# Theory of motion for monopole-dipole singularities of classical Yang-Mills-Higgs fields. I. Laws of motion

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In two recent papers, the general form of the laws of motion for point particles which are multipole sources of the classical coupled Yang-Mills-Higgs fields was determined by Havas, and for the special case of monopole singularities of a Yang-Mills field an iteration procedure was developed by Drechsler and Rosenblum to obtain the equations of motion of mass points, i.e., the laws of motion including the explicit form of the fields of all interacting particles. In this paper we give a detailed derivation of the laws of motion of monopole-dipole singularities of the coupled Yang-Mills-Higgs fields for point particles with mass and spin, following a procedure first applied by Mathisson and developed by Havas. To obtain the equations of motion, a systematic approximation method is developed in the following paper for the solution of the nonlinear field equations and determination of the fields entering the laws of motion found here to any given order in the coupling constant $g$.

## I. INTRODUCTION

In recent years, the study of non-Abelian gauge theories has been at the center of interest in elementary particle physics. An early example of such a theory investigated in some depth (which will be discussed in more detail below) was that of the interacting mesic and electromagnetic fields. A theory with some similar features which was formally more satisfactory was proposed by Yang and Mills in 1954,[1] and the structure of general gauge theories was investigated by Utiyama.[2] In 1964 Higgs[3] proposed a mechanism by which non-Abelian local gauge symmetry could be broken without introducing Goldstone bosons,[4] and a few years later it was shown by 't Hooft[5] that this mechanism does not destroy the renormalizability of a gauge theory.

Because of the great complexity of quantum-mechanical investigations of non-Abelian gauge theories, there has been renewed interest in attempts to gain information about such theories by first studying their classical counterparts, and several extensive reviews of these studies are available.[6,7] Our own interest in such studies arose from two different motivations. First, one of us had been investigating the problem whether a non-Abelian charge can radiate away charge, and a preliminary investigation of the corresponding classical problem seemed to indicate that such a phenomenon cannot occur.[8] Second, two of us had investigated the classical equations of motion of point singularities of interacting (nonlinear) mesic and electromagnetic fields for some time[9–12] and had noted that the methods developed could be applied to other gauge theories such as the Yang-Mills field, and possibly to the general theory of relativity.

These methods were recently applied to determining the "laws of motion" of multipole singularities of arbitrary order of the coupled Yang-Mills-Higgs fields[13] in close analogy to a similar study of the nonlinear meson theory.[9,10] These laws, for any particular mass point, can be determined exactly, and depend on the undetermined fields due to other particles. To obtain explicit expressions for these fields (i.e., determine the "equations of motion"[14]) is only possible by an approximation method. Such a method was developed originally for the corresponding problem in general relativity,[15] and a similar method was used to find the equations of motion of monopole singularities of the nonlinear meson field.[11] These equations of motion, to any given order $n$, can be considered as integrability conditions of the field equations of the $n+1$st order, and only require an explicit knowledge of the $n$th-order fields. These fields can be obtained by various methods, most simply by that of Riesz potentials,[16] as discussed in some detail in Refs. 15 and 11.

Although this paper is in several respects closely analogous to M and deals with a special case of the laws of motion obtained in H, no knowledge of those papers is assumed. However, to facilitate comparison with the results of those papers, we use the same notation. A brief discussion of the fundamental field equations is presented in Sec. II; the laws of motion are determined for particles with mass and spin which are monopole-dipole singularities of the coupled Yang-Mills-Higgs fields.[17] The derivation of these laws is outlined both to avoid the need for a detailed study of H by the reader (where the necessity of dealing with the problem of general multipoles obscures the simplicity of the method used and the ease with which the results can be obtained for the interactions considered here), and in preparation for the following paper, where we will determine the equations of motion.

Since we are only dealing with the laws of motion in this paper, we do not have to consider any solutions of the field equations, and in particular do not have to choose between retarded and time-symmetric fields. Such a choice will have to be made in the following paper, however, once the general approximation method has been developed. The nonlinear field equations will be solved there under the assumption of retarded fields by an iteration procedure using the Riesz method (compare also Ref. 18). Because of the complexity of the calculations, this will be done only for the case of monopole interactions. The results are given in integral form. In the absence of Higgs fields and of spin, the results reduce to those obtained earlier by two of us.[19]

Since the equations of motion resulting from the laws of motion given here and the fields obtained in the following paper are only approximate, we cannot follow the procedure used for linear theories[20] to obtain a general action-at-a-distance formulation of the theory presented here. However, it is still possible to obtain the approximate field-theoretical equations of motion with time-symmetric interactions and the corresponding action-at-a-distance equations of motion. The derivation is outlined in Appendix C of the following paper.

## II. FIELD EQUATIONS AND LAWS OF MOTION

We consider a four-space with coordinates $x^\rho$, Greek letters taking the values 0,1,2,3, where $x^0$ is the time coordinate. Repetition of an index implies summation over this range. The velocity of light is taken as unity. The metric tensor $\eta_{\mu\nu}$ is given by

$$\eta_{\mu\nu}=0 \quad \text{if } \mu\neq\nu\,, \qquad \eta_{00}=-\eta_{11}=-\eta_{22}=-\eta_{33}=1\,. \tag{2.1}$$

We shall use the abbreviations

$$\partial_\mu\equiv\frac{\partial}{\partial x^\mu}\,,\quad \partial_{\alpha\beta\cdots}\equiv\partial_\alpha\partial_\beta\cdots\,,\quad \Box\equiv\partial^\mu\partial_\mu\,. \tag{2.2}$$

All field quantities are taken as three-component vectors in "charge space" or isospace, referred to a local set of orthonormal vectors $\vec\alpha,\vec\beta,\vec\gamma$ as

$$\vec\Psi\equiv(\Psi_1,\Psi_2,\Psi_3)\equiv(\vec\alpha\Psi_1+\vec\beta\Psi_2+\vec\gamma\Psi_3)\,. \tag{2.3}$$

Scalar and vector products in isospace are denoted by a centered dot and $\wedge$, respectively.

The Yang-Mills theory with a Higgs field can be obtained from the Lagrangian [Eq. (H29)]

$$4\pi\mathscr{L}\equiv\frac{1}{2}\left\{\vec{\mathrm F}_\mu\cdot\vec{\mathrm F}^\mu-\chi^2\phi^2\left[1+V\left[\frac{g^2\phi^2}{\chi^2}\right]\right]\right\} +4\pi\vec\rho\cdot\vec\phi-\tfrac{1}{4}\vec{\mathrm F}_{\mu\nu}\cdot\vec{\mathrm F}^{\mu\nu}-4\pi\vec{\mathrm j}_\mu\cdot\vec{\mathrm A}^\mu\,. \tag{2.4}$$

Here

$$\vec{\mathrm F}_{\mu\nu}\equiv\partial_\mu\vec{\mathrm A}_\nu-\partial_\nu\vec{\mathrm A}_\mu-g\vec{\mathrm A}_\mu\wedge\vec{\mathrm A}_\nu \tag{2.5}$$

is the field tensor of the Yang-Mills field expressed in terms of its potentials $\vec{\mathrm A}_\mu$, where $g$ is a dimensionless constant, and

$$\vec{\mathrm F}_\mu\equiv D_\mu\vec\phi \tag{2.6}$$

is the Higgs field strength expressed in terms of the Higgs potential $\vec\phi$. The Lagrangian (2.4) differs from the one apparently first used by 't Hooft[21] (apart from notation) by the inclusion of Yang-Mills sources $\vec{\mathrm j}_\mu$ and by allowing an arbitrary function $V$ of $\phi^2\equiv\vec\phi\cdot\vec\phi$ times $g^2/\chi^2$ rather than one proportional to $\phi^2$ alone. It is understood that this function does not contain a constant term, since the term $-\frac{1}{2}\chi^2\phi^2$ has been included explicitly in Eq. (2.4). If that term alone is taken into account the constant $\chi$ is inversely proportional to the range of the static force due to the Higgs field, and proportional to the mass of the field in the quantized version of the theory. The constant $g^2/\chi^2$ in the argument of $V$ is introduced to make it dimensionless through the factor $1/\chi^2$ and to allow the possibility of expansion in $g$ in applications. $\vec\rho$ is the source density of the Higgs field.

The "covariant derivative" $D^\mu$ is defined as

$$D^\mu\vec\Psi\equiv(\partial^\mu-g\vec{\mathrm A}^\mu\wedge)\vec\Psi \tag{2.7}$$

for any vector in isospace; this operator satisfies the commutation relation

$$(D^\mu D^\nu-D^\nu D^\mu)\vec\Psi=-g\vec{\mathrm F}^{\mu\nu}\wedge\vec\Psi\,. \tag{2.8}$$

Variation of $\int\mathscr{L}\,d^4x$ with respect to the potentials yields the field equations

$$D_\mu\vec{\mathrm F}^{\mu\nu}-g\vec\phi\wedge D^\nu\vec\phi=4\pi\vec{\mathrm j}^{\,\nu}\,, \tag{2.9}$$

$$D_\mu\vec{\mathrm F}^\mu+\chi^2\vec\phi+F\vec\phi=4\pi\vec\rho\,, \tag{2.10}$$

$$F\equiv\frac{1}{2}\frac{\partial}{\partial\phi^2}\left[\chi^2\phi^2V\left[\frac{g^2\phi^2}{\chi^2}\right]\right],$$

which can be written in the equivalent form

$$\Box\vec{\mathrm A}^\nu-\partial^\nu{}_\mu\vec{\mathrm A}^\mu-g\partial_\mu(\vec{\mathrm A}^\mu\wedge\vec{\mathrm A}^\nu)-g\vec{\mathrm A}_\mu\wedge(\partial^\mu\vec{\mathrm A}^\nu-\partial^\nu\vec{\mathrm A}^\mu-g\vec{\mathrm A}^\mu\wedge\vec{\mathrm A}^\nu)-g\vec\phi\wedge D^\nu\vec\phi=4\pi\vec{\mathrm j}^{\,\nu}\,, \tag{2.11}$$

$$D^\mu D_\mu\vec\phi+\chi^2\vec\phi+F\vec\phi=4\pi\vec\rho\,. \tag{2.12}$$

Equations (2.9) or (2.11) imply

$$D_\nu(\vec{\mathrm j}^{\,\nu}+\vec{\mathrm J}^{\,\nu})=0\,, \tag{2.13}$$

where the charge-current density associated with the Higgs field is given by

$$4\pi\vec{\mathrm J}^{\,\nu}\equiv g\vec\phi\wedge D^\nu\vec\phi \tag{2.14}$$

with

$$D_\nu\vec{\mathrm J}^{\,\nu}=g\vec\phi\wedge\vec\rho \tag{2.15}$$

as a consequence of the field equations. It should be noted that the "continuity equation" (2.13) involves covariant derivatives, and thus does not have the same simple physical meaning as the continuity equations encountered, e.g., in electrodynamics and continuum mechanics.

It is clear from Eqs. (2.9), (2.14), and (2.15) that the classical Higgs field $\vec{\phi}$ generated by a pointlike source produces (in the non-Abelian case, as discussed later) an extended Yang-Mills charge distribution with corresponding dynamical effects in close analogy to the electromagnetic effects of the charged meson field considered in M. The effective size of such a charge distribution is determined by the magnitude of the constant $\chi^{-1}$.

The energy-momentum tensor can be obtained from the Lagrangian (2.4) by standard methods as

$$4\pi T_f^{\mu\nu} \equiv \vec{F}^{\mu}\cdot\vec{F}^{\nu} - \tfrac{1}{2}\eta^{\mu\nu}\vec{F}_\rho\cdot\vec{F}^\rho + \tfrac{1}{2}\eta^{\mu\nu}\chi^2\phi^2 + \eta^{\mu\nu}\chi^4\phi^4V^2 + \vec{F}^{\mu\rho}\cdot\vec{F}_\rho{}^\nu + \tfrac{1}{4}\eta^{\mu\nu}\vec{F}^{\rho\sigma}\cdot\vec{F}_{\rho\sigma} . \tag{2.16}$$

A straightforward calculation, using the field equations (2.11) and (2.12) and the Jacobi identity (also called Bianchi identity)

$$D_\lambda\vec{F}_{\mu\nu} + D_\mu\vec{F}_{\nu\lambda} + D_\nu\vec{F}_{\lambda\mu} = 0 \tag{2.17}$$

following from the definition (2.5) yields

$$\partial_\mu T_f^{\mu\nu} = \vec{\rho}\cdot D^\nu\vec{\phi} + \vec{j}_\mu\cdot\vec{F}^{\mu\nu} . \tag{2.18}$$

We shall be concerned with $N$ particles, which are taken as mass points with intrinsic dipole moments, with coordinates $z_i^\rho(\tau_i)$, where $\tau_i$ is the proper time of the $i$th particle, defined by

$$d\tau_i \equiv (\eta_{\mu\nu}dz_i^\mu dz_i^\nu)^{1/2} . \tag{2.19}$$

The corresponding matter energy-momentum tensor is taken as

$$T_m^{\mu\nu} \equiv \sum_i \int_{-\infty}^{\infty} \{p_i^{\mu\nu}(\tau_i)\delta^4(s_i^\sigma) + \partial_\rho[p_i^{\rho\mu\nu}(\tau_i)\delta^4(s_i^\sigma)]\}d\tau_i , \tag{2.20}$$

where

$$p_i^{\mu\nu} = p_i^{\nu\mu}, \quad p_i^{\rho\mu\nu} = p_i^{\rho\nu\mu} \tag{2.21}$$

are the matter monopole and dipole moments, whose exact form will be determined below, $\delta^4$ is the fourfold product of $\delta$ functions, and

$$s_i^\sigma \equiv x^\sigma - z_i^\sigma(\tau_i) . \tag{2.22}$$

Similarly, we take as the source densities for the Yang-Mills and the Higgs field

$$\vec{j}^{\,\mu} \equiv \sum_i \int_{-\infty}^{\infty} \{\vec{Q}_i^\mu(\tau_i)\delta^4(s_i^\sigma) + D_\rho[\vec{S}_i^{\rho,\mu}(\tau_i)\delta^4(s_i^\sigma)]\}d\tau_i \tag{2.23}$$

and

$$\vec{\rho} \equiv \sum_i \int_{-\infty}^{\infty} \{\vec{S}_i(\tau_i)\delta^4(s_i^\sigma) + D_\rho[\vec{S}_i^\rho(\tau_i)\delta^4(s_i^\sigma)]\}d\tau_i , \tag{2.24}$$

respectively, where $\vec{Q}_i^\mu$, $\vec{S}_i^{\rho,\mu}$, and $\vec{S}_i, \vec{S}_i^\rho$ are the monopole and dipole moments of the sources for the Yang-Mills and the Higgs field, respectively. It should be noted that, unlike Eq. (2.20), Eqs. (2.23) and (2.24) involve the covariant divergence $D_\rho$ rather than the ordinary divergence appearing in Eq. (2.20).

Either as a consequence of the general theory of relativity[22,23] or as a postulate we have a differential conservation law for the total energy-momentum tensor

$$\partial_\mu(T_m^{\mu\nu} + T_f^{\mu\nu}) = 0 . \tag{2.25}$$

To obtain the laws of motion we use a method described some time ago,[22,23] which is a development of one due to Mathisson.[24] We multiply Eq. (2.25) by a function $\xi_\nu(x^\rho)$, substitute the expressions (2.20), (2.18), (2.23), and (2.24), and integrate over all $x$ to obtain

$$\sum_i \int \{(p_i^{\mu\nu}\partial_\mu\delta^4 + p_i^{\rho\mu\nu}\partial_{\mu\rho}\delta^4) + D^\nu\vec{\phi}\cdot[\vec{S}_i\delta^4 + D_\rho(\vec{S}_i^\rho\delta^4)] + \vec{F}_\mu{}^\nu\cdot[\vec{Q}_i^\mu\delta^4 + D_\rho(\vec{S}_i^{\rho,\mu}\delta^4)]\}\xi_\nu d\tau_i d^4x = 0 . \tag{2.26}$$

Similarly[10,13,19,25] we multiply Eq. (2.13) by a function $\vec{\xi}(x^\rho)$, substitute the expressions (2.15), (2.23), and (2.24), and integrate over all $x$ to obtain

$$\sum_i \int \{D_\nu[\vec{Q}_i^\nu\delta^4 + D_\rho(\vec{S}_i^\rho\delta^4)] + g\vec{\phi}\wedge[\vec{S}_i\delta^4 + D_\rho(\vec{S}_i^\rho\delta^4)]\}\cdot\vec{\xi}\, d\tau_i d^4x = 0 . \tag{2.27}$$

Here $\xi_\nu$ and $\vec{\xi}$ are completely arbitrary except for vanishing at the limits of the $\tau_i$ and $x$ integrations.

Because of its simpler structure, we shall first consider Eq. (2.27). We can remove the covariant derivatives from the $\delta$ functions by successive integrations by parts. Because of their form (2.7) all terms in (2.27) involving such derivatives contain mixed triple vector products; due to the properties of such products the $D$'s can be treated just like $\partial$'s in these integrations, except that due account has to be taken of their order because of the commutation relation (2.8). Thus we obtain

$$\sum_i \int [-\vec{Q}_i^\nu\cdot D_\nu\vec{\xi} + \vec{S}_i^{\rho,\nu}\cdot D_\rho D_\nu\vec{\xi} + g(\vec{\phi}\wedge\vec{S}_i)\cdot\vec{\xi} - g(D_\rho\vec{\xi}\wedge\vec{\phi} + \vec{\xi}\wedge D_\rho\vec{\phi})\cdot\vec{S}_i^\rho]\delta^4 d\tau_i d^4x = 0 . \tag{2.28}$$

We can now carry out the $x$ integrations, which yield

$$\sum_i \int [-\vec{Q}_i^\nu\cdot D_\nu\vec{\xi} + \vec{S}_i^{\rho,\nu}\cdot D_\rho D_\nu\vec{\xi} + g(\vec{\phi}\wedge\vec{S}_i)\cdot\vec{\xi} - g(D_\rho\vec{\xi}\wedge\vec{\phi} + \vec{\xi}\wedge D_\rho\vec{\phi})\cdot\vec{S}_i^\rho]d\tau_i = 0 , \tag{2.29}$$

where it is understood that all quantities are to be evaluated at $z_i^\sigma(\tau_i)$.

Because of the arbitrariness of $\vec{\xi}$ and its derivatives the coefficients of $\vec{\xi}$ and of each of its derivatives must vanish separately. However, this rule cannot be applied to Eq. (2.29) as it stands. We first consider the term $\vec{S}_i^{\rho,\nu}\cdot D_\rho D_\nu \vec{\xi}$. We introduce the four-velocity of the $i$th particle,

$$v_i^\rho \equiv \frac{dz_i^\rho}{d\tau_i}, \quad v_i^\rho v_{i\rho}=1 , \tag{2.30}$$

and note that

$$v_i^\nu D_\nu \vec{\Psi} = v_i^\nu \partial_\nu \vec{\Psi} - g v_i^\nu \vec{A}_\nu \wedge \vec{\Psi} = \frac{d\vec{\Psi}}{d\tau_i} - g v_i^\nu \vec{A}_\nu \wedge \vec{\Psi} \equiv \frac{D\vec{\Psi}}{d\tau_i} \tag{2.31}$$

for any vector in isospace. Breaking up the dipole moment $\vec{S}_i^{\rho,\nu}$ in components parallel and perpendicular to the four-velocity in the first index as

$$\vec{S}_i^{\rho,\nu} = v_i^\rho \vec{T}_i^\nu + {}^*\vec{S}_i^{\rho,\nu}, \quad {}^*\vec{S}_i^{\rho,\nu} v_{i\rho}=0 , \tag{2.32}$$

we can write the second term in the integral (2.29) as

$$\begin{aligned}\int \vec{S}_i^{\rho,\nu}\cdot D_\rho D_\nu \vec{\xi}\, d\tau_i &= \int \vec{T}_i^\nu v_i^\rho D_\rho D_\nu \vec{\xi}\, d\tau_i + \int {}^*\vec{S}_i^{\rho,\nu} D_\rho D_\nu \vec{\xi}\, d\tau_i \\ &= \int \vec{T}_i^\nu \frac{D}{d\tau_i}(D_\nu \vec{\xi}) d\tau_i + \int {}^*\vec{S}_i^{\rho,\nu} D_\rho D_\nu \vec{\xi}\, d\tau_i \\ &= -\int \frac{D\vec{T}_i^\nu}{d\tau_i} D_\nu \vec{\xi} d\tau_i + \int {}^*\vec{S}_i^{\rho,\nu} D_\rho D_\nu \vec{\xi}\, d\tau_i ,\end{aligned} \tag{2.33}$$

using Eq. (2.31) and a subsequent integration by parts. Therefore the term originating in $v_i^\rho \vec{T}_i^\nu$ does not contribute to the coefficient of the second derivative of $\vec{\xi}$, but only of the first one; thus $DT_i^\nu/d\tau_i$ can be absorbed in the as-yet-undetermined $\vec{Q}_i^\nu$ without loss of generality, i.e., we can take

$$\vec{T}_i^\nu = 0 . \tag{2.34}$$

Now we break up ${}^*\vec{S}_i^{\rho,\nu}$ into

$${}^*\vec{S}_i^{\rho,\nu} = \vec{S}_i^{(\rho\nu)} + \vec{S}_i^{[\rho\nu]}, \quad \vec{S}_i^{(\rho\nu)} v_{i\rho} = \vec{S}_i^{[\rho\nu]} v_{i\rho} = 0 , \tag{2.35}$$

where the parentheses and square brackets, as usual, denote symmetry and antisymmetry, respectively, of an index pair. Then we have

$$\begin{aligned}\vec{S}_i^{\rho,\nu}\cdot D_\rho D_\nu \vec{\xi} &= \tfrac{1}{2}\vec{S}_i^{(\rho\nu)}\cdot(D_\rho D_\nu + D_\nu D_\rho)\vec{\xi} + \tfrac{1}{2}\vec{S}_i^{[\rho\nu]}\cdot(D_\rho D_\nu - D_\nu D_\rho)\vec{\xi} \\ &= \tfrac{1}{2}\vec{S}_i^{(\rho\nu)}\cdot(D_\rho D_\nu + D_\nu D_\rho)\vec{\xi} - \tfrac{1}{2} g \vec{S}_i^{[\rho\nu]}\cdot(\vec{F}_{\rho\nu}\wedge\vec{\xi}) ,\end{aligned} \tag{2.36}$$

where the last equality follows from Eq. (2.8). Thus the vanishing of the coefficient of the second derivatives of $\vec{\xi}$ only requires

$$\vec{S}_i^{(\rho\nu)} = 0 . \tag{2.37}$$

Now we similarly break up $\vec{Q}_i^\nu$ and $\vec{S}_i^\nu$ into components parallel and perpendicular to $v_{i\nu}$:

$$\begin{aligned}\vec{Q}_i^\nu &= {}^*\vec{Q}_i^\nu + \vec{Q}_i v_i^\nu, \quad {}^*\vec{Q}_i^\nu v_{i\nu}=0 , \\ \vec{S}_i^\nu &= {}^*\vec{S}_i^\nu + {}^*\vec{S}_i v_i^\nu, \quad {}^*\vec{S}_i^\nu v_{i\nu} = 0 .\end{aligned} \tag{2.38}$$

However, without loss of generality we can take

$${}^*\vec{S}_i = 0 \tag{2.39}$$

by the same argument as used above to arrive at Eq. (2.34). Inserting Eqs. (2.38) and (2.39) and the term remaining from Eq. (2.36) into (2.29) we obtain, using (2.31),

$$\int \left[ -{}^*\vec{Q}_i^\nu\cdot D_\nu \vec{\xi} - \vec{Q}_i\cdot\frac{D\vec{\xi}}{d\tau_i} - \tfrac{1}{2} g \vec{S}_i^{[\rho\nu]}\cdot(\vec{F}_{\rho\nu}\wedge\vec{\xi}) + g(\vec{\phi}\wedge\vec{S}_i)\cdot\vec{\xi} - g(D_\rho\vec{\xi}\wedge\vec{\phi} + \vec{\xi}\wedge D_\rho\vec{\phi})\cdot{}^*\vec{S}_i^\rho \right] d\tau_i = 0 , \tag{2.40}$$

where we have omitted the summation over $i$, since $\vec{\xi}$ and its derivatives can be chosen so that (2.40) must hold on each world line. [This fact was already used implicitly in arriving at Eq. (2.35).] Using Eq. (2.39) in the last product and making use of the properties of the mixed triple vector product, we can write Eq. (2.40) as

$$\int \left[ -{}^*\vec{Q}_i^{\nu}\cdot D_\nu \vec{\xi} - \vec{Q}_i \cdot \frac{D\vec{\xi}}{d\tau_i} - \tfrac{1}{2} g(\vec{S}_i^{[\rho\nu]} \wedge \vec{F}_{\rho\nu})\cdot\vec{\xi} + g(\vec{\phi}\wedge\vec{S}_i)\cdot\vec{\xi} - g(\vec{\phi}\wedge {}^*\vec{S}_i^{\rho})\cdot D_\rho\vec{\xi} - g(D_\rho\vec{\phi}\wedge {}^*\vec{S}_i^{\rho})\cdot\vec{\xi} \right] d\tau_i = 0 . \tag{2.41}$$

In the second term the $\tau_i$ derivative can be removed from $\vec{\xi}$ by an integration by parts. Requiring the coefficients of $\vec{\xi}$ and $D_\rho\vec{\xi}$ to vanish separately yields the conditions

$${}^*\vec{Q}_i^{\rho} = g\vec{\phi}\wedge {}^*\vec{S}_i^{\rho} \tag{2.42}$$

and

$$\frac{D\vec{Q}_i}{d\tau_i} = g(\tfrac{1}{2}\vec{S}_i^{[\rho\nu]}\wedge\vec{F}_{\rho\nu} + \vec{S}_i\wedge\vec{\phi} - {}^*\vec{S}_i^{\rho}\wedge D_\rho\vec{\phi}) . \tag{2.43}$$

Equation (2.43) determines the time variation of the non-Abelian charge $\vec{Q}_i$, while Eq. (2.42) establishes a relation allowing us to eliminate ${}^*\vec{Q}_i^{\rho}$ from the laws of motion, which will now be derived from Eq. (2.26).

Proceeding as with Eq. (2.27), we first remove all derivatives from the $\delta$ function by integrations by parts to obtain

$$\sum_i \int [p_i^{\rho\mu\nu}\partial_{\mu\rho}\xi_\nu - p_i^{\mu\nu}\partial_\mu\xi_\nu + (D^\nu\vec{\phi}\cdot\vec{S}_i - \vec{F}^{\nu\rho}\cdot\vec{Q}_{i\rho})\xi_\nu - D_\rho(\xi_\nu D^\nu\vec{\phi})\cdot\vec{S}_i^{\rho} + D_\sigma(\vec{F}^{\nu\rho}\xi_\nu)\cdot\vec{S}_i^{\sigma,}{}_{\rho}]\delta^4 d\tau_i d^4x = 0 . \tag{2.44}$$

Now we can carry out the $x$ integrations; as before, from now on all quantities have to be evaluated at $z_i^\sigma(\tau_i)$. Furthermore, by the same arguments as given above, we can omit the summation over $i$. We also note that for any vector in isospace, since $\xi_\nu$ is a scalar in that space, we have

$$D_\rho(\vec{\Psi}\xi_\nu) = \xi_\nu D_\rho\vec{\Psi} + \vec{\Psi}\partial_\rho\xi_\nu . \tag{2.45}$$

Thus Eq. (2.44) reduces to

$$\int [p_i^{\rho\mu\nu}\partial_{\mu\rho}\xi_\nu - p_i^{\mu\nu}\partial_\mu\xi_\nu + (D^\nu\vec{\phi}\cdot\vec{S}_i - \vec{F}^{\nu\rho}\cdot\vec{Q}_{i\rho})\xi_\nu - \vec{S}_i^{\rho}\cdot D^\nu\vec{\phi}\partial_\rho\xi_\nu - \vec{S}_i^{\rho}\cdot D_\rho D^\nu\vec{\phi}\xi_\nu + \vec{S}_i^{\sigma,}{}_{\rho}\cdot\vec{F}^{\nu\rho}\partial_\sigma\xi_\nu + \vec{S}_i^{\sigma,}{}_{\rho}\cdot D_\sigma\vec{F}^{\nu\rho}\xi_\nu] d\tau_i = 0 . \tag{2.46}$$

Now we break up the $p_i^{\mu\nu}$ and $p_i^{\rho\mu\nu}$ into components parallel and perpendicular to the $v_{i\nu}$:[22–24]

$$p_i^{\mu\nu} = {}^*p_i^{\mu\nu} + \tfrac{1}{2}(n_i^\mu v_i^\nu + n_i^\nu v_i^\mu) + M_i v_i^\mu v_i^\nu ,$$
$${}^*p_i^{\mu\nu} = {}^*p_i^{\nu\mu},\quad {}^*p_i^{\mu\nu}v_{i\nu} = 0,\quad n_i^\nu v_{i\nu} = 0 , \tag{2.47}$$

$$p_i^{\rho\mu\nu} = {}^*p_i^{\rho\mu\nu} + \tfrac{1}{2}(B_i^{\mu\rho}v_i^\nu + B_i^{\nu\rho}v_i^\mu) + p_i^\rho v_i^\mu v_i^\nu ,$$
$${}^*p_i^{\rho\mu\nu} = {}^*p_i^{\rho\nu\mu},\quad {}^*p_i^{\rho\mu\nu}v_{i\rho} = 0,\quad {}^*p_i^{\rho\mu\nu}v_{i\nu} = 0 , \tag{2.48}$$
$$B_i^{\mu\rho}v_{i\rho} = 0,\quad B_i^{\mu\rho}v_{i\mu} = 0,\quad p_i^\rho v_{i\rho} = 0 .$$

It is not necessary to include a component containing a factor $v_i^\rho$ in $p_i^{\rho\mu\nu}$, since the term in Eq. (2.46) corresponding to this component can be transformed by an integration by parts to a form which, having the same symmetry properties, can be included in the monopole term described by (2.47). Furthermore

$$\int p_i^\rho v_i^\mu v_i^\nu D_\rho D_\mu \xi_\nu d\tau_i = -\int \frac{d}{d\tau_i}(p_i^\mu v_i^\nu)\partial_\mu\xi_\nu d\tau_i = \int \frac{d}{d\tau_i}(p_i^{[\nu}v_i^{\mu]} - p_i^{(\nu}v_i^{\mu)})\partial_\mu\xi_\nu d\tau_i ; \tag{2.49}$$

thus the last term again has the same symmetry properties as the monopole term and can be included in it. Introducing the abbreviation

$$D_i^{\mu\nu} = 2p_i^{[\nu}v_i^{\mu]} \to p_i^\mu = -D_i^{\mu\nu}v_{i\nu} , \tag{2.50}$$

and performing an integration by parts in Eq. (2.46) in the part containing $B_i^{\nu\rho}v_i^\mu$ we can write this equation as

$$\sum_i \int \{({}^*p_i^{\rho\mu\nu} + \tfrac{1}{2}B_i^{\mu\rho}v_i^\nu)\partial_{\rho\mu}\xi_\nu - [{}^*p_i^{\mu\nu} + \tfrac{1}{2}(n_i^\mu v_i^\nu + n_i^\nu v_i^\mu + \dot{B}_i^{\nu\mu} - \dot{D}_i^{\mu\nu}) + M_i v_i^\mu v_i^\nu + \vec{S}_i^{\mu}\cdot D^\nu\vec{\phi} - \vec{S}_i^{\mu,}{}_{\rho}\cdot\vec{F}^{\nu\rho}]\partial_\mu\xi_\nu$$
$$+ (\vec{S}_i\cdot D^\nu\vec{\phi} - \vec{Q}_{i\rho}\cdot\vec{F}^{\nu\rho} - \vec{S}_i^{\rho}\cdot D_\rho D^\nu\vec{\phi} + \vec{S}_i^{\sigma,}{}_{\rho}\cdot D_\sigma\vec{F}^{\nu\rho})\xi_\nu\} d\tau_i = 0 , \tag{2.51}$$

where an overdot denotes differentiation with respect to $\tau_i$. The vanishing of the coefficient of $\partial_{\rho\mu}\xi_\nu$ implies

$${}^*p_i^{\rho\mu\nu} + {}^*p_i^{\mu\rho\nu} + \tfrac{1}{2}(B_i^{\mu\rho} + B_i^{\rho\mu})v_i^\nu = 0 , \tag{2.52}$$

from which we obtain by contraction with $v_{i\nu}$

$$B_i^{\mu\rho} = -B_i^{\rho\mu} \ . \tag{2.53}$$

But then Eq. (2.52) requires

$$^*p_i^{\rho\mu\nu} + {^*p_i^{\mu\rho\nu}} = 0 \ , \tag{2.54}$$

which in turn implies

$$^*p_i^{\rho\mu\nu} = 0 \ , \tag{2.55}$$

since no third rank tensor can be symmetric in one pair of indices and antisymmetric in another.

We now break up the factors of $\partial_\mu \xi_\nu$ in Eq. (2.51) involving the fields into symmetric and antisymmetric parts:

$$\vec{S}_i{}^{\mu}\cdot D^\nu \vec{\phi} - \vec{S}_i{}^{\mu,}{}_{\rho}\cdot \vec{F}^{\nu\rho} = \vec{S}_i{}^{(\mu}\cdot D^{\nu)}\vec{\phi} + \vec{S}_i{}^{[\mu}\cdot D^{\nu]}\vec{\phi} - \vec{S}_i{}^{(\mu,}{}_{\rho}\cdot \vec{F}^{\nu)\rho} - \vec{S}_i{}^{[\mu,}{}_{\rho}\cdot \vec{F}^{\nu]\rho} \ . \tag{2.56}$$

The symmetric part can be included in the as yet undetermined $^*p_i^{\mu\nu}$. The tensor

$$Y_i^{\mu\nu} \equiv \tfrac{1}{2}(\dot{B}_i^{\mu\nu} + \dot{D}_i^{\mu\nu}) + \vec{S}_i{}^{[\mu}\cdot D^{\nu]}\vec{\phi} - \vec{S}_i{}^{[\mu,}{}_{\rho}\cdot \vec{F}^{\nu]\rho} \tag{2.57}$$

is antisymmetric by Eqs. (2.50) and (2.53); we break it up as

$$\begin{aligned} Y_i^{\mu\nu} &= {^*Y^{\mu\nu}} + Y_i^{\mu} v_i^{\nu} - Y_i^{\nu} v_i^{\mu} \ , \\ {^*Y_i^{\mu\nu}} &= -{^*Y_i^{\nu\mu}}, \quad {^*Y_i^{\mu\nu}} v_{i\nu} = 0, \quad Y_i^{\nu} v_{i\nu} = 0 \ . \end{aligned} \tag{2.58}$$

Introducing this into Eq. (2.51), integrating all terms containing a factor $v_i^\mu$ by parts, and omitting the term involving $\partial_{\rho\mu}\xi_\nu$ as discussed above, we obtain

$$\int \left[ -({^*p_i^{\mu\nu}} - {^*Y_i^{\mu\nu}} + \tfrac{1}{2} n_i^\mu v_i^\nu - Y_i^\mu v_i^\nu)\partial_\mu \xi_\nu + \left\{ \vec{S}_i \cdot D^\nu \vec{\phi} - \vec{Q}_{i\rho}\cdot \vec{F}^{\nu\rho} - \vec{S}_i{}^{\rho}\cdot D_\rho D^\nu \vec{\phi} + \vec{S}_i{}^{\sigma,}{}_{\rho}\cdot D_\sigma \vec{F}^{\nu\rho} + \frac{d}{d\tau_i}(M_i v_i^\nu + \tfrac{1}{2} n_i^\nu + Y_i^\nu) \right\} \xi_\nu \right] d\tau_i = 0 \ . \tag{2.59}$$

The vanishing of the coefficients of $\partial_\nu \xi_\nu$ and of $\xi_\nu$ yields

$$^*p_i^{\mu\nu} - {^*Y_i^{\mu\nu}} + \tfrac{1}{2} n_i^\mu v_i^\nu - Y_i^\mu v_i^\upsilon = 0 \ , \tag{2.60}$$

$$\vec{S}_i \cdot D^\nu \vec{\phi} - \vec{Q}_{i\rho}\cdot \vec{F}^{\nu\rho} - \vec{S}_i{}^{\rho}\cdot D_\rho D^\nu \vec{\phi} + \vec{S}_i{}^{\sigma,}{}_{\rho}\cdot D_\sigma \vec{F}^{\nu\rho} + \frac{d}{d\tau_i}(M_i v_i^\nu + \tfrac{1}{2} n_i^\nu + Y_i^\nu) = 0 \ . \tag{2.61}$$

Contracting Eq. (2.60) with $v_{i\nu}$ we obtain

$$\tfrac{1}{2} n_i^\mu = Y_i^\mu \tag{2.62}$$

and thus Eq. (2.61) reduces to the translational law of motion

$$\frac{dA_i^\nu}{d\tau_i} = -\vec{S}_i \cdot D^\nu \vec{\phi} + \vec{S}_i{}^{\rho}\cdot D_\rho D^\nu \vec{\phi} + \vec{Q}_{i\rho}\cdot \vec{F}^{\nu\rho} - \vec{S}_i^{\mu,}{}_{\rho}\cdot D_\mu \vec{F}^{\nu\rho} \ , \tag{2.63}$$

where

$$A_i^\nu = M_i v_i^\nu + n_i^\nu \ . \tag{2.64}$$

Since in the absence of the forces due to the fields $A_i^\nu$ is constant, it is natural to interpret it as the four-momentum of the $i$th particle; it should be noted that in general it is not parallel to the four-velocity.

From Eqs. (2.60) and (2.62) we conclude because of the different symmetry properties of $^*p_i^{\mu\nu}$ and $^*Y_i^{\mu\nu}$ that

$$^*p_i^{\mu\nu} = 0, \quad {^*Y_i^{\mu\nu}} = 0 \ . \tag{2.65}$$

Inserting Eq. (2.58) with (2.62) and (2.63) into (2.57) we obtain the rotational law of motion

$$\frac{d}{d\tau_i}(B_i^{\mu\nu} + D_i^{\mu\nu}) = 2(\vec{S}_i{}^{[\mu}\cdot D^{\nu]}\vec{\phi} - \vec{S}_i{}^{[\mu,}{}_{\rho}\cdot \vec{F}^{\nu]\rho}) + n_i^\mu v_i^\nu - n_i^\nu v_i^\mu \ . \tag{2.66}$$

Since in the absence of fields in Eqs. (2.63) and (2.66) the left-hand side equals minus the rate of change of orbital angular momentum, it is natural to interpret $B_i^{\mu\nu} + D_i^{\mu\nu}$ similarly as some form of angular momentum associated with the $i$th particle; this interpretation will be further discussed below.

Equations (2.63) and (2.66) are special cases of Eqs. (48) and (49) of H; in that paper the laws of motion were derived for the general case of arbitrary multipole singularities of the field satisfying Eqs. (2.9) and (2.10) rather than just the monopole-dipole terms contained in the sources (2.23) and (2.24). Here we have given the derivation of our simpler case in full detail, while in H it was only indicated that the derivation was fully analogous to the treatment of the general case of arbitrary fields given in Refs. 22 and 23. Similarly, the law (2.43) describing the variation of the non-Abelian charge is a special case of Eq. (73) of H; a full derivation was given there, however, and here the details for our special case were given only for completeness.

The laws of motion (2.63) and (2.66) are written in terms of arbitrary monopole and dipole moments of the fields. We now have to take into account the decompositions (2.32), (2.35), and (2.38) of the moments and the restrictions (2.34), (2.37), (2.39), and (2.42) on some of their parts. Inserting these into Eq. (2.63) we obtain

$$\frac{dA_i^\nu}{d\tau_i}=-\vec{S}_i\cdot D^\nu\vec{\phi}+{}^*\vec{S}_i{}^\rho\cdot D_\rho D^\nu\vec{\phi}+\vec{Q}_i\cdot\vec{F}^{\nu\rho}v_{i\rho}+g(\vec{\phi}\wedge{}^*\vec{S}_{i\rho})\cdot\vec{F}^{\nu\rho}-\vec{S}_i{}^{[\mu\rho]}\cdot D_\mu\vec{F}^\nu{}_\rho\ . \tag{2.67}$$

Combining the second and fourth terms on the right-hand side and using Eq. (2.8) and the properties of the mixed triple vector product we get

$$\begin{aligned}{}^*\vec{S}_i{}^\rho\cdot D_\rho D^\nu\vec{\phi}+g(\vec{\phi}\wedge{}^*\vec{S}_{i\rho})\cdot\vec{F}^{\nu\rho}&={}^*\vec{S}_{i\rho}\cdot(D^\nu D^\rho\vec{\phi}+g\vec{F}^{\nu\rho}\wedge\vec{\phi})+g(\vec{\phi}\wedge{}^*\vec{S}_{i\rho})\cdot\vec{F}^{\nu\rho}\\&=\vec{S}_{i\rho}\cdot D^\nu D^\rho\vec{\phi}\ ,\end{aligned} \tag{2.68}$$

where the asterisk on the final expression has been omitted because of Eq. (2.39). Thus Eq. (2.67) reduces to

$$\frac{dA_i^\nu}{d\tau_i}=-\vec{S}_i\cdot D^\nu\vec{\phi}+\vec{S}_{i\rho}\cdot D^\nu D^\rho\vec{\phi}+\vec{Q}_i\cdot\vec{F}^{\nu\rho}v_{i\rho}-\vec{S}_i{}^{[\mu\rho]}\cdot D_\mu\vec{F}^\nu{}_\rho\ . \tag{2.69}$$

Similarly, Eq. (2.66) reduces to

$$\frac{d}{d\tau_i}(B_i^{\mu\nu}+D_i^{\mu\nu})=(\vec{S}_i{}^\mu\cdot D^\nu\vec{\phi}-\vec{S}_i^\nu\cdot D^\mu\vec{\phi})-(\vec{S}_i{}^{[\mu\rho]}\cdot\vec{F}^\nu{}_\rho-\vec{S}_i{}^{[\nu\rho]}\cdot\vec{F}^\mu{}_\rho)+(n_i^\mu v_i^\nu-n_i^\nu v_i^\mu)\ . \tag{2.70}$$

For easier reference we repeat the restrictions imposed on the quantities entering Eqs. (2.69) and (2.70) by Eqs. (2.39), (2.48), and (2.64):

$$\vec{S}_i{}^\nu\cdot v_{i\nu}=0,\quad B_i^{\mu\nu}v_{i\nu}=0,\quad n_i^\nu v_{i\nu}=0\ . \tag{2.71}$$

Furthermore, the variation of $\vec{Q}_i$ is determined by Eq. (2.43).

Equations (2.69), (2.70), and (2.43) are the *laws of motion for the momentum, angular momentum, and non-Abelian charge of the ith particle.* Although they appear to determine the motion of this particle, this is actually not the case. As was shown for an analogous case in Ref. 22, Eqs. (2.69) and (2.70) are compatible with *arbitrary* motion, due to the presence of the term $\dot{D}_i^{\mu\nu}$. For a system of interacting particles it was shown in H that $D_i^{\mu\nu}$ can be interpreted as an induced angular momentum, and its presence permits a variety of multipole moments to be induced by the motion of the particles. To exclude this arbitrariness, we therefore require

$$D_i^{\mu\nu}=0\ . \tag{2.72}$$

On the other hand, $B_i^{\mu\nu}$ can be shown to be the limit of the angular momentum of an extended body,[24,22] and thus will be taken to describe the intrinsic angular momentum or spin, whose magnitude will be taken as a constant of the motion:

$$B_i^{\mu\nu}\cdot\dot{B}_{i\mu\nu}=0\ . \tag{2.73}$$

Similarly, the magnitudes of the monopole and dipole moments will be required to be constants, or

$$\begin{aligned}\frac{d}{d\tau_i}(\vec{S}_i\cdot\vec{S}_i)&=\frac{d}{d\tau_i}(\vec{S}_{i\rho}\cdot\vec{S}_i{}^\rho)\\&=\frac{d}{d\tau_i}(\vec{Q}_{i\rho}\cdot\vec{Q}_i^\rho)\\&=\frac{d}{d\tau_i}(\vec{S}_{i[\mu\nu]}\cdot\vec{S}_i{}^{[\mu\nu]})=0\end{aligned} \tag{2.74}$$

as a consequence of the laws of motion, as is frequently appropriate for elementary particles (but not if, e.g., a classical description of exchange forces is desired). Finally, we shall impose the even more stringent requirement that the momenta $A_i^\nu$ as well as the multipole singularities of the fields for the $i$th particle should be expressible in terms of the quantities $v_i^\mu$ and $B_i^{\mu\nu}$ describing the translational and rotational motions of a non-Abelian charged particle characterized by a single vector in isospace, the isospin vector $\vec{\tau}_i$, of constant magnitude

$$\vec{\tau}_i\cdot\vec{\tau}_i=1,\quad \vec{\tau}_i\cdot\frac{d\vec{\tau}_i}{d\tau_i}=0\ , \tag{2.75}$$

and of a number of constants.

We note that for any scalar $\xi$ in isospace

$$\frac{d\xi(\tau_i)}{d\tau_i}=\frac{D\xi(\tau_i)}{d\tau_i} \tag{2.76}$$

and therefore Eqs. (2.73)—(2.75) imply that all the magnitudes are covariantly constant as well.

The general case of multipoles of arbitrary order was treated in H. Here we shall find the general form of the monopole and dipole moments satisfying the requirements stated.

The non-Abelian charge can be taken as

$$\vec{Q}_i=I_i\vec{\tau}_i\ , \tag{2.77}$$

where $I_i$ is the magnitude of the Yang-Mills charge, with the monopole moment of the singularity of the field given by

$$\vec{Q}_i^\mu=\vec{Q}_i v_i^\mu=I_i v_i^\mu\vec{\tau}_i\ . \tag{2.78}$$

The corresponding dipole moment can be taken without loss of generality as (compare H)

$$\vec{S}_i{}^{[\mu\nu]}=f_i B_i^{\mu\nu}\vec{\tau}_i\ . \tag{2.79}$$

Similarly, the monopole and dipole moments of the Higgs field can be taken as

$$\begin{aligned}&\vec{S}_i=h_{i1}\vec{\tau}_i\ ,\\&\vec{S}_i{}^\mu=h_{i2}S_i{}^\mu\vec{\tau}_i\ ,\\&S_i{}^\mu\equiv\epsilon^{\mu\nu\rho\sigma}B_{i\nu\rho}v_{i\sigma}\ ,\end{aligned} \tag{2.80}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor density with $\epsilon^{0123}=1$. The form of $S_i^{\mu}$ establishes Eq. (2.39) because of the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$. In the above, $f_i$, $h_{i1}$, and $h_{i2}$ characterize the strength of the coupling; while they, as well as $I_i$, are scalars which may at first be taken as undetermined functions of the proper times, it will be shown below that they must all be constants.

Slight modifications of the initial assumptions on the monopole and dipole moments and the corresponding changes in the allowed forms of these moments will be discussed in Sec. III.

Inserting Eqs. (2.77), (2.79), and (2.80) into Eq. (2.43), we obtain

$$\frac{D\vec{Q}_i}{d\tau_i}=\vec{\tau}_i\frac{dI_i}{d\tau_i}+I_i\frac{D\vec{\tau}_i}{d\tau_i}$$
$$=g\vec{\tau}_i\wedge(\tfrac{1}{2}f_iB_i^{\rho\sigma}\vec{F}_{\rho\sigma}+h_{i1}\vec{\phi}-h_{i2}S_i^{\rho}D_\rho\vec{\phi})\ . \quad (2.81)$$

Scalar multiplication with $I_i\vec{\tau}_i$ yields

$$\vec{Q}_i\cdot\frac{D\vec{Q}_i}{d\tau_i}=\tau_i^2I_i\frac{dI_i}{d\tau_i}+I_i^2\vec{\tau}_i\cdot\frac{D\vec{\tau}_i}{d\tau_i}=0\ , \quad (2.82)$$

and thus Eq. (2.75) holds as required, provided that $I_i$ is a constant, and Eq. (2.81) reduces to

$$I_i\frac{D\vec{\tau}_i}{d\tau_i}=g\vec{\tau}_i\wedge(\tfrac{1}{2}f_iB_i^{\rho\sigma}\vec{F}_{\rho\sigma}+h_{i1}\vec{\phi}-h_{i2}S_i^{\rho}D_\rho\vec{\phi})\ . \quad (2.83)$$

From Eq. (2.78), Eq. (2.82) implies

$$\vec{Q}_{i\rho}\cdot\frac{D\vec{Q}_i^{\rho}}{d\tau_i}=\vec{Q}_iv_i^{\rho}\cdot\left[v_{i\rho}\frac{D\vec{Q}_i}{d\tau_i}+\vec{Q}_i\frac{dv_{i\rho}}{d\tau_i}\right]=0 \quad (2.84)$$

as required by Eq. (2.74). Similarly, from Eqs. (2.75) and (2.80)

$$\vec{S}_i\cdot\frac{D\vec{S}_i}{d\tau_i}=h_{i1}^2\vec{\tau}_i\cdot\frac{D\vec{\tau}_i}{d\tau_i}+\vec{\tau}_i^2h_{i1}\frac{dh_{i1}}{d\tau_i}=0\ , \quad (2.85)$$

provided that $h_{i1}$ is constant.

Inserting Eqs. (2.72) and (2.77)–(2.81) into the laws of motion (2.69) and (2.70) yields

$$\frac{dA_i^{\nu}}{d\tau_i}=-\vec{\tau}_i\cdot(h_{i1}D^{\nu}\vec{\phi}-h_{i2}S_{i\rho}D^{\nu}D^{\rho}\vec{\phi}$$
$$-I_i\vec{F}^{\nu\rho}v_{i\rho}+f_iB_i^{\mu\rho}D_\mu\vec{F}^{\nu}{}_{\rho}) \quad (2.86)$$

and

$$\frac{dB_i^{\mu\nu}}{d\tau_i}=\vec{\tau}_i\cdot[h_{i2}(S_i^{\mu}D^{\nu}\vec{\phi}-S_i^{\nu}D^{\mu}\vec{\phi})$$
$$-f_i(B_i^{\mu\rho}\vec{F}^{\nu}{}_{\rho}-B_i^{\nu\rho}\vec{F}^{\mu}{}_{\rho})]+(n_i^{\mu}v_i^{\nu}-n_i^{\nu}v_i^{\mu})\ . \quad (2.87)$$

Contracting Eq. (2.87) with $B_{i\mu\nu}$ and using (2.71) we get

$$B_{i\mu\nu}\frac{dB_i^{\mu\nu}}{d\tau_i}=2\vec{\tau}_i\cdot(h_{i2}B_{i\mu\nu}S_i^{\mu}D^{\nu}\vec{\phi}-f_iB_{i\mu\nu}B_i^{\mu\rho}\vec{F}^{\nu}{}_{\rho})\ . \quad (2.88)$$

The last term vanishes because $B_{i\mu\nu}B_i^{\mu}{}_{\rho}$ is symmetric in $\nu$ and $\rho$, whereas $\vec{F}^{\nu\rho}$ is antisymmetric in these indices. The first term vanishes because

$$B_{i\mu\nu}S_i^{\mu}=0\ , \quad (2.89)$$

which can be most easily seen by evaluating this expression in the "standard rest system," in which

$$v_i^1=v_i^2=v_i^3=0,\quad v_i^0=1\ ,$$
$$\text{all } B_{i\mu\nu}=0 \text{ except } B_{i12}=-B_{i21}\ . \quad (2.90)$$

Thus Eq. (2.73) holds as required. By similar considerations it follows that

$$S_{i\rho}S_i^{\rho}=-2B_{i\mu\nu}B_i^{\mu\nu}=\text{const}\ . \quad (2.91)$$

Furthermore, from Eqs. (2.81) and (2.79),

$$\vec{S}_{i\rho}\cdot\vec{S}_i^{\rho}=h_{i2}^2S_{i\rho}S_i^{\rho}=-2\frac{h_{i2}^2}{f_i^2}\vec{S}_{i[\mu\nu]}\cdot\vec{S}_i^{[\mu\nu]}\ . \quad (2.92)$$

These expressions are constant, as required by Eq. (2.74), as a consequence of (2.91), provided that $h_{i2}$ and $f_i$ are constant. This together with Eqs. (2.84) and (2.85) completes the proof of (2.74). From the consideration of the values of the two dipole moments introduced in (2.79) in the standard rest system it is clear that $f_i$ is the Yang-Mills analog of the gyromagnetic ratio.

Now we contract Eq. (2.87) with $v_{i\nu}$ and obtain

$$n_i^{\mu}=-\vec{\tau}_i\cdot\left[h_{i2}S_i^{\mu}\frac{D\vec{\phi}}{d\tau_i}-f_iB_i^{\mu\rho}v_i^{\sigma}\vec{F}_{\sigma\rho}\right]+\dot{B}_i^{\mu\rho}v_{i\rho}\ ; \quad (2.93)$$

contracting this with $\dot{v}_{i\mu}$ and using Eqs. (2.71) we get

$$\dot{n}_i^{\mu}v_{i\mu}=\vec{\tau}_i\cdot\left[h_{i2}\dot{v}_{i\mu}S_i^{\mu}\frac{D\vec{\phi}}{d\tau_i}-f_i\dot{v}_{i\mu}B_i^{\mu\rho}v_i^{\sigma}\vec{F}_{\sigma\rho}\right]\ . \quad (2.94)$$

On the other hand, we get from Eq. (2.86) by contraction with $v_{i\nu}$ and use of the decomposition (2.64)

$$v_{i\nu}\dot{n}_i^{\nu}+\frac{dM_i}{d\tau_i}=-\vec{\tau}_i\cdot\left[h_{i1}\frac{D\vec{\phi}}{d\tau_i}-h_{i2}S_{i\rho}\frac{D}{d\tau_i}(D^{\rho}\vec{\phi})+f_iB_i^{\mu\rho}D_\mu\vec{F}_{\nu\rho}v_i^{\nu}\right]\ . \quad (2.95)$$

Subtracting Eq. (2.94) from (2.95) we obtain

$$\dot{M}_i=\vec{\tau}_i\cdot\left[-h_{i1}\frac{D\vec{\phi}}{d\tau_i}+h_{i2}S_{i\rho}\left[\frac{D}{d\tau_i}D^{\rho}\vec{\phi}-\dot{v}_i^{\rho}\frac{D\vec{\phi}}{d\tau_i}\right]-f_iB_i^{\mu\rho}(-\dot{v}_{i\mu}v_i^{\nu}\vec{F}_{\nu\rho}+D_\mu\vec{F}_{\nu\rho}v_i^{\nu})\right]\ , \quad (2.96)$$

which must be shown to be integrable. To establish this result we add the expression

$$0=-\frac{\dot S_{i\mu}\dot B_i^{\mu\nu}S_{i\nu}}{S_{i\rho}S_i^{\rho}}=\vec\tau_i\cdot\left[h_{i2}\dot S_{i\mu}D^{\mu}\vec\phi+f_i\frac{\dot S_{i\mu}B_i^{\mu\sigma}S_{i\nu}\vec F^{\nu}{}_{\sigma}}{S_{i\rho}S_i^{\rho}}+h_{i2}\dot S_i^{\mu}v_{i\mu}\frac{D\vec\phi}{d\tau_i}\right], \tag{2.97}$$

where the form of the last expression follows from Eqs. (2.87) and (2.93), and which vanishes by (2.89). We then obtain

$$\dot M_i=\frac{D}{d\tau_i}[-\vec\tau_i\cdot(h_{i1}\vec\phi-h_{i2}S_{i\rho}D^{\rho}\vec\phi+\tfrac12 f_iB_i^{\mu\rho}\vec F_{\mu\rho})]+\frac{D\vec\tau_i}{d\tau_i}\cdot(h_{i1}\vec\phi-h_{i2}S_{i\rho}D^{\rho}\vec\phi+\tfrac12 f_iB_i^{\mu\rho}\vec F_{\mu\rho})$$
$$+f_i\vec\tau_i\cdot\left[B_i^{\mu\rho}\left[\dot v_{i\mu}v_i^{\nu}\vec F_{\nu\rho}-D_{\mu}\vec F_{\nu\rho}v_i^{\nu}+\frac12\frac{D\vec F_{\mu\rho}}{d\tau_i}\right]+\tfrac12\dot B_i^{\mu\rho}\vec F_{\mu\rho}+\frac{\dot S_{i\mu}B_i^{\mu\sigma}S_{i\nu}\vec F^{\nu}{}_{\sigma}}{S_{i\rho}S_i^{\rho}}\right]. \tag{2.98}$$

The second term on the right-hand side vanishes from Eq. (2.81), since it is proportional to the scalar product of the right-hand side of that equation with the expression in square brackets. The vanishing of the last term of Eq. (2.98) follows most easily by evaluating it in the standard rest system and noting that in that system for any field quantity $d/d\tau_i$ equals $\partial_0$. Thus we can integrate Eq. (2.98) to obtain

$$M_i=m_i-\vec\tau_i\cdot(h_{i1}\vec\phi-h_{i2}S_{i\rho}D^{\rho}\vec\phi+\tfrac12 f_iB_i^{\mu\rho}\vec F_{\mu\rho}), \tag{2.99}$$

where $m_i$ is a constant of integration, which will be taken as positive and interpreted as the mass of the particle.

Substitution of Eqs. (2.99) and (2.93) into (2.86) yields the final form of the translational law of motion:

$$\frac{d}{d\tau_i}\left[[m_i-\vec\tau_i\cdot(h_{i1}\vec\phi-h_{i2}S_{i\rho}D^{\rho}\vec\phi+\tfrac12 f_iB_i^{\mu\rho}\vec F_{\mu\rho})]v_i^{\nu}+\vec\tau_i\cdot\left[h_{i2}S_i^{\nu}\frac{D\vec\phi}{d\tau_i}-f_iB_i^{\nu\rho}v_i^{\sigma}\vec F_{\sigma\rho}\right]+\dot B_i^{\nu\rho}v_{i\rho}\right]$$
$$=\vec\tau_i\cdot(-h_{i1}D^{\nu}\vec\phi+h_{i2}S_{i\rho}D^{\nu}D^{\rho}\vec\phi+I_i\vec F^{\nu\rho}v_{i\rho}+f_iB_{i\rho\sigma}D^{\rho}\vec F^{\nu\sigma}). \tag{2.100}$$

Similarly, substitution of Eq. (2.93) into (2.87) yields the final form of the rotational law of motion

$$\dot B_i^{\mu\nu}-\dot B_i^{\mu\rho}v_{i\rho}v_i^{\nu}+\dot B_i^{\nu\rho}v_{i\rho}v_i^{\mu}=\vec\tau_i\cdot\left\{h_{i2}\left[S_i^{\mu}\left[D^{\nu}\vec\phi-v_i^{\nu}\frac{D\vec\phi}{d\tau_i}\right]-S_i^{\nu}\left[D^{\mu}\vec\phi-v_i^{\mu}\frac{D\vec\phi}{d\tau_i}\right]\right]\right.$$
$$\left.-f_i[B_i^{\mu\rho}(\vec F^{\nu}{}_{\rho}+v_i^{\nu}\vec F_{\rho\sigma}v_i^{\sigma})-B_i^{\nu\rho}(\vec F^{\mu}{}_{\rho}+v_i^{\mu}\vec F_{\rho\sigma}v_i^{\sigma})]\right\}. \tag{2.101}$$

Equations (2.100), (2.101), and (2.83) determine the variation of the independent variables $v_i^{\nu}$, $B_i^{\mu\nu}$, and $\vec\tau_i$.

## III. DISCUSSION

In this paper, we have derived the laws of translational and rotational motions (2.100) and (2.101) and the law of variation of non-Abelian charge (2.83). To obtain the equations of motion, we will have to determine the fields to be inserted into these laws in terms of their sources. This will be done in the following paper, where we shall develop an approximation method for this purpose. In the case of retarded interactions, these fields involve not only the contributions of all particles other than the $i$th one, but also radiation-reaction terms due to the $i$th particle itself. Except in lowest order, this yields extremely complicated expressions. Although the approximation method itself is the same regardless of the number of multipole terms included in the sources $\vec j^{\,\mu}$ and $\vec\rho$ of the field equations (2.9) and (2.10), the complexity of calculation of the fields increases by an order of magnitude for each additional multipole as well as for each successive order of approximation. Therefore, once we have described the approximation method, we will restrict ourselves to the case of monopole singularities of the fields for which the exact laws of motion (2.100), (2.101), and (2.83) reduce to

$$\frac{d}{d\tau_i}[(m_i-h_{i1}\vec\tau_i\cdot\vec\phi)v_i^{\nu}+\dot B_i^{\nu\rho}v_{i\rho}]=-\vec\tau_i\cdot(h_{i1}D^{\nu}\vec\phi-I_i\vec F^{\nu\rho}v_{i\rho}), \tag{3.1}$$

$$\dot B_i^{\mu\nu}-\dot B_i^{\mu\rho}v_{i\rho}v_i^{\nu}+\dot B_i^{\nu\rho}v_{i\rho}v_i^{\nu}=0, \tag{3.2}$$

$$I_i\frac{D\vec\tau_i}{d\tau_i}=gh_{i1}\vec\tau_i\wedge\vec\phi. \tag{3.3}$$

It should be noted that even in this simpler case, in which there are no torques due to fields acting on the particles, the translational and rotational motions remain coupled, and $\vec\tau_i$, and thus the non-Abelian charge $\vec Q_i$, is not necessarily covariantly constant [unlike the case considered in Ref. 19, which is obtained from Eqs. (3.1)–(3.3) by taking $B_i^{\mu\nu}=h_{i1}=0$]. Furthermore, the inclusion of the Higgs field allows us to obtain gauge-invariant results more easily and in a physically more satisfactory manner than in its absence, as will be discussed in detail in the following paper.

If we take [see Eq. (2.3)]

$$\vec A^{\nu}=\vec\gamma A^{\nu}, \tag{3.4}$$

restricting the gauge field everywhere to be in a particular direction in isospace, the Yang-Mills field reduces to the electromagnetic one, and Eqs. (2.100), (2.101), and (2.83) reduce to the case of a charge-symmetric spin zero meson field interacting with the electromagnetic field, involving point particles with spin which are monopole-dipole singularities of these fields.

Instead of taking the non-Abelian charge $\vec{Q}_i$ to be given by Eq. (2.77), we could more generally take

$$\vec{Q}_i=\tfrac{1}{2}\vec{Y}_i+I_i\vec{\tau}_i \;, \tag{3.5}$$

where $\vec{Y}_i$ is a constant vector corresponding to a classical representation of hypercharge. Then we can maintain all the conditions (2.74) by taking $\vec{Q}_i^{\mu}$, $\vec{S}_i^{[\mu\nu]}$, $\vec{S}_i$, and $\vec{S}_i^{\mu}$ to be proportional to this $\vec{Q}_i$ rather than to $\vec{\tau}_i$ as in Eqs. (2.78)—(2.80); the subsequent proofs of the conditions (2.74) are only trivially modified, and Eqs. (2.100), (2.101), and (2.81)—(2.83) are modified only by replacement of $I_i\vec{\tau}_i$ by the expression (3.5) everywhere, and similarly for Eqs. (3.1)—(3.3).

Instead, we can maintain the forms (2.79) and (2.80) of $\vec{S}_i^{[\mu\nu]}$, $\vec{S}_i$, and $\vec{S}_i^{\mu}$ and only take $\vec{Q}_i^{\mu}$ to be proportional to the expression (3.5). Then Eqs. (2.101) and (2.83) remain unchanged, as do their special cases (3.2) and (3.3); only the translational equations of motion (2.100) and (3.1) are modified by addition of an extra term $-\frac{1}{2}\vec{Y}_i\cdot\vec{F}^{\nu\rho}v_{i\rho}$ on the right-hand side. Furthermore, now $\vec{Q}_i\cdot\vec{Q}_i$ and $\vec{Q}_{i\mu}\cdot\vec{Q}_i^{\mu}$ are no longer constant, while the other conditions (2.74) are maintained.

While the choice of the form of the multipole moments suggested in the preceding paragraph appears arbitrary, it seems to be the physically necessary one for the case (3.4) of nonlinear meson theory. In particular, if we take

$$B_i^{\mu\nu}=f_i=h_{i2}=0,\quad \vec{\gamma}_i=\vec{\gamma},\quad I_i=1 \;,$$

and use Eq. (3.4), (3.1) and (3.3) reduce to (2.39) and (2.35) of M.

In the following paper we shall maintain the simpler choices (2.78)—(2.80). However, all resulting expressions require only trivial modifications to accommodate either of the two other choices discussed above.

After this paper was completed, a recent article by Ragusa[26] came to our attention. In that article, which was stimulated by our recent paper on the Yang-Mills theory with monopole singularities,[19] the results of that paper were extended to a particle with spin and a dipole moment, but not including a Higgs field. The laws of motion obtained by a method similar to ours agree with our laws (2.100), (2.101), and (2.83) if the Higgs field is omitted in these equations.

As noted in the Introduction, all the results of this paper are special cases of the results of H (Ref. 13) for multipoles of arbitrary order. However, that paper required knowledge of several earlier papers and omitted many of the details of the derivation. As noted in H, many of its general results (and thus also the results of this paper) can be extended to gauge theories involving other gauge groups. Research on this is in progress.

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