}$ and could fix the particular Abelian direction throughout space-time by means of the Abelian electromagnetic interaction. Now, the zero-order Abelian solution for a single particle is characterized by $\dot{\vec{Q}}=0$ as a consequence of the Lorentz condition (2.18) as well as of the charge conservation (2.14). In going to the next order of approximation the constraints originating from the zero-order Lorentz condition and charge conservation are relaxed, as explained before, and the condition $\dot{\vec{Q}}=0$ is dropped. We shall determine the dynamic effects of a changing direction of the particle's non-Abelian charge during the motion along its first- or higher-order trajectory against the chosen background, i.e., against the choice of gauge implied by the direction of the zero-order fields; or, expressed in more physical terms, against the direction determined by electromagnetism. Thus in this gauge it does make sense to add up field contributions originating from different space-time points. This will, indeed, occur in a situation where the source charge $\vec{Q}(\tau)$ on the trajectory does not point into the same direction in isospace for all τ, and the computation of the fields at a position x involves the integration over contributions originating from all points y in the past light cone of x [compare Eqs. (4.24) and (4.25) below].

One may now ask: what happened to the freedom of performing arbitrary gauge transformations, i.e., arbitrary space- and time-dependent rotations in isospace? By the LW and BH *Ansätze* this freedom is restricted here to allowing only "global gauge transformations" for the zero-order input fields. It is apparent that the full gauge freedom of the theory can only be discussed for a physical system of several interacting particles after carrying the approximation scheme to arbitrarily high orders and comparing the results of the limit $n\to\infty$ for different choices adopted for the Abelian input data. Since this is impossible in practice, we adopt the gauge associated with the LW and BH *Ansätze* and work up from there considering the nonlinearities arising step by step. Clearly, we end up with an approximate answer to the solution of the nonlinear problem formulated in a gauge where proper time-dependent gauge transformations along a particle's trajectory are not allowed, and where the retarded input fields spread on light cones without change of direction in isospace. It is not clear whether one can construct the most general solution in this way. All we are aiming at here is an approximate solution of special type showing truly non-Abelian features.

We now turn to the problem of solving Eqs. (4.13) and (4.14). All the source terms (4.16) and (4.17) are quadratic in the fields; moreover, *each* of the quadratic terms contains four distinct contributions of different structure, since we have in the notation of Sec. III

$$ {}_0\vec{A}^{\nu}(x)={}_0\vec{A}^{\nu(\alpha)}+{}_0\vec{A}^{\nu}_{\rm ex}\,, \tag{4.18}$$

$$ {}_0\vec{\phi}(x)={}_0\vec{\phi}^{(\alpha)}+{}_0\vec{\phi}_{\rm ex}\,, \tag{4.19}$$

where the first term on the RHS of each of these equations is due to the particle whose equations of motion are under consideration, and the second one is due to all other particles. Thus the source terms (4.16) and (4.17) are, with the subscript x standing for "external" and the subscript r standing for "retarded,"

$$ {}_1\vec{K}^{\nu}={}_1\vec{K}^{\nu}_{xx}+{}_1\vec{K}^{\nu}_{xr}+{}_1\vec{K}^{\nu}_{rr}\,, \tag{4.20a}$$

$$ {}_1\vec{K}^{\nu}_{xx}=\partial^{\mu}{}_0\vec{A}_{\mu\rm ex}\wedge{}_0\vec{A}^{\nu}_{\rm ex}+2{}_0\vec{A}_{\mu\rm ex}\wedge\partial^{\mu}{}_0\vec{A}^{\nu}_{\rm ex}-{}_0\vec{A}_{\mu\rm ex}\wedge\partial^{\nu}{}_0\vec{A}^{\mu}_{\rm ex}+{}_0\vec{\phi}_{\rm ex}\wedge\partial^{\nu}{}_0\vec{\phi}_{\rm ex}\,, \tag{4.20b}$$

$$ \begin{aligned}{}_1\vec{K}^{\nu}_{xr}=\;&\partial^{\mu}{}_0\vec{A}_{\mu\rm ex}\wedge{}_0\vec{A}^{\nu(\alpha)}+\partial^{\mu}{}_0\vec{A}^{(\alpha)}_{\mu}\wedge{}_0\vec{A}^{\nu}_{\rm ex}+2{}_0\vec{A}_{\mu\rm ex}\wedge\partial^{\mu}{}_0\vec{A}^{\nu(\alpha)}+2{}_0\vec{A}^{(\alpha)}_{\mu}\wedge\partial^{\mu}{}_0\vec{A}^{\nu}_{\rm ex}\\&-{}_0\vec{A}_{\mu\rm ex}\wedge\partial^{\nu}{}_0\vec{A}^{\mu(\alpha)}-{}_0\vec{A}^{(\alpha)}_{\mu}\wedge\partial^{\nu}{}_0\vec{A}^{\mu}_{\rm ex}+{}_0\vec{\phi}_{\rm ex}\wedge\partial^{\nu}{}_0\vec{\phi}^{(\alpha)}+{}_0\vec{\phi}^{(\alpha)}\wedge\partial^{\nu}{}_0\vec{\phi}_{\rm ex}\,,\end{aligned} \tag{4.20c}$$

$$ {}_1\vec{K}^{\nu}_{rr}=\partial^{\mu}{}_0\vec{A}^{(\alpha)}_{\mu}\wedge{}_0\vec{A}^{\nu(\beta)}+2{}_0\vec{A}^{(\alpha)}_{\mu}\wedge\partial^{\mu}{}_0\vec{A}^{\nu(\beta)}-{}_0\vec{A}^{(\alpha)}_{\mu}\wedge\partial^{\nu}{}_0\vec{A}^{\mu(\beta)}+{}_0\vec{\phi}^{(\alpha)}\wedge\partial^{\nu}{}_0\vec{\phi}^{(\beta)}\,, \tag{4.20d}$$

and

$$ {}_1\vec{\Lambda}={}_1\vec{\Lambda}_{xx}+{}_1\vec{\Lambda}_{xr}+{}_1\vec{\Lambda}_{rr}\,, \tag{4.21a}$$

$$ {}_1\vec{\Lambda}_{xx}=2{}_0\vec{A}^{\mu}_{\rm ex}\wedge\partial_{\mu 0}\vec{\phi}_{\rm ex}\,, \tag{4.21b}$$

$$ {}_1\vec{\Lambda}_{xr}=2{}_0\vec{A}^{\mu}_{\rm ex}\wedge\partial_{\mu 0}\vec{\phi}^{(\alpha)}+2{}_0\vec{A}^{\mu(\alpha)}\wedge\partial_{\mu 0}\vec{\phi}_{\rm ex}\,, \tag{4.21c}$$

$$ {}_1\vec{\Lambda}_{rr}=2{}_0\vec{A}^{\mu(\alpha)}\wedge\partial_{\mu 0}\vec{\phi}^{(\beta)}\,. \tag{4.21d}$$

The solutions for Eqs. (4.14) and (4.15) corresponding to the various source terms will be denoted by

$$ {}_1\vec{A}^{\nu} = {}_1\vec{A}^{\nu}_{xx} + {}_1\vec{A}^{\nu}_{xr} + {}_1\vec{A}^{\nu}_{rr} , \tag{4.22} $$

and

$$ {}_1\vec{\phi} = {}_1\vec{\phi}_{xx} + {}_1\vec{\phi}_{xr} + {}_1\vec{\phi}_{rr} ; \tag{4.23} $$

a similar notation will be used for the various parts of the corresponding fields (4.9).

The regularized solutions of Eqs. (4.14) and (4.15) are (cf. Ref. 3)

$$ {}_1\vec{A}^{\nu(\alpha)}(x) = \frac{1}{2^{\alpha+1}\pi\Gamma(\alpha/2)\Gamma(\alpha/2+1)} \int_{D_{-\infty}} {}_1\vec{K}^{\nu}(y) R'^{\alpha-2} d^4y , \tag{4.24} $$

$$ {}_1\vec{\phi}^{(\alpha)}(x) = \frac{1}{2^{\alpha/2+2}\pi\Gamma(\alpha/2+1)} \int_{D_{-\infty}} {}_1\vec{\Lambda}(y) \left[\frac{R'}{\chi} \right]^{(\alpha-2)/2} J_{(\alpha-2)/2}(\chi R') d^4y , \tag{4.25} $$

where the integration is over the past light cone $D_{-\infty}$ of the point x (field point); the source point is denoted by y. We have to evaluate these integrals after substituting the appropriate source terms from Eqs. (4.20) and (4.21) and take the limit $\alpha \to 0$.

For some of the integrals it will be convenient to use retarded null coordinates.[21] The volume element is then given by

$$ d^4y = \kappa^2 d\kappa\, d^2\Omega\, dt , \tag{4.26} $$

and we will use the notations

$$ \begin{aligned} R'^{\mu} &= x^{\mu} - y^{\mu} = R^{\mu} - \kappa k^{\mu} , \\ y^{\mu} &= z^{\mu}(\tau) + \kappa k^{\mu} , \\ \kappa &= [y^{\mu} - z^{\mu}(\tau)] v_{\mu} , \\ R' &= (R^2 - 2\kappa k \cdot R)^{1/2} , \\ k \cdot R &= k_{\mu} R^{\mu} . \end{aligned} \tag{4.27} $$

Here and in the following the retarded times associated with the points x^{μ} and y^{μ} are called τ_0 and τ, respectively. The singularities of a number of integrals arising from ${}_1\vec{K}^{\nu}(y)$ and ${}_1\vec{\Lambda}(y)$ for $y^{\mu} \to z^{\mu}(\tau)$, i.e., for $\kappa \to 0$, are proportional to κ^{-2} or κ^{-1}: multiplying them with the volume element (4.26) thus yields a finite expression. Thus for these terms no regularization is needed, and we can use the standard rather than the Riesz zero-order potentials in those integrals.[22] This will be noted explicitly in each case in the following.

2. The purely external fields

We first consider the purely external fields, i.e., the first terms in the potentials (4.22) and (4.23) and the corresponding derivatives of these potentials, which are due to the source terms (4.20b) and (4.21b). No analytic continuation is required for these source terms, since the standard expressions are finite on the world line of the particle under consideration. From Eqs. (4.24) and (4.25) we obtain for the potentials for x on the world line, i.e., for $x^{\mu} = z^{\mu}(\tau_0)$ (using null coordinates and taking the limit $\alpha \to 0$, compare Appendix A)

$$ {}_1\vec{A}^{\nu}_{xx}(x = z(\tau_0)) = \int_{-\infty}^{\tau_0} d\tau \int \frac{d^2\Omega}{4\pi} \frac{R^4}{4(k\cdot R)^3} {}_1\vec{K}^{\nu}_{xx} , \tag{4.28} $$

and

$$ {}_1\vec{\phi}_{xx}(x = z(\tau_0)) = \int_{-\infty}^{\tau_0} d\tau \int \frac{d^2\Omega}{4\pi} \frac{R^4}{4(k\cdot R)^3} {}_1\vec{\Lambda}_{xx} , \tag{4.29} $$

where ${}_1\vec{K}^{\nu}_{xx}$ and ${}_1\vec{\Lambda}_{xx}$ are given by Eqs. (4.20b) and (4.21b), respectively. In Eqs. (4.28) and (4.29) we used the fact that the effective external source terms (4.20b) and (4.21b) are independent of the retarded distance κ.

Introducing the appropriate differentiations of the integrands[3,4] in Eqs. (4.24) and (4.25) and proceeding as before, we obtain for the purely external part of the fields defined by Eq. (4.9)

$$ {}_1\vec{F}^{\nu\rho}_{xx}(x = z(\tau_0)) = \int_{-\infty}^{\tau_0} d\tau \int \frac{d^2\Omega}{4\pi} \frac{R^4}{4(k\cdot R)^3} (\partial'^{\nu}{}_1\vec{K}^{\rho}_{xx} - \partial'^{\rho}{}_1\vec{K}^{\nu}_{xx}) \tag{4.30} $$

and

$$ {}_1\vec{F}^{\nu}_{xx}(x = z(\tau_0)) = \int_{-\infty}^{\tau_0} d\tau \int \frac{d^2\Omega}{4\pi} \frac{R^4}{4(k\cdot R)^3} \partial'^{\nu}{}_1\vec{\Lambda}_{xx} . \tag{4.31} $$

In Eqs. (4.30) and (4.31) an integration by parts has been performed observing that the contribution of the boundary term is zero in both cases; moreover, $\partial'_{\mu} = \partial/\partial y^{\mu}$.

3. The mixed fields

Now we consider the contribution to the field due to the mixed source terms ${}_1\vec{K}^{\nu}_{xr}$ and ${}_1\vec{\Lambda}_{xr}$ given by Eqs. (4.20c) and (4.21c), respectively. These are given by

$$ {}_1\vec{A}^{\nu}_{xr}(x = z(\tau_0)) = \lim_{\alpha\to 0} \alpha \int_{-\infty}^{\tau_0} d\tau \int \frac{d^2\Omega}{4\pi} \int_0^{\bar{\kappa}(\tau)} {}_1\vec{K}^{\nu}_{xr} R'^{\alpha-2} \kappa^2 d\kappa \tag{4.32} $$

and

$$ {}_1\vec{\phi}_{xr}(x=z(\tau_0))=\lim_{\alpha=0}\chi\int_{-\infty}^{\tau_0}d\tau\int\frac{d^2\Omega}{4\pi}\int_0^{\tilde{\kappa}(\tau)}{}_1\vec{A}_{xr}J_{(\alpha-2)/2}(\chi R')R'^{(\alpha-2)/2}\kappa^2 d\kappa\ . \tag{4.33}$$

In the expressions for the source term we now can substitute the standard zero-order potentials (2.22) or (2.25) and the corresponding derivatives off the world line

$$\partial^\mu{}_0\vec{A}^\nu=\frac{\vec{Q}}{\kappa}(k^\mu a^\nu-k^\mu v^\nu a\cdot k)+\frac{\vec{Q}}{\kappa^2}n^\mu v^\nu+\frac{\dot{\vec{Q}}}{\kappa}k^\mu v^\nu \tag{4.34}$$

and

$$ {}_0\vec{F}^\mu=\partial^\mu{}_0\vec{\phi}=\frac{\dot{\vec{S}}k^\mu}{\kappa}-\frac{1}{\kappa^2}\vec{S}k^\mu a\cdot k+\frac{\vec{S}}{\kappa^2}n^\mu-\frac{\chi^2}{2}\vec{S}k^\mu+\chi^2\int_{-\infty}^{\tau}\vec{S}\bar{R}^\mu\frac{J_2(\chi\bar{R})}{\bar{R}^2}d\bar{\tau}\ , \tag{4.35}$$

and make use of Eq. (A2) and

$$\frac{d}{d\tau}\left[\frac{\vec{Q}v^\nu R^\mu}{\kappa}\right]=\frac{\dot{\vec{Q}}v^\nu R^\mu}{\kappa}+\frac{\vec{Q}a^\nu R^\mu}{\kappa}+\frac{\vec{Q}v^\nu R^\mu}{\kappa^2}(1-a_\rho R^\rho)\ . \tag{4.36}$$

Here and in the following we use the notation

$$\bar{R}^\mu=z^\mu(\tau)-z^\mu(\bar{\tau}),\quad \bar{R}=(\bar{R}^\mu\bar{R}_\mu)^{1/2}\ . \tag{4.37}$$

Then Eq. (4.32) becomes

$$\begin{aligned} {}_1\vec{A}^{\nu}_{xr}(x=z(\tau_0))=\int_{-\infty}^{\tau_0}d\tau\int\frac{d^2\Omega}{4\pi}\frac{R^2}{2(k\cdot R)^2}\Bigg[&2\vec{Q}v_\mu\wedge\partial^\mu{}_0\vec{A}^{\nu}_{\text{ex}}+2{}_0\vec{A}_{\mu\text{ex}}\wedge(\dot{\vec{Q}}v^\nu k^\mu+\vec{Q}a^\nu k^\mu-\vec{Q}v^\nu k^\mu a\cdot k)\\ &-\frac{4k\cdot R}{R^2}{}_0\vec{A}_{\mu\text{ex}}\wedge\vec{Q}v^\mu n^\nu+\partial^\mu{}_0\vec{A}_{\mu\text{ex}}\wedge\vec{Q}v^\nu+\dot{\vec{Q}}\wedge{}_0\vec{A}^{\nu}_{\text{ex}}-\vec{Q}v_\mu\wedge\partial^\nu{}_0\vec{A}^{\mu}_{\text{ex}}\\ &-{}_0\vec{A}_{\mu\text{ex}}\wedge(\dot{\vec{Q}}v^\mu k^\nu+\vec{Q}a^\mu k^\nu-\vec{Q}v^\mu k^\nu a\cdot k)-\frac{k\cdot R}{2R^2}{}_0\vec{A}_{\mu\text{ex}}\wedge\vec{Q}v^\mu n^\nu\\ &+\vec{S}\wedge\partial^\nu{}_0\vec{\phi}_{\text{ex}}+\frac{R^2}{2k\cdot R}\chi\int_{-\infty}^{\tau}\partial^\nu{}_0\vec{\phi}_{\text{ex}}\wedge\vec{S}\frac{J_0(\chi\bar{R})}{\bar{R}}d\bar{\tau}\\ &+\frac{k\cdot R}{R^2}{}_0\vec{\phi}_{\text{ex}}\wedge\vec{S}n^\nu+{}_0\vec{\phi}_{\text{ex}}\wedge(\dot{\vec{S}}k^\nu-\vec{S}a\cdot kk^\nu)\\ &+\frac{R^2\chi^2}{2k\cdot R}{}_0\vec{\phi}_{\text{ex}}\wedge\left[-\tfrac{1}{2}\vec{S}k^\nu+\int_{-\infty}^{\tau}\vec{S}\bar{R}^\nu\frac{J_2(\chi\bar{R})}{\bar{R}^2}d\bar{\tau}\right]\Bigg]\ . \end{aligned}\tag{4.38}$$

To evaluate (4.33) we use Eq. (A3) and obtain after carrying out the κ integration and taking the limit $\alpha\to 0$

$$\begin{aligned} &{}_1\vec{\phi}_{xr}(x=z(\tau_0))\\ &=\int_{-\infty}^{\tau_0}d\tau\int\frac{d^2\Omega}{4\pi}\Bigg\{\frac{R^2}{2(k\cdot R)^2}\left[2\vec{Q}\wedge v_\mu\partial^\mu{}_0\vec{\phi}_{\text{ex}}+2k_{\mu 0}\vec{A}^{\mu}_{\text{ex}}\wedge(\dot{\vec{S}}-\vec{S}a\cdot k)+\frac{4k\cdot R}{R^2}{}_0\vec{A}^{\mu}_{\text{ex}}\wedge\vec{S}n_\mu-\frac{R^2\chi^2}{k\cdot R}{}_0\vec{A}^{\mu}_{\text{ex}}\wedge\vec{S}k_\mu\right.\\ &\qquad\left.+2\chi^2{}_0\vec{A}^{\mu}_{\text{ex}}\wedge\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})\bar{R}_\mu\frac{J_2(\chi\bar{R})}{\bar{R}^2}d\bar{\tau}\right.\\ &\qquad\left.-2\int_0^{\tilde{\kappa}}\kappa d\kappa\frac{J_1(\chi R)}{R}(\vec{Q}\wedge v_\mu\partial^\mu{}_0\vec{\phi}_{\text{ex}}+\vec{S}\wedge k_{\mu 0}\vec{A}^{\mu}_{\text{ex}}a\cdot k-\dot{\vec{S}}\wedge k_{\mu 0}\vec{A}^{\mu}_{\text{ex}})-2\int_0^{\tilde{\kappa}}d\kappa\frac{J_1(\chi R)}{R}{}_0\vec{A}^{\mu}_{\text{ex}}\wedge\vec{S}n_\mu\right.\\ &\qquad\left.+\int_0^{\tilde{\kappa}}\kappa^2 d\kappa\frac{J_1(\chi R)}{R}\left[\chi^2 k_{\mu 0}\vec{A}^{\mu}_{\text{ex}}\wedge\vec{S}-2\chi^2{}_0\vec{A}^{\mu}_{\text{ex}}\wedge\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})\bar{R}_\mu\frac{J_2(\chi\bar{R})}{\bar{R}^2}d\bar{\tau}\right]\right]\Bigg\}\ , \end{aligned}\tag{4.39}$$

where $\tilde{\kappa}=R^2/(2k^\mu R_\mu)$.

It remains to evaluate the mixed part of the fields (4.9)

$$ {}_1\vec{F}^{\nu\rho}_{xr}=\partial^\nu{}_1\vec{A}^{\rho}_{xr}-\partial^\rho{}_1\vec{A}^{\nu}_{xr}-{}_0\vec{A}^{\nu}_{\text{ex}}\wedge{}_0\vec{A}^{\rho(\alpha)}-{}_0\vec{A}^{\nu(\alpha)}\wedge{}_0\vec{A}^{\rho}_{\text{ex}}\ , \tag{4.40}$$

$$ {}_1\vec{F}^{\,\nu}_{xr} = \partial^\nu {}_1\vec{\phi}_{xr} - {}_0\vec{A}^{\,\nu}_{ex} \wedge {}_0\vec{\phi}^{(\alpha)} - {}_0\vec{A}^{\,\nu(\alpha)} \wedge {}_0\vec{\phi}_{ex} \tag{4.41} $$

on the world line of the particle. To do this we must substitute Eqs. (4.20c) and (4.21c) for the source terms in the integrals (4.24) and (4.25) and then take derivatives under the integral sign. We then obtain

$$ {}_1\vec{F}^{\,\nu\rho}_{xr}(x=z(\tau_0)) = \lim_{\alpha\to 0}\left\{ \frac{1}{2^{\alpha+1}\pi\Gamma(\alpha/2)\Gamma(\alpha/2+1)} \int_{D_{-\infty}} ({}_1\vec{K}^{\,\rho}_{xr}\partial^\nu R'^{\alpha-2} - {}_1\vec{K}^{\,\nu}_{xr}\partial^\rho R'^{\alpha-2}) d^4y - {}_0\vec{A}^{\,\nu}_{ex}\wedge {}_0\vec{A}^{\rho(\alpha)} - {}_0\vec{A}^{\,\nu(\alpha)}\wedge {}_0\vec{A}^{\,\rho}_{ex} \right\}, \tag{4.42} $$

$$ {}_1\vec{F}^{\,\nu}_{xr}(x=z(\tau_0)) = \lim_{\alpha\to 0}\left\{ \frac{1}{2^{(\alpha+4)/2}\pi\Gamma(\alpha/2+1)} \int_{D_{-\infty}} \vec{\Lambda}_{xr}\partial^\nu\left[\left[\frac{R'}{\chi}\right]^{(\alpha-2)/2} J_{(\alpha-2)/2}(\chi R')\right] d^4y - {}_0\vec{A}^{\,\nu}_{ex}\wedge {}_0\vec{\phi}^{(\alpha)} - {}_0\vec{A}^{\,\nu(\alpha)}\wedge {}_0\vec{\phi}_{ex} \right\}, \tag{4.43} $$

where ${}_0\vec{A}^{\,\nu(\alpha)}$ and ${}_0\vec{\phi}^{(\alpha)}$ are given by Eqs. (3.1) and (3.2), respectively. Since the external zero-order potentials are finite at the position of the particle under consideration, Eqs. (3.1) and (3.2) can be replaced by their limit $\alpha\to 0$ given by Eqs. (3.6) and (3.7), respectively, in the last two terms of both Eqs. (4.42) and (4.43). For the remaining integrals, no simplification such as the one discussed in Sec. IV B 1 is possible since some of the terms in the integrands diverge as κ^{-3}.

We note that

$$ \partial^\nu R'^{\alpha-2} = (\alpha-2)R'^{\alpha-4}R'^\nu, \tag{4.44} $$

and (cf. Ref. 8)

$$ \int_{D_{-\infty}} {}_1\vec{\Lambda}_{xr}\partial^\nu[R'^{(\alpha-2)/2}J_{(\alpha-2)/2}(\chi R')]d^4y = \chi \int_{D_{-\infty}} {}_1\vec{\Lambda}_{xr} R'^\nu R'^{(\alpha-4)/2} J_{(\alpha-4)/2}(\chi R')d^4y . \tag{4.45} $$

Thus Eqs. (4.42) and (4.43) become

$$ {}_1\vec{F}^{\,\nu\rho}_{xr}(x=z(\tau_0)) = \lim_{\alpha\to 0} \frac{\alpha-2}{2^{\alpha+1}\pi\Gamma(\alpha/2)\Gamma((\alpha+2)/2)} \int_{D_{-\infty}} R'^{\alpha-4}(R'^\nu {}_1\vec{K}^{\,\rho}_{xr} - R'^\rho {}_1\vec{K}^{\,\nu}_{xr})d^4y + {}_0\vec{A}^{\,\nu}_{ex}\wedge \frac{d}{d\tau}(\vec{Q}v^\rho) - {}_0\vec{A}^{\,\rho}_{ex}\wedge\frac{d}{d\tau}(\vec{Q}v^\nu), \tag{4.46} $$

$$ {}_1\vec{F}^{\,\nu}_{xr}(x=z(\tau_0)) = \lim_{\alpha\to 0} \frac{\chi^{(4-\alpha)/2}}{2^{(\alpha+4)/2}\pi\Gamma((\alpha+2)/2)} \int_{D_{-\infty}} R'^\nu R'^{(\alpha-4)/2}J_{(\alpha-4)/2}(\chi R') {}_1\vec{\Lambda}_{xr} d^4y + {}_0\vec{A}^{\,\nu}_{ex}\wedge\left[\frac{d\vec{S}}{d\tau} - \chi\int_{-\infty}^{\tau_0} \vec{S}\frac{J_1(\chi R)}{R}d\tau\right] + \frac{d}{d\tau}(\vec{Q}v^\nu)\wedge {}_0\vec{\phi}_{ex} . \tag{4.47} $$

These are, as far as we can evaluate, the expressions in general.

4. *The regularized self-fields*

We now have to evaluate the self-fields of the particles in first order. With the help of the formulas assembled in Appendices A and B the κ integrals over the source terms ${}_1\vec{K}^{\,\mu}_{rr}$ and ${}_1\vec{\Lambda}_{rr}$, defined by Eqs. (4.20d) and (4.21d) can be carried out by making use of the simplifications discussed in Sec. IV B 1 above; one obtains

$$ \begin{aligned} {}_1\vec{A}^{\,\nu}_{rr}&(x=z(\tau_0)) \\ &= \int_{-\infty}^{\tau_0} d\tau \Bigg\{ \dot{\vec{Q}}\wedge\vec{Q}\int\frac{d^2\Omega}{4\pi}\frac{n^\nu}{k\cdot R} - \dot{\vec{S}}\wedge\vec{S}\int\frac{d^2\Omega}{4\pi}\frac{k^\nu}{k\cdot R} \\ &\quad + \vec{S}\wedge\int\frac{d^2\Omega}{4\pi}\frac{n^\nu}{k\cdot R}\vec{A}(\tau,\theta,\varphi) - \vec{S}\wedge\tfrac{1}{2}R^2a^\mu\int\frac{d^2\Omega}{4\pi}\frac{k_\mu k^\nu}{(k\cdot R)^2}\vec{A}(\tau,\theta,\varphi) + \dot{\vec{S}}\wedge\tfrac{1}{2}R^2\int\frac{d^2\Omega}{4\pi}\frac{k^\nu}{(k\cdot R)^2}\vec{A}(\tau,\theta,\varphi) \\ &\quad + \vec{S}\wedge\tfrac{1}{2}R^2\int\frac{d^2\Omega}{4\pi}\left[\frac{1}{(k\cdot R)^2}\vec{B}^{\,\nu}(\tau,\theta,\varphi) + \frac{R^2}{2(k\cdot R)^3}k^\nu\vec{B}(\tau,\theta,\varphi)\right] - \vec{S}\wedge\tfrac{1}{4}R^4\int\frac{d^2\Omega}{4\pi}\frac{k^\nu}{(k\cdot R)^3}\chi^2\vec{A}(\tau,\theta,\varphi) \\ &\quad - \tfrac{1}{4}R^4\int\frac{d^2\Omega}{4\pi}\frac{1}{(k\cdot R)^3}\vec{A}(\tau,\theta,\varphi)\wedge\left[\vec{B}(\tau,\theta,\varphi) + \frac{R^2}{2k\cdot R}k^\nu\vec{B}(\tau,\theta,\varphi)\right]\Bigg\} \end{aligned} \tag{4.48} $$

and

$$ {}_1\vec{\phi}_{rr}(x=z(\tau_0))=2\int_{-\infty}^{\tau_0}d\tau\Bigg\{\vec{Q}\wedge\dot{\vec{S}}\int\frac{d^2\Omega}{4\pi}\frac{1}{k\cdot R}+\vec{Q}\wedge v^{\mu}\tfrac{1}{2}R^2\int\frac{d^2\Omega}{4\pi}\frac{1}{(k\cdot R)^2}\vec{B}_{\mu}(\tau,\theta,\varphi) $$

$$ +\vec{Q}\wedge v^{\mu}\tfrac{1}{4}R^4\int\frac{d^2\Omega}{4\pi}\frac{k_{\mu}}{(k\cdot R)^3}\vec{B}(\tau,\theta,\varphi) $$

$$ -\chi\int\frac{d^2\Omega}{4\pi}\int_0^{\bar{\kappa}}\Bigg[\vec{Q}\wedge\dot{\vec{S}}+\kappa\vec{Q}\wedge v^{\mu}\chi^2\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})[\bar{R}_{\mu}+\kappa k_{\mu}] $$

$$ \times\frac{J_2(\chi(\bar{R}^2+2\kappa k\cdot\bar{R})^{1/2})}{\bar{R}^2+2\kappa k\cdot\bar{R}}d\bar{\tau}\Bigg]\frac{J_1(\chi(R^2-2\kappa k\cdot R)^{1/2})}{(R^2-2\kappa k\cdot R)^{1/2}}d\kappa\Bigg\}. \tag{4.49} $$

Here we have used the following definitions:

$$ \vec{A}(\tau,\theta,\varphi)=\chi\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})\frac{J_1\left[\chi\bar{R}\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]^{1/2}\right]}{\bar{R}\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]^{1/2}}d\bar{\tau}, \tag{4.50} $$

$$ \vec{B}(\tau,\theta,\varphi)=\chi^2\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})\frac{J_2\left[\chi\bar{R}\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]^{1/2}\right]}{\bar{R}^2\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]}d\bar{\tau}, \tag{4.51} $$

and

$$ \vec{B}^{\nu}(\tau,\theta,\varphi)=\chi^2\int_{-\infty}^{\tau}\vec{S}(\bar{\tau})\bar{R}^{\nu}\frac{J_2\left[\chi\bar{R}\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]^{1/2}\right]}{\bar{R}^2\left[1+\dfrac{R^2}{\bar{R}^2}\dfrac{k\cdot\bar{R}}{k\cdot R}\right]}d\bar{\tau}, \tag{4.52} $$

with $\bar{R}^{\mu}$ as given by Eq. (4.37) and R^{μ} for x on the world line, i.e., for $x^{\mu}=x^{\mu}(\tau_0)$, as defined in Eq. (3.8). A centered dot between two four-vectors denotes the scalar product in Minkowski space. Furthermore, n^{μ} is the spacelike unit vector introduced in Eq. (3.17). The κ integrals used in the derivation of Eqs. (4.48) and (4.49) are discussed in Appendix A. In Appendix B we collect some useful results concerning the angular integrals appearing in Eqs. (4.48) and (4.49). As regards the limit $\alpha\to 0$ we remark that this limit has already been taken in the integrands of the τ integrals. This is possible for $x=z(\tau_0)$. However, for x off the world line we have to keep the factors R^{α} under the integrals in order to properly regularize these expressions for $\tau\to\tau_0$. This is also to be remembered in computing $\partial^{\mu}{}_1\vec{A}_{\mu}^{(\alpha)}(x)$ and ${}_1\vec{F}_{\mu\nu}^{(\alpha)}(x)$ with the limit $\alpha\to 0$ taken at the end.

It only remains to evaluate the self-part of the fields (4.9)

$$ {}_1\vec{F}_{rr}^{\nu\rho}(x)=\partial^{\nu}{}_1\vec{A}_{rr}^{\rho}-\partial^{\rho}{}_1\vec{A}_{rr}^{\nu}-{}_0\vec{A}_r^{\nu(\alpha)}\wedge{}_0\vec{A}_r^{\rho(\beta)}, \tag{4.53} $$

$$ {}_1\vec{F}_{rr}^{\nu}(x)=\partial^{\nu}{}_1\vec{\phi}^{(\alpha)}-{}_0\vec{A}^{\nu(\alpha)}\wedge{}_0\vec{\phi}^{(\beta)} \tag{4.54} $$

for points x on the world line of the particle. Substituting the self-parts of the source terms (4.20d) and (4.21d) in the expressions (4.24) and (4.25), we obtain

$$ {}_1\vec{F}_{rr}^{\nu\rho}(z)=\lim_{\alpha\to 0}\frac{1}{2^{\alpha+1}\pi\Gamma(\alpha/2)\Gamma(\alpha/2+1)}\int_{D_{-\infty}}({}_1\vec{K}_{rr}^{\rho}\partial^{\nu}R'^{\alpha-2}-{}_1\vec{K}_{rr}^{\nu}\partial^{\rho}R'^{\alpha-2})d^4y-\lim_{\substack{\alpha\to 0\\ \beta\to 0}}{}_0\vec{A}^{\nu(\alpha)}\wedge{}_0\vec{A}^{\rho(\beta)}, \tag{4.55} $$

$$ {}_1\vec{F}_{rr}^{\nu}(z)=\lim_{\alpha\to 0}\frac{1}{2^{(\alpha/2)+2}\pi\Gamma(\alpha/2+1)}\int_{D_{-\infty}}{}_1\vec{\Lambda}_{rr}\partial^{\nu}\left[\left[\frac{R'}{\chi}\right]^{(\alpha-2)/2}J_{(\alpha-2)/2}(\chi R')\right]d^4y-\lim_{\substack{\alpha\to 0\\ \beta\to 0}}{}_0\vec{A}^{\nu(\alpha)}\wedge{}_0\vec{\phi}^{(\beta)}. \tag{4.56} $$

The last term of each of these expressions presents no difficulty, since it is the product of two finite expressions of the forms (3.6) or (3.7). The integrals of the first terms, however, just as those of Eqs. (4.42) and (4.43), do not allow any simplification; they are actually significantly more complicated than those encountered before, since they are products of three Riesz-type expressions (just like the corresponding terms of Ref. 8).

Using Eqs. (4.44), (4.45), (3.6), and (3.7) the self-fields become

$$ {}_1\vec{F}^{\nu\rho}_{rr}(z)=\lim_{\alpha\to 0}\frac{\alpha-2}{2^{\alpha+1}\pi\Gamma(\alpha/2)\Gamma(\alpha/2+1)}\int_{D_{-\infty}}(R'^{\nu}{}_1\vec{K}^{\rho}_{rr}-R'^{\rho}{}_1\vec{K}^{\nu}_{rr})R'^{\alpha-4}d^4y+\vec{Q}\wedge\dot{\vec{Q}}(v^{\nu}a^{\rho}-v^{\rho}a^{\nu}) , \tag{4.57}$$

$$ {}_1\vec{F}^{\nu}_{rr}(z)=\lim_{\alpha\to 0}\frac{1}{2^{(\alpha/2)+2}\pi\Gamma(\alpha/2+1)}\int_{D_{-\infty}}{}_1\vec{\Lambda}_{rr}R'^{\nu}J_{(\alpha-4)/2}(\chi R')R'^{(\alpha-4)/2}d^4y-\frac{d}{d\tau}(\vec{Q}v^{\nu})\wedge\left[\frac{d\vec{S}}{d\tau}+\chi\int_{-\infty}^{\tau_0}\vec{S}\frac{J_1(\chi R)}{R}d\tau\right] . \tag{4.58}$$

C. The first-order equations of motion

We now have to substitute the various parts of the fields into the first-order laws (4.10), (3.21), and (4.7).

As noted before, Eq. (3.21) does not contain any fields at all, and thus already is the first-order rotational equation of motion. The law of variation of the charge vector (4.7) only contains the zero-order fields (4.18) and (4.19) with the corresponding self-fields (3.6) and (3.7). Labeling in Eqs. (3.6) and (3.7) the point on the world line by τ instead of τ_0 and calling the integration variable $\bar{\tau}$, the first-order equation for the variation of charge is

$$ I\frac{d\vec{\tau}}{d\tau}=g\vec{\tau}\wedge\left\{\left[-h_1{}^2\frac{d\vec{\tau}}{d\tau}-h_1{}^2\chi\int_{-\infty}^{\tau}\vec{\bar{\tau}}\frac{J_1(\chi\bar{R})}{R}d\bar{\tau}+h_1{}_0\vec{\phi}_{\text{ex}}\right]+\frac{d\vec{\tau}}{d\tau}I^2-I_0\vec{A}^{\nu}_{\text{ex}}v_{\nu}\right\} . \tag{4.59}$$

We now have to substitute the various parts of the zero- and first-order external and self-fields into Eq. (4.10). It should be noted that in obtaining the zero-order equation of motion (3.20) we could simplify some of the zero-order self-fields because of the constancy of the charge vector $\vec{Q}$ in that order [see Eq. (3.15)]. This is no longer permissible, and thus even the "zero-order" terms in the translational equation of motion are more complicated than before. We obtain

$$ \begin{aligned} &\frac{d}{d\tau}\left\{\left[m-h_1\vec{\tau}\cdot({}_0\vec{\phi}_{\text{ex}}+g[{}_1\vec{\phi}_{xx}+{}_1\vec{\phi}_{xr}+{}_1\vec{\phi}_{rr}])+h_1\chi\vec{\tau}\cdot\int_{-\infty}^{\tau}\vec{\bar{\tau}}\frac{J_1(\chi\bar{R})}{\bar{R}}d\bar{\tau}\right]v^{\nu}+\dot{B}^{\nu\rho}v_{\rho}\right\} \\ &\quad=-\vec{\tau}\cdot\Big\{h_1[\partial^{\nu}{}_0\vec{\phi}_{\text{ex}}+g({}_1\vec{F}^{\nu}_{xx}+{}_1\vec{F}^{\nu}_{xr}+{}_1\vec{F}^{\nu}_{rr})]-h_1{}^2[\tfrac{1}{3}\vec{\tau}(a^2v^{\nu}+\dot{a}^{\nu})+\ddot{\vec{\tau}}v^{\nu}] \\ &\qquad-\tfrac{1}{2}h_1{}^2\chi^2\vec{\tau}v^{\nu}+h_1{}^2\chi^2\int_{-\infty}^{\tau}\vec{\bar{\tau}}\bar{R}^{\nu}\frac{J_2(\chi\bar{R})}{\bar{R}^2}d\bar{\tau}-\tfrac{2}{3}\vec{\tau}I^2(a^2v^{\nu}+\dot{a}^{\nu})-I[{}_0\vec{F}^{\nu\rho}_{\text{ex}}+g({}_1\vec{F}^{\nu\rho}_{xx}+{}_1\vec{F}^{\nu\rho}_{xr}+\vec{F}^{\nu\rho}_{rr})]v_{\rho}\Big\} . \end{aligned} \tag{4.60}$$

We still have to substitute Eqs. (4.29), (4.39), and (4.49) for ${}_1\vec{\phi}_{xx}$, ${}_1\vec{\phi}_{xr}$, and ${}_1\vec{\phi}_{rr}$, Eqs. (4.31), (4.47), and (4.56) for ${}_1\vec{F}^{\nu}_{xx}$, ${}_1\vec{F}^{\nu}_{xr}$, and ${}_1\vec{F}^{\nu}_{rr}$, and Eqs. (4.30), (4.46), and (4.55) for ${}_1\vec{F}^{\nu\rho}_{xx}$, ${}_1\vec{F}^{\nu\rho}_{xr}$, and ${}_1\vec{F}^{\nu\rho}_{rr}$. This will not be done explicitly because of the very lengthy resulting expression.

V. DISCUSSION

In the preceding paper we had obtained the laws of motion of a spinning particle which is a monopole-dipole singularity of a Yang-Mills-Higgs field. In Sec. II of this paper, an approximation method was developed for solving the nonlinear field equations to any given order in a dimensionless constant g, together with the equations of motion consistent with these solutions to that order. The laws of motion are special cases of those obtained in Ref. 16 for multipoles of arbitrary order. The reason for rederiving them in the preceding paper was first, that the general case is so complex that the simplicity of the procedure for monopoles and dipoles is obscured, and second, that the same method is needed in finding the approximate laws of motion in Secs. II and IV A.

In Sec. II we obtained the laws of motion in zeroth order for the monopole-dipole case. In Sec. IV A, we obtained them in first order for the case of a monopole singularity, omitting the dipole term, but maintaining the intrinsic angular momentum of the particle. In Sec. III we calculated the fields to be inserted into the zero-order laws of motion to obtain the zero-order equations of motion. These fields had to be regularized to avoid infinities on the world line of the particle; this was achieved by means of the method of Riesz potentials.[2,3] The same method was used to obtain the first-order fields in Sec. IV B.

Our approximation method is a systematic linearization method, leading to a series of inhomogeneous wave equations. The method of Riesz was originally developed for such a system of linear equations, and thus at each stage of approximation it guarantees the existence of solutions finite on the world line of the particle, provided certain conditions are satisfied by the sources. However, it has not been rigorously proved that this succession of solutions actually approximates the exact solution of the nonlinear problem, a defect common to all approximation methods for such problems. Furthermore, in (partially) evaluating the first-order Riesz integrals (which involve integrals over the zero-order integrals and, depending on the particular terms, are 8- to 12-fold integrals), we freely interchanged the order of integration wherever we found it convenient, without attempting to prove that this is permissible.

The method used in Secs. II and IV is very similar to the one used in Ref. 11 for finding the laws and equations of motion of interacting monopole singularities in general relativity, and in Ref. 8 for finding them for the corresponding problem in nonlinear meson theory. In the absence of the Higgs field and of spin, the final results obtained here reduce to those of Ref. 6, except that the contributions of the zero-order external fields to the first-order field (obtained in Secs. IV B 2 and IV B 3) were not included there. Furthermore, Eq. (45) of Ref. 6 can be simplified by using null coordinates as discussed in Sec. IV B 1 and is only an approximation because of an expansion used in the regularizing factors defining the integrand; in fact this regularization is not necessary in this order, as noted before.

Our final first-order translational equation of motion is given by Eq. (4.60) (with the fields to be inserted in that equation indicated there), the rotational one by Eq. (3.21), and the equation for the variation of the charge vector by Eq. (4.59). These equations are complicated coupled integro-differential equations for the trajectory $z^{\mu}(\tau)$, the spin quantities $B_{\mu\nu}$, and the normalized non-Abelian charge vector $\vec{\tau}$. It can be seen from the discussion below that for a single particle in the absence of external fields a straight-line motion with constant spin $B_{\mu\nu}$ and constant charge vector $\vec{\tau}$ is an exact solution, as expected. For a single particle without dipole moment or spin in a given external field (but no Higgs field) some simple cases have been discussed by Säring.[23] The case of two interacting particles is currently being investigated. The problem is considerably simplified if the Higgs field is taken as massless (i.e., $\chi=0$), in which case the integrals (4.39), (4.49), (4.47), (4.56), (4.46), and (4.55) for ${}_1\vec{\phi}_{xr}$, ${}_1\vec{\phi}_{rr}$, ${}_1\vec{\mathbf{F}}^{\,\nu}_{xr}$, ${}_1\vec{\mathbf{F}}^{\,\nu}_{rr}$, ${}_1\vec{\mathbf{F}}^{\,\nu\rho}_{xr}$, and ${}_1\vec{\mathbf{F}}^{\,\nu\rho}_{rr}$, respectively, can be almost completely evaluated for *any* motion, as will be shown elsewhere.

As mentioned in the Introduction, one of our main motivations for this work was the problem whether non-Abelian charge can be radiated away. In zeroth order this is not possible in the absence of external fields, since the charge vector is constant [see Eq. (3.15)]. Equation (4.59) determines the change of the vector $\vec{\tau}$ in first order. This equation only contains the zero-order fields. Since we have already determined the functional form of the *first*-order fields, we can with only minimal effort include the variation of the charge vector in *second* order in our discussion. Proceeding as in the preceding paper and keeping in mind the discussion at the beginning of Sec. IV A, we obtain from covariant charge conservation in that order instead of Eq. (4.59).

$$I\frac{d\vec{\tau}}{d\tau}=g\vec{\tau}\wedge\{-[{}_0\vec{\mathbf{A}}^{\mu}+g\,{}_1\vec{\mathbf{A}}^{\mu}_{rr}]Iv_{\mu}+[{}_0\vec{\phi}+g\,{}_1\vec{\phi}_{rr}]h_1+[{}_0\vec{\mathbf{A}}^{\mu}_{\text{ex}}+g\,{}_1\vec{\mathbf{A}}^{\mu}_{\text{ex}}]Iv_{\mu}+[{}_0\vec{\phi}_{\text{ex}}+g\,{}_1\vec{\phi}_{\text{ex}}]h_1\} \tag{5.1}$$

with

$${}_1\vec{\mathbf{A}}^{\mu}_{\text{ex}}={}_1\vec{\mathbf{A}}^{\mu}_{xx}+{}_1\vec{\mathbf{A}}^{\mu}_{xr},\quad {}_1\vec{\phi}_{\text{ex}}={}_1\vec{\phi}_{xx}+{}_1\vec{\phi}_{xr}\;, \tag{5.2}$$

where all fields are to be considered as functionals of the trajectories in second order. Using the expressions (3.6), (3.7), (4.48), and (4.49) in Eq. (5.1) and taking the scalar product with $\dot{\vec{\tau}}$, we obtain

$$\begin{aligned}\dot{\vec{\tau}}^{2}=&-g\chi\frac{{h_1}^2}{I}\dot{\vec{\tau}}\cdot\vec{\tau}\wedge\int_{-\infty}^{\tau}\vec{\tau}\frac{J_1(\chi\bar{R})}{\bar{R}}d\bar{\tau}\\ &-g^2\dot{\vec{\tau}}\cdot\left\{v^{\mu}\vec{\tau}\wedge\int_{-\infty}^{\tau}\dot{\vec{\tau}}\wedge\vec{\tau}\left[I^2\int\frac{d^2\Omega}{4\pi}\frac{n_{\mu}}{k\cdot\bar{R}}-{h_1}^2\int\frac{d^2\Omega}{4\pi}\frac{k_{\mu}}{k\cdot\bar{R}}\right]d\bar{\tau}+2{h_1}^2\vec{\tau}\wedge\int_{-\infty}^{\tau}\dot{\vec{\tau}}\wedge\vec{\tau}\int\frac{d^2\Omega}{4\pi}\frac{1}{k\cdot\bar{R}}d\bar{\tau}\right\}+\cdots\\ &+g\dot{\vec{\tau}}\cdot\left\{v_{\mu}({}_0\vec{\mathbf{A}}^{\mu}_{\text{ex}}+g\,{}_1\vec{\mathbf{A}}^{\mu}_{\text{ex}})-\frac{h_1}{I}({}_0\vec{\phi}_{\text{ex}}+g\,{}_1\vec{\phi}_{\text{ex}})\right\}\wedge\vec{\tau}\,.\end{aligned} \tag{5.3}$$

Here the ellipsis represents contributions proportional to $g^2\chi$ and $g^2\chi^2$ originating from the χ part of the scalar field. These terms are difficult to discuss even in a numerical evaluation of the integrals using a predetermined trajectory in this order. We, therefore, investigate Eq. (5.3) for the case of a "zero-mass" scalar field by dropping the terms involving χ. With the help of Eqs. (B18) and (B19), Eq. (5.3) can then be written in the form

$$\dot{\vec{\tau}}^{2}=-g^2I^2\dot{\vec{\tau}}\cdot\vec{\tau}\wedge\int_{-\infty}^{\tau}\dot{\vec{\tau}}(\bar{\tau})\wedge\vec{\tau}(\bar{\tau})f(\tau,\bar{\tau})d\bar{\tau}+g\dot{\vec{\tau}}\cdot\left\{v_{\mu}({}_0\vec{\mathbf{A}}^{\mu}_{\text{ex}}+g\,{}_1\vec{\mathbf{A}}^{\mu}_{\text{ex}})-\frac{h_1}{I}({}_0\vec{\phi}_{\text{ex}}+g\,{}_1\vec{\phi}_{\text{ex}})\right\}\wedge\vec{\tau}\,. \tag{5.4}$$

Here $\vec{\tau}=\vec{\tau}(\tau)$ and $f(\tau,\bar{\tau})$ is a functional of the second-order trajectory given by

$$f(\tau,\bar{\tau})=\left[1-\frac{{h_1}^2}{I^2}\right]v^{\mu}(\tau)\left[\frac{\bar{R}_{\mu}(\bar{\tau})+\bar{\kappa}v_{\mu}(\bar{\tau})}{\bar{\kappa}^2}\right]\sum_{m=0}^{\infty}\frac{1}{2m+3}\left[1-\frac{\bar{R}^2}{\bar{\kappa}^2}\right]^m+\frac{{h_1}^2}{I^2}[2-v^{\mu}(\tau)v_{\mu}(\bar{\tau})]\frac{1}{\bar{\kappa}}\sum_{m=0}^{\infty}\frac{1}{2m+1}\left[1-\frac{\bar{R}^2}{\bar{\kappa}^2}\right]^m. \tag{5.5}$$

The notation used in Eq. (5.5) is the same as in Appendix B except for the replacement of τ_0 and τ there by τ and $\overline{\tau}$ here; furthermore, $\overline{\kappa}=\overline{\kappa}(\overline{\tau})=\overline{R}^{\mu}(\overline{\tau})v_{\mu}(\overline{\tau})$.

The expression (5.4) does not necessarily vanish. It can only be evaluated in conjunction with solving for the motion of the system under consideration. It does vanish, of course, in the Abelian case which, as noted before, is a possible solution of our equations in all orders.

Even in the non-Abelian case, the *magnitude* of the charge vector must still be constant, i.e., we must have

$$\vec{Q}\cdot\dot{\vec{Q}}=I^2\vec{\tau}\cdot\dot{\vec{\tau}}=0 \tag{5.6}$$

in the exact law of motion [Eq. (I2.83)] as well as in all orders of approximation. This follows trivially by scalar multiplication of the appropriate equation for $\dot{\vec{\tau}}$ with $\vec{\tau}$, since the RHS of any such equations is of the form of a vector product containing a factor $\vec{\tau}$.

In first order, the motion of a system of two particles, say, is determined by Eqs. (3.21), (4.59), and (4.60), which must be solved together. Although the magnitude of the charge vector of each particle remains constant, their relative orientation may change. It is this aspect of the motion of the particles which must be studied in determining the possible loss of charge from the system, as well as the question of the appropriate definition of the charge in the field. We shall return to these questions elsewhere.

The fields used in this paper always were purely retarded. For the zero-order fields this implied that at the position of each particle we obtain contributions from all other particles emanating on and inside the past light cone of the particle under consideration, as well as (for the Higgs field) an integral over the past world line of this particle. The first-order fields have a much more complicated structure.[24] The nonlinear fields allow the particle to produce a Yang-Mills field and a Higgs field at the same or at different times. Their effects may be pictured to interact in space at a later time and then return to affect the particle.

The effects just described involve integrations on the surface (or in the interior) of the past light cone of effects emanating from the past world line directed toward the future. For the time-symmetric field theory and the action-at-a-distance theory considered in Appendix C, the first-order field can be interpreted similarly, except that we now may have similar integrations over the past light cone of effects emanating from the entire world line directed toward the past as well as the future; for the advanced Riesz potential we have similar integrals over the future rather than the past line cone. No violation of causality is implied by any such expression, as we are dealing with the description of the motion of a closed system of particles.[25]

Apart from any intrinsic interest, the time-symmetric field theory discussed in Appendix C is of importance because its equation of motion allow stable motions (without radiation loss). It is possible to obtain exact solutions for such motions, which can then be used as a starting point for finding approximate solutions for corresponding problems with retarded interactions, as will be discussed elsewhere. The time-symmetric action-at-a-distance theory discussed in Appendix C differs slightly from the corresponding field theory, just as in the case of the meson theory (linear[26] as well as nonlinear[8]), a difference which could be used to distinguish between the two points of view observationally. The effects of this difference have been investigated for some simple cases in meson theory, both in classical[27] and in quantum theories;[28] similar effects may be expected to arise in Yang-Mills-Higgs theory.

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APPENDIX A: EVALUATION OF THE FIRST-ORDER FIELDS

To arrive at the first-order fields for points on the world line which were given in Sec. IV B the following formulas for the κ integrals taken from $\kappa=0$ to $\kappa=\widetilde{\kappa}=\widetilde{\kappa}(\tau)=R^2/(2k\cdot R)$ have been used:

$$\int_0^{\widetilde{\kappa}} R'^{\alpha-2}d\kappa=\int_0^{\widetilde{\kappa}}(R^2-2\kappa k\cdot R)^{(\alpha-2)/2}d\kappa =\frac{R^\alpha}{\alpha k\cdot R}\,. \tag{A1}$$

For arbitrary real n one has the more general formula derived from Lebesgue's criteria of convergence[23]

$$\lim_{\alpha\to 0}\alpha\int_0^{\widetilde{\kappa}}\kappa^n(R^2-2\kappa k\cdot r)^{(\alpha-2)/2}d\kappa=\frac{\widetilde{\kappa}^n}{k\cdot R}\,. \tag{A2}$$

For the integrals involving Bessel functions one has

$$\lim_{\alpha\to 0}\alpha\int_0^{\widetilde{\kappa}}\kappa^n(R^2-2\kappa k\cdot R)^{(\alpha-2)/4}J_{\alpha/2}(\chi(R^2-2\kappa k\cdot R)^{1/2})d\kappa =\frac{\widetilde{\kappa}^n}{k\cdot R} \tag{A3}$$

following from the expansion of the Bessel function in powers of the argument $\chi R'$ together with Eq. (A2) and $J_0(0)=1$. Similarly, for a function $f(\tau,\kappa)$ which can be expanded into a convergent series in powers of κ one has

$$\lim_{\alpha\to 0}\alpha\int_0^{\widetilde{\kappa}}\kappa^n(R^2-2\kappa k\cdot R)^{(\alpha-2)/2}f(\tau,\kappa)d\kappa=\frac{\widetilde{\kappa}^n f(\tau,\widetilde{\kappa})}{k\cdot R}\,. \tag{A4}$$

We also need the formulas

$$\int_0^{\widetilde{\kappa}} R'^{\alpha-4}d\kappa=\frac{1}{k\cdot R}\frac{R^{\alpha-2}}{\alpha-2} \tag{A5}$$

and

$$\int_0^{\widetilde{\kappa}} \kappa R'^{\alpha-4}d\kappa=\frac{1}{(k\cdot R)^2}\frac{R^\alpha}{\alpha(\alpha-2)}\,. \tag{A6}$$

APPENDIX B: ANGULAR INTEGRALS

Here we discuss various angular integrals which appear in the evaluation of expressions like (4.24) and (4.25), using retarded null coordinates. (Compare Appendix B of

Ref. 21, and Ref. 23.) In these coordinates the point with Minkowski coordinates y^μ is characterized by the retarded proper time $\tau_{\rm ret}=\tau$, the retarded distance κ from the world line, and two angles θ and φ. For $x^\mu=z^\mu(\tau_0)$, which is the case treated in Sec. IV, we have (see Fig. 1)

$$R^\mu(\tau)=z^\mu(\tau_0)-z^\mu(\tau)\ , \tag{B1}$$

$$R^2=R^\mu(\tau)R_\mu(\tau)\ , \tag{B2}$$

$$y^\mu=z^\mu(\tau)+\kappa k^\mu(\tau,\theta,\varphi)\ , \tag{B3}$$

$$\kappa(\tau)=R^\mu(\tau)v_\mu(\tau)\ , \tag{B4}$$

$$n^\mu(\tau,\theta,\varphi)=k^\mu(\tau,\theta,\varphi)-v^\mu(\tau)\ . \tag{B5}$$

$k^\mu(\tau,\theta,\varphi)=(y^\mu-z^\mu(\tau))\kappa^{-1}$ is a lightlike vector satisfying

$$k^2=0,\quad k\cdot v=1\ , \tag{B6}$$

and n^μ is a spacelike unit vector.

In Sec IV an integration over all points in the past light cone of the field point x^μ (here *on* the world line) was performed, involving various directional integrals which can be reduced to the type

$$\int \frac{d^2\Omega}{4\pi} n^{\mu_1} n^{\mu_2}\cdots n^{\mu_m}\ ,\quad m=1,2,3,\ldots\ , \tag{B7}$$

by suitably expanding the expressions involved. Here $d^2\Omega$ is the angular part of the volume element d^4y expressed in retarded null coordinates

$$d^4y=\kappa^2 d\kappa\, d^2\Omega\, d\tau\ , \tag{B8}$$

which is given by

$$d^2\Omega=\epsilon_{\alpha\beta\gamma\delta} v^\alpha k^\beta k_\theta^\gamma k_\varphi^\delta d\theta\, d\varphi\ , \tag{B9}$$

where

$$k_\theta^\mu=\frac{\partial k^\mu}{\partial\theta},\quad k_\varphi^\mu=\frac{\partial k^\mu}{\partial\varphi}\ , \tag{B10}$$

and $\epsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor density.

The angles θ and φ determining a direction on the light cone are chosen in such a manner that for fixed θ and φ the vectors k^μ at two neighboring points on the world line are obtained without a rotation. This implies that

$$\frac{\partial k^\mu}{\partial\tau}=-(a\cdot k)k^\mu\ . \tag{B11}$$

Moreover, Eqs. (B6) imply the orthogonality properties

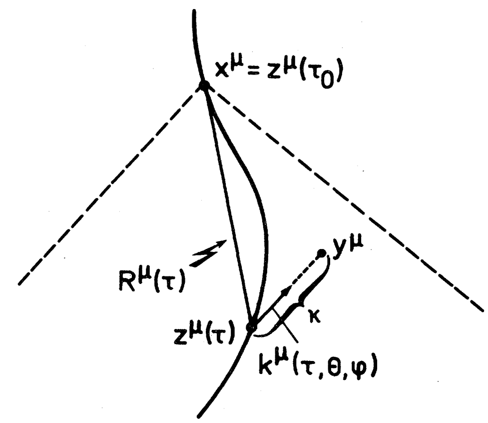

FIG. 1. Domain of integration for the computation of the first-order fields.

$$v\cdot k_\theta=v\cdot k_\varphi=0\ , \tag{B12}$$

$$k\cdot k_\theta=k\cdot k_\phi=0\ . \tag{B13}$$

One can, furthermore, demand that

$$k_\theta\cdot k_\varphi=0\ , \tag{B14}$$

since if (B14) is true for one value of τ, it is true for all τ because of (B11) and (B13), i.e.,

$$\frac{\partial}{\partial\tau}(k_\theta\cdot k_\varphi)=-2(a\cdot k)(k_\theta\cdot k_\varphi)\ . \tag{B15}$$

Returning now to (B7) one can show[21,23] that for odd values of m the integrals vanish:

$$\int\frac{d^2\Omega}{4\pi} n^{\mu_1} n^{\mu_2}\cdots n^{\mu_{2l+1}}=0\ , \tag{B16}$$

while for even m they are given by

$$\int\frac{d^2\Omega}{4\pi} n^{\mu_1} n^{\mu_2}\cdots n^{\mu_{2l}} =\frac{(-1)^l}{(2l+1)!}\sum_{\sigma\in S_{2l}} I^{\mu_{\sigma(1)}\mu_{\sigma(2)}}\cdots I^{\mu_{\sigma(2l-1)}\mu_{\sigma(2l)}}\ . \tag{B17}$$

Here the sum extends over the permutations forming the symmetric group of order $2l$, and the projection operator $I^{\mu\nu}=I^{\nu\mu}$ is given by Eq. (3.16) with (3.17). Replacing k^μ according to (B5) by $n^\mu+v^\mu$, expanding the denominators, and using Eqs. (B16) and (B17) and the properties of $I^{\mu\nu}$ one can resum the series and find that

$$\int\frac{d^2\Omega}{4\pi}\frac{1}{k\cdot R}=\frac{1}{\kappa}\sum_{m=0}^{\infty}\frac{1}{2m+1}\left[1-\frac{R^2}{\kappa^2}\right]^m\ , \tag{B18}$$

$$\int\frac{d^2\Omega}{4\pi}\frac{n^\nu}{k\cdot R}=\frac{R^\nu-\kappa v^\nu}{\kappa^2}\sum_{m=0}^{\infty}\frac{1}{2m+3}\left[1-\frac{R^2}{\kappa^2}\right]^m\ , \tag{B19}$$

$$\int\frac{d^2\Omega}{4\pi}\frac{1}{(k\cdot R)^2}=\frac{1}{R^2}\ , \tag{B20}$$

$$\int \frac{d^2\Omega}{4\pi}\frac{n^\nu}{(k\cdot R)^2}=2\frac{R^\nu-\kappa v^\nu}{\kappa^3}\sum_{m=0}^{\infty}\frac{m+1}{2m+3}\left[1-\frac{R^2}{\kappa^2}\right]^m, \tag{B21}$$

$$\int \frac{d^2\Omega}{4\pi}\frac{n^\mu n^\nu}{(k\cdot R)^2}=-\frac{I^{\mu\nu}}{\kappa^2}\sum_{m=0}^{\infty}\frac{1}{2m+3}\left[1-\frac{R^2}{\kappa^2}\right]^m+\left[\frac{R^\mu-\kappa v^\mu}{\kappa^2}\right]\left[\frac{R^\nu-\kappa v^\nu}{\kappa^2}\right]\sum_{m=0}^{\infty}\frac{2m}{2m+3}\left[1-\frac{R^2}{\kappa^2}\right]^m. \tag{B22}$$

The series in these equations converge for

$$\left|1-\frac{R^2}{\kappa^2}\right|<1\,. \tag{B23}$$

Furthermore $R^\nu-\kappa v^\nu$ is a spacelike vector since it is orthogonal to the timelike vector v^ν. Therefore

$$(R^\mu-\kappa v^\mu)^2<0\,. \tag{B24}$$

This, together with the fact that R^μ is timelike, is equivalent to

$$0\le\frac{R^2}{\kappa^2}<1\,. \tag{B25}$$

The quantity $1-R^2/\kappa^2$ can thus be unity only for $R^2=0$ and $\kappa^2\neq 0$, i.e., for x off the world line and τ approaching τ_0.[29] For $x=z(\tau_0)$ one has

$$\lim_{\tau\to\tau_0}\frac{R^2}{\kappa^2}=1\,, \tag{B26}$$

as can be immediately checked by expanding R^2 and κ^2 around τ_0. One thus sees that the expansions (B18)—(B22) converge for all values of $\tau\le\tau_0$ for which the result of the angular integrations are needed in the integrals of Sec. IV. The integrals in Eqs. (4.48) and (4.49) for the case of a nonvanishing χ, however, are more complicated since they involve a Bessel function of first or second order with a direction dependent argument.

APPENDIX C: TIME-SYMMETRIC FIELD THEORY AND ACTION-AT-A-DISTANCE THEORY

The derivation in the preceding paper of the exact laws of motion (I2.100), (I2.101), and (I2.83) is independent of any assumptions made on the solutions to the field equations (I2.11) and (I2.12); the same is true for the derivation in Secs. II and IV of the zeroth- and first-order laws of motion (2.16), (2.17), (2.15), and (4.10), (3.21), (4.7), respectively. The form of these laws is the same regardless of these assumptions; it is only the choice of the fields to be inserted into these laws which depends on the assumptions. In the following we shall only be concerned with the first-order laws. The zero-order fields chosen in Sec. IV were of the form (4.18) and (4.19), where the external field consisted of the sum of the retarded fields of all other particles, and we were then concerned with the calculation of the retarded solutions of the first-order field equations (4.14) and (4.15) which contained the zero-order fields (4.18) and (4.19). Instead we can choose the symmetric (half-retarded, half-advanced) zero-order fields

$$_0\vec{A}^{\nu}={}_0\vec{A}^{\nu}_{s}+{}_0\vec{A}^{\nu}_{xs}\,,\quad {}_0\vec{\phi}={}_0\vec{\phi}_s+{}_0\vec{\phi}_{xs}\,, \tag{C1}$$

where s stands for the symmetric Riesz solution of the zero-order field equations for the particle under consideration and the external field consists of the sum of the symmetric fields of all other particles; the first-order field equations (4.14) and (4.15) will then contain the expressions (C1) rather than (4.18) and (4.19), and it is their symmetric solutions which must be inserted into the first-order laws of motion (4.10) and (4.7) to obtain the first-order equations of motion of the time-symmetric field theory.

The zero-order symmetric Riesz potentials are

$$\begin{aligned}{}_0\vec{A}^{\nu(\alpha)}_{s}&=\tfrac{1}{2}({}_0\vec{A}^{\nu(\alpha)}_{r}+{}_0\vec{A}^{\nu(\alpha)}_{a})\,,\\ {}_0\vec{\phi}^{(\alpha)}_{s}&=\tfrac{1}{2}({}_0\vec{\phi}^{(\alpha)}_{r}+{}_0\vec{\phi}^{(\alpha)}_{a})\,.\end{aligned} \tag{C2}$$

Here r and a indicate that the integration must be carried out from $-\infty$ to the retarded point of x, and from the advanced point of x to $+\infty$, respectively. It should be noted that *on* the world line (but *not* off it) ${}_0\vec{A}^{\nu}_{s}$ vanishes; ${}_0\vec{\phi}_s$ reduces to an integral over the entire world line of the particle, analogous to the case of meson theory. Since the integral corresponds to a self-action which has no meaning from the point of view of action at a distance, it was proposed in Ref. 20 to omit this term to obtain the basic equation of motion of the theory of action at a distance for combined electromagnetic and mesic interactions. This new equation can be derived from a variational principle which also allows the definition of an adjunct field theory and of detailed conservation laws in terms of particle quantities, as discussed in Refs. 20 and 26. We can proceed in complete analogy in the case of Young-Mills-Higgs interactions. Radiation effects can be included by application of the Wheeler-Feynman condition[30]

$$\sum_{\text{all }k}{}_0\vec{A}^{\nu}_{kr}=\sum_{\text{all }k}{}_0\vec{A}^{\nu}_{ka}\,,\quad \sum_{\text{all }k}{}_0\vec{\phi}_{kr}=\sum_{\text{all }k}{}_0\vec{\phi}_{ka}\,, \tag{C3}$$

for the total field of all particles, since then

$$\begin{aligned}\sum_{k\neq i}{}_0\vec{A}^{\nu}_{ks}&=\sum_{k\neq i}{}_0\vec{A}^{\nu}_{kr}+{}_0\vec{A}^{\nu}_{id}\,,\\ {}_0\vec{A}^{\nu}_{id}&=\tfrac{1}{2}({}_0\vec{A}^{\nu}_{ir}-{}_0\vec{A}^{\nu}_{ia})\,,\\ \sum_{k\neq i}{}_0\vec{\phi}_{ks}&=\sum_{k\neq i}{}_0\vec{\phi}_{kr}+{}_0\vec{\phi}_{id}\,,\\ {}_0\vec{\phi}_{id}&=\tfrac{1}{2}({}_0\vec{\phi}_{ir}-{}_0\vec{\phi}_{ia})\,.\end{aligned} \tag{C4}$$

The sums on the LHS are ${}_0\vec{A}^{\nu}_{xs}$ and ${}_0\vec{\phi}_{xs}$ of Eq. (C1), while the sums on the RHS are precisely ${}_0\vec{A}^{\nu}_{\text{ex}}$ and ${}_0\vec{\phi}_{\text{ex}}$ of Eqs. (4.18) and (4.19). The last terms (evaluated at the position of particle i) are the radiation-reaction terms of the theory of action at a distance. The reaction terms of field theory differ from them only by the above-mentioned integral over the world line which we had proposed to omit. Since, as noted before, ${}_0\vec{A}^{\,\nu}_{is}$ vanishes, ${}_0\vec{A}^{\,\nu}_{id}$ equals ${}_0\vec{A}^{\,\nu}_{ir}$. As a consequence of this and (C4), the zero-order fields to be inserted into the zero-order law of motion (2.16) (with the dipole terms omitted) are

$$ {}_0\vec{A}^{\,\nu}={}_0\vec{A}^{\,\nu}_r+{}_0\vec{A}^{\,\nu}_{\rm ex}\,,\quad {}_0\vec{\phi}={}_0\vec{\phi}_d+{}_0\vec{\phi}_{\rm ex}\,. \tag{C5}$$

The resultant equation is identical with Eq. (3.20) except for containing $\frac{1}{2}\{\int_{-\infty}^{\tau}-\int_{\tau}^{\infty}\}$ instead of $\int_{-\infty}^{\tau}$.

In the following we shall extend the corresponding considerations to the next order. However, because of the complexity of the equations of motion we shall not attempt to find a variational principle, but only consider the first-order equations of motion following after application of the conditions corresponding to (C3) and (C4) for the sum of the zero- and first-order fields. The fact that it is indeed possible to extend the considerations to nonlinear theories order-by-order is discussed in Ref. 11 (see Appendix II and especially footnote 71).

The only problem arising is the proper handling of the Wheeler-Feynman condition (C3) as one proceeds from one order to the next. It is clear that at each order the condition has to be applied to the sum of the fields up to that order. However, the fields of higher order contain integrals over the total fields of lower order. It seems appropriate, in analogy with our treatment of the Lorentz condition discussed in Sec. IV, to maintain the lower-order condition (2.18) within the integrand, but allow for the actual (higher-order) motion. Equivalently, the first-order field equations to be solved are the *same* whether we are dealing with the retarded or the time-symmetric field theory, since the total zero-order fields entering these equations are the same in both cases because of the condition (C3); the difference only arises in taking as the *solutions* of Eqs. (4.14) and (4.15) the retarded and the time-symmetric fields, respectively, and in the interpretation of what is to be considered as an external field.

If we insert the first-order time-symmetric fields thus obtained into the first-order time-symmetric field-theoretical equations of motion, the form of the conditions (C3) (applied to the sum of the zero-order and first-order fields) guarantees that we obtain precisely the equations of motion of the retarded field theory, i.e., in our case Eqs. (4.59) and (4.60) with the fields listed at the end of Sec. IV, without any need for additional calculations. If instead we insert the first-order time-symmetric fields into the first-order time-symmetric action-at-a-distance equations of motion, application of (C3) does *not* yield the equations of motion of the retarded field theory; the resulting equations differ from the field-theoretical ones by containing everywhere half the difference of the retarded and the advanced Riesz field instead of the retarded one.

As an example we consider the first-order correction to the Higgs field. Instead of ${}_1\vec{\phi}$ determined by Eq. (4.25) as an integral over the past light cone over the external and retarded zero-order fields, we have a corresponding expression

$$ {}_1\vec{\phi}_d\equiv{}_1\vec{\phi}_{xx}+{}_1\vec{\phi}_{xd}+{}_1\vec{\phi}_{dd}\,, \tag{C6}$$

in terms of the external and radiation fields. Furthermore, each integral extending over the past light cone is replaced by half the difference between the integrals over the past and the future light cones. Considering the term ${}_1\vec{\phi}_{dd}$ as an example, we can proceed as we did to Eq. (4.49), but now get

$$ {}_1\vec{\phi}_{dd}=\left\{\int_{-\infty}^{\tau_0}-\int_{\tau_0}^{\infty}\right\}d\tau\int\cdots\,. \tag{C7}$$

Each of these integrals contains contributions both from ${}_0\vec{\phi}_r$ and from ${}_0\vec{\phi}_a$. Therefore we get eight types of integrals corresponding to the four different products in the two integrals of (C7).

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[10]This is generally the case for special-relativistic equations of motion; see P. Havas, in *Recent Developments in General Relativity* (Pergamon, New York, 1962), p. 259, and Ref. 5. It is also the case for the Lorentz-invariant approximation in general relativity developed in Ref. 11.

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[15]For the definition of these quantities compare Eqs. (I2.64) and (I2.48).

[16]P. Havas (unpublished).

[17]These definitions differ from those given in Refs. 2, 3, and 8 by replacement of α by $\alpha+2$. This is more convenient in allowing us to consider the limit $\alpha\to 0$ rather than $\alpha\to 2$.

[18]As usual $a_{[\mu}k_{\nu]}=\frac{1}{2}(a_\nu k_\nu-a_\nu k_\mu)$; a_μ is the acceleration, $a_\mu=\dot{v}_\mu=dv_\mu/d\tau$, $a^2=a_\mu a^\mu$, and $k_\mu=k_\mu(\tau,\theta,\varphi)$ is the light-cone vector at the retarded point x, i.e., $x^\mu=z^\mu(\tau_{\rm ret})+\kappa k^\mu$,

with $k_\mu k^\mu = 0$ and $\kappa = \kappa(\tau_{ret}) = R_\mu v^\mu |_{\tau=\tau_{ret}}$. Moreover, to obtain the first term on the RHS of Eq. (3.11b) the appropriate analytic continuation of the Bessel function has to be used, applying the formula $J_{\alpha/2}(\chi R) \approx (\frac{1}{2}\chi R)^{\alpha/2}$ for small values of χR.

[19]This vector n^ν is not to be confused with the vector n_i^ν appearing in the preceding paper.

[20]For a similar shift in the case of classical meson fields, see P. Havas, Phys. Rev. 93, 882 (1954), and references given there.

[21]See, e.g., C. Teitelboim, D. Villaroel, and Ch. G. van Weert, Riv. Nuovo Cimento 3, 9 (1980). The ρ of their paper is denoted here by κ, in agreement with standard usage (see, e.g., Refs. 13).

[22]Similar simplifications arise in the corresponding integrals of Ref. 8, Sec. IV, if they are evaluated by the present method. However, in higher orders than the first the integrands will involve singularities of higher order than κ^{-2} as κ approaches zero, thus requiring the use of appropriately regularized expressions in the effective source terms ${}_n\vec{K}^\nu(y)$ and ${}_n\vec{\Lambda}(y)$ for $n \geq 2$ given in terms of the Riesz potentials (3.1) and (3.2).

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[24]For related discussions see B. Bertotti and J. Plebański, Ann. Phys. (N.Y.) 11, 169 (1960); and H. Goenner, J. Math. Phys. 11, 1645 (1970) (who analyzed the higher-order particle fields in general relativity in terms of Green's functions) and Ref. 8.

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[29]As mentioned in Sec. IV, the integrals for x off the world line still contain a regularizing factor R^α, with the limit $\alpha \to 0$ to be taken at the end.

[30]These conditions were suggested in electrodynamics by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945), and extended to mesodynamics in Refs. 26. For a discussion of the derivation and its limitations see also P. Havas, Phys. Rev. 74, 456 (1948); 86, 974 (1952).