Effects of quantum fields on singularities and particle horizons in the early universe. II

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The back-reaction problem for conformally invariant free quantum fields in homogeneous and isotropic spacetimes containing classical radiation is solved for spacetimes with nonzero spatial curvatures and/or nonzero cosmological constants. The solutions depend upon two regularization parameters which we call α and β. Only solutions which at late times approach the appropriate solution to the field equations of general relativity are considered. The results are much the same as those found previously for spatially flat spacetimes with zero cosmological constants. Thus, if $\beta > 3\alpha > 0$, there is always one solution which undergoes a "time-symmetric bounce" and which contains no singularities, if $\alpha,\beta > 0$ there is a family of solutions with particle horizons and no singularities, and if $\alpha > 0$ there is always at least one solution with an initial singularity but no particle horizons. The differences caused by the spatial curvature and cosmological constant include the initial behavior of the time-symmetric bounce solution and, if the spatial curvature is nonzero, the initial behavior of many solutions for the cases $\beta = 3\alpha > 0$ and $\beta \approx 3\alpha < 0$.

I. INTRODUCTION

Quantum fields may have profoundly influenced the dynamical behavior of the early universe. Their possible effects include the inflationary universe scenario,[1–3] the damping of anisotropy,[4–6] and the removal of particle horizons and singularities.[7–9]

Much of the evidence for the removal of particle horizons and singularities comes from studies of conformally invariant free quantum fields in homogeneous and isotropic spacetimes containing classical radiation. Ruzmaikina and Ruzmaikin,[10] Gurovich and Starobinsky,[11] and Frenkel and Brecher[12] undertook such investigations in the context of higher-derivative theories of gravity. They found that the initial singularity predicted by classical general relativity can be removed, but it is replaced by a different initial singularity at an earlier epoch. Fischetti, Hartle, and Hu,[8] using the semiclassical approximation to quantum gravity, found additional evidence that the singularity predicted by classical general relativity can be removed. They also found one solution to the semiclassical back-reaction equation with an initial singularity, but no particle horizons. Frenkel and Brecher pointed out that this solution also occurs for higher-derivative theories of gravity. In Ref. 9, hereafter referred to as paper I, the author extended the investigation of Fischetti *et al.* to include all possible values of the regularization parameters α and β. In the process, several additional solutions to the semiclassical back-reaction equation with initial singularities but no particle horizons were found. Also, a family of solutions with particle horizons but no singularities and one time-symmetric bounce solution with neither particle horizons nor singularities was found.

In this paper, the investigation of paper I is extended to include homogeneous and isotropic spacetimes with positive and negative spatial curvatures and nonzero cosmological constants. The assumptions and approximations which are used in this investigation are outlined in the next few paragraphs. A more detailed discussion of them appears in paper I.

We assume that the universe is homogeneous and isotropic and that it contains classical radiation. The classical radiation is included to support the expansion at late times compared with the Planck time, when quantum effects are small. The line element for homogeneous and isotropic spacetimes is called the Robertson-Walker (RW) line element and it has the form[13]

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1.1}$$

where $a(\eta)$ is the scale factor, $k = 0, \pm 1$, and the spatial curvature (intrinsic curvature of a surface of constant η) equals k/a^2.

We also assume that the only quantum fields in the universe are conformally invariant free fields. These include photons, massless neutrinos, and massless conformally coupled scalar fields. Since homogeneous and isotropic spacetimes are conformally flat,[14] it is possible to solve the wave equations for these fields by conformally transforming them to flat spacetime, solving the wave equations there and transforming the solutions back to curved spacetime.

Our final assumption is that an appropriate way to modify classical general relativity so that quantum effects may be taken into account is to modify Einstein's equations to read[15]

$$G_{ab} + \Lambda g_{ab} = \frac{l^2}{2}(T_{ab}^{\rm Cl} + \langle T_{ab}^{\rm QM}\rangle)\,. \tag{1.2}$$

Here G_{ab} is the Einstein tensor, Λ is the cosmological constant, g_{ab} is the metric tensor, $l=(16\pi G)^{1/2}$ is the Planck length, T_{ab}^{Cl} is the stress-energy tensor for any classical fields present, and $\langle T_{ab}^{\mathrm{QM}}\rangle$ is the expectation value of the stress-energy tensor operator for the quantum fields. Equation (1.2) is called the semiclassical back-reaction equation and results from the semiclassical approximation to quantum gravity.

For conformally invariant free fields in homogeneous and isotropic spacetimes

$$\langle T_{ab}^{\mathrm{QM}}\rangle = T_{ab}^{\mathrm{CR}} + \langle 0 | T_{ab}^{\mathrm{QM}} | 0 \rangle \ , \tag{1.3}$$

where T_{ab}^{CR} has the same form as the stress-energy tensor for classical radiation[16] and $\langle 0 | T_{ab}^{\mathrm{QM}} | 0\rangle$ is the regularized vacuum expectation value of T_{ab}^{QM}. Various authors find that[17,18]

$$\langle 0 | T_{ab}^{\mathrm{QM}} | 0 \rangle = \frac{\alpha}{3}(g_{ab}R^{;c}{}_{;c} - R_{;ab} + RR_{ab} - \tfrac{1}{4}g_{ab}R^2) + \beta(\tfrac{2}{3}RR_{ab} - R_a{}^cR_{bc} + \tfrac{1}{2}g_{ab}R_{cd}R^{cd} - \tfrac{1}{4}g_{ab}R^2) \ , \tag{1.4}$$

where R_{ab} is the Ricci tensor, $R=g^{ab}R_{ab}$ is the scalar curvature, and α and β are constants which depend upon the number and types of fields present and, in the case of the photon, on the regularization scheme used. For example, dimensional regularization gives $\alpha=12(2880\pi^2)^{-1}$, $\beta=62(2880\pi^2)^{-1}$ for the photon, while ζ-function regularization gives $\alpha=-18(2880\pi^2)^{-1}$, $\beta=62(2880\pi^2)^{-1}$.[19]

If Eqs. (1.1), (1.3), and (1.4) are substituted into (1.2), the result is an ordinary differential equation for $a(\eta)$. We wish to solve this equation and examine the physical properties of its solutions.

Since we do not yet know what fields were present in the early universe or which regularization scheme, if any, is the correct one, we do not know what the values of α and β were for the early universe. Therefore, we take a phenomenological approach and consider all reasonable[20] values of α and β.

We are interested in models of the early universe which evolve into universes which look like ours at late times compared with the Planck time. Thus we impose the boundary condition that solutions of (1.2) must approach the appropriate solution to the field equations of classical general relativity at late times. We call such solutions asymptotically classical solutions (ACS).

To summarize: We wish to find the asymptotically classical solutions of (1.2) for homogeneous and isotropic spacetimes containing classical radiation, when the only quantum fields present are conformally invariant free fields. All reasonable values of the regularization parameters α and β will be considered.

Once the initial behaviors of the ACS are known, it can be determined if particle horizons and singularities are present. This allows us to address the issues of whether our universe began with an initial singularity and whether it has particle horizons.

Our results are very similar to those of paper I. We find that for $\alpha \leq 0$ there is one ACS for each value of α and β, while for $\alpha > 0$, there are many ACS.

If $\beta > 3\alpha > 0$, we find one ACS always exists which describes a universe which bounces and expands in the same way after the bounce that it contracted before the bounce. If this universe is closed, then its scale factor oscillates between a minimum and maximum value. If it is open, then its scale factor, which is initially infinite, decreases to some minimum value and then increases forever. In both cases, we call this solution a "time-symmetric bounce solution." It has no singularities and, if $\Lambda=0$ or the universe is closed, it has no particle horizons.

If $\alpha>0$, $\beta>0$, we also find a one-parameter family of ACS which begin with an infinite scale factor. The scale factor initially decreases exponentially as a function of proper time, $dt = a\,d\eta$. Later, it bounces once and then approaches the classical solution. These solutions have particle horizons, but no singularities. There are no other ACS without singularities.

If $\alpha>0$, there is always at least one ACS without particle horizons for each α and β. If $\alpha \leq 0$, the results are mixed. The above results are summarized in Table I.

Only if $\beta \approx 3\alpha$ does the spatial curvature significantly influence the initial behavior of the ACS, with the exception of the time-symmetric bounce solutions discussed above. The cosmological constant never significantly af-

TABLE I. Summary of the asymptotically classical solutions found in Sec. III and their physical properties.

α	β	k	Number of parameters solutions needed to specify a solution	Number of solutions without a singularity	Number of solutions without a horizon	Equation describing initial behavior
$\alpha>0$	$\beta\geq 3\alpha$	$0,\pm1$	1	Family	Family	(3.10),(3.17),(3.18)
	$0<\beta<3\alpha$	$0,\pm1$	1	Family	1	(3.10),(3.17),(3.20)
	$\beta\leq 0$	$0,\pm1$	1	None	1	(3.11),(3.12),(3.17),(3.20)
$\alpha=0$	$\beta\geq 0$	$0,\pm1$	None	None	None	(2.11),(3.22),(3.24)
	$\beta<0$	$0,\pm1$	None	None	1	(3.25)
$\alpha<0$	$\beta>3\alpha$	$0,-1$	None	None	None	(3.29)
	$\beta>3\alpha[1-B^2(1+E)^{-1}]$	1	None	None	None	(3.29)
	$3\alpha<\beta\leq 3\alpha[1-B^2(1+E)^{-1}]$	1	None	None	Unknown	(3.29),(3.31),(3.32)
	$\beta\leq 3\alpha$	$0,\pm1$	None	None	1	(3.28)

fects the behavior of solutions when the scale factor is small.

In Sec. II, we derive and discuss the dynamical equation of motion for the scale factor. If the spatial curvature and cosmological constant do not vanish, then we find that quantum effects persist, to a small extent, even at late times. In Sec. III, we find both the early- and late-time behaviors of the ACS for all reasonable values of α and β and we determine which ACS have singularities and particle horizons.

II. DERIVATION AND DISCUSSION OF THE DYNAMICAL EQUATION OF MOTION

The dynamical behavior of homogeneous and isotropic spacetimes consists of uniform expansions and contractions which can be completely described by the scale factor $a(\eta)$. Our goal is to derive an equation for $a(\eta)$ using Eqs. (1.1), (1.2), (1.3), and (1.4) and then to solve this equation for various values of k, Λ, α, and β.

To derive an equation for $a(\eta)$, we must have expressions for the stress-energy tensors on the right-hand side of (1.2). The expression for $\langle 0 | T_{ab}^{\rm QM} | 0 \rangle$ is given by (1.4). The stress-energy tensor for classical radiation is

$$T_{ab}^{\rm CR} = (\rho_r + p_r) U_a U_b + p g_{ab} \ , \tag{2.1}$$

where U_a is the four-velocity of the radiation, ρ_r is its energy density, and p_r its pressure. The equation of state is $p_r = \frac{1}{3}\rho_r$ and ρ_r varies with a as

$$\rho_r = \frac{\widetilde{\rho}_r}{a^4} \ , \tag{2.2}$$

where $\widetilde{\rho}_r$ is a constant.

To conform with the notation of paper I, we write our equation in terms of the variables

$$b = l^{-1} \widetilde{\rho}_r^{\,-1/4} a \ , \qquad \chi = 6^{-1/2} \widetilde{\rho}_r^{\,1/4} \eta \ . \tag{2.3}$$

For given values of k, $\widetilde{\rho}_r$, and Λ, the only variable in RW spacetimes is the scale factor $a(\eta)$. Thus all of the nontrivial components of (1.2) must be linearly dependent. For convenience we choose the "00" component which, combined with Eqs. (1.1), (1.3), (1.4), (2.1), (2.2), and (2.3), gives the following equation for scale factor:

$$b'^2 + \frac{6kb^2}{\widetilde{\rho}_r^{\,1/2}} - 2l^2 \Lambda b^4 = 1 + \frac{\alpha}{3}\left[\frac{b'''b'}{2b^2} - \frac{b''b'^2}{b^3} - \frac{1}{4}\left[\frac{b''}{b}\right]^2 - \frac{3k}{\widetilde{\rho}_r^{\,1/2}}\left[\frac{b'}{b}\right]^2 + \frac{9k^2}{\widetilde{\rho}_r}\right] + \beta\left[\frac{1}{12}\left[\frac{b'}{b}\right]^4 + \frac{k}{\widetilde{\rho}_r^{\,1/2}}\left[\frac{b'}{b}\right]^2 + \frac{3k^2}{\widetilde{\rho}_r}\right] , \tag{2.4}$$

where $b' \equiv db/d\chi$. This is a third-order ordinary differential equation which does not explicitly depend on the independent variable χ. This implies that its solutions will be invariant under translations in χ. It is also easy to check that Eq. (2.4) is invariant under the transformation $\chi \to -\chi$, although its solutions in general are not.

Because Eq. (2.4) is explicitly independent of χ, it can be reduced to a second-order differential equation. One way to do this[10,21] is to define the new variables

$$y = |\alpha|^{-3/4} b^3 \ , \qquad f = |b'|^{3/2} \ . \tag{2.5}$$

In terms of f and y, Eq. (2.4) becomes

$$\frac{d^2 f}{dy^2} = \frac{-\beta}{12\alpha}\frac{f}{y^2} + \frac{\alpha}{|\alpha|}\left[\frac{1}{y^{2/3} f^{1/3}}\left[1 - \frac{B}{y^{2/3}}\right] - \frac{1}{y^{2/3} f^{5/3}}(1 + E + A y^{2/3} + \widetilde{\Lambda} y^{4/3})\right] , \tag{2.6}$$

where we have introduced the constants

$$E = \frac{3k^2(\alpha+\beta)}{\widetilde{\rho}_r}, \quad A = \frac{-6|\alpha|^{1/2} k}{\widetilde{\rho}_r^{\,1/2}} \ ,$$

$$B = \frac{A(\alpha-\beta)}{6|\alpha|}, \quad \widetilde{\Lambda} = 2l^2 |\alpha| \Lambda \ .$$

Some discussion of these equations is in order. If $k=0$, then $A=B=E=0$ and Eqs. (2.4) and (2.6) are both independent of $\widetilde{\rho}_r$. This comes from the fact that b is invariant under changes of scale, while $\widetilde{\rho}_r$ is not. Once b is known, the scale can be set by $\widetilde{\rho}_r$ and $a(\eta)$ can be determined.

The spacetimes of interest are those which approach our universe at times much larger than the Planck time. Since quantum effects should be negligible at such times, these spacetimes should contain the same amount of radiation and have the same value of the cosmological constant as general relativity and current observations predict our universe had at those times.

From observations of the dynamics of clusters of galaxies, it can be deduced that today[22]

$$|\Lambda| \lesssim 10^{-57} \ {\rm cm}^{-2} \ ,$$

which implies

$$|\widetilde{\Lambda}| \lesssim |\alpha| \times 10^{-121} \ . \tag{2.7}$$

It is possible to put limits on the current value of $\widetilde{\rho}_r$, if $k \neq 0$, from observations of the temperature of the microwave background radiation T_b, the present value of Hubble's constant H_0, the present value of the deceleration parameter q_0, and the present density of matter (not radiation) ρ_{m0}. For $k \neq 0$ one finds[23]

$$\tilde{\rho}_r = 9.3\times 10^{115}\left[\frac{T_b}{2.7\text{ K}}\right]^4 \times k^2 h^{-4}[0.8h^{-2}(\rho_{m0}/(10^{-29}\text{ g/cm}^3))-q_0-1]^{-2}, \quad (2.8)$$

where $h = H_0(100(\text{km/sec})/\text{Mpc})^{-1}$.

According to current observations, the expressions in parentheses in (2.8) are both equal to unity to within one or two orders of magnitude, as are h and q_0.[24] For spatially flat spacetimes, the expression in square brackets vanishes, so

$$\tilde{\rho}_r \gtrsim 10^{116} \quad (2.9)$$

is a reasonable limit to put on the present value of $\tilde{\rho}_r$, if the universe has nonzero spatial curvature. This gives the following limits on A, B, and E:

$$|A| \lesssim 10^{-58}|\alpha|^{1/2},$$
$$|B| \lesssim 10^{-58}|\alpha-\beta||\alpha|^{-1/2}, \quad (2.10)$$
$$|E| \lesssim 10^{-116}|\alpha+\beta| .$$

The reason $\tilde{\rho}_r$ is so large and therefore A, B, and E are so small, is that through the scaling (2.3) the spatial curvatures, energy densities, etc., have been expressed in Planck units. Compared to the natural Planck scales, our universe is remarkably flat. This is yet another example of the flatness problem.[1]

A glance at Eq. (2.6) shows that terms containing A and $\tilde{\Lambda}$ only make significant contributions to d^2f/dy^2 when y is very large. Since b is zero at the initial singularity predicted by classical general relativity and since $y \propto b^3$, the qualitative behavior of the early universe should be independent of A and $\tilde{\Lambda}$ unless the universe did not begin with a small scale factor. In Sec. III, it is found that this is indeed the case.

The term containing B comes from $\langle 0|T_{00}^{\text{QM}}|0\rangle$ and it makes a significant contribution to d^2f/dy^2 only when y is near zero. Even then, it turns out that the qualitative behavior of solutions is only affected by this term if $\beta \approx 3\alpha$. This means that most of the types of behaviors found in paper I for universes with $k=\Lambda=0$ occur also for universes with $k\neq 0$ and/or $\Lambda\neq 0$.

As discussed in the Introduction, the solutions of interest are the asymptotically classical solutions, that is, solutions which at late times, compared with the Planck time, approach the appropriate solutions of Einstein's equations, which in this case are the radiation-dominated Friedmann universes. In the variables f and y, these solutions are

$$f = (1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4} . \quad (2.11)$$

There are two basic criteria which are applied to solutions of (2.6) to determine if they are ACS. The first is that the classical solution, (2.11), should be approached long before the terms containing A and $\tilde{\Lambda}$ make significant contributions to (2.11). This is because, even today, these terms are so insignificant that observations have not determined if they are positive or negative. The second criterion is that an ACS should continue to approach the classical solution, once it has begun to do so, for all large values of y.

In Sec. III, the late-time behavior of one of the ACS is displayed. It has the general form

$$f = (1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}\{1+O(y^{-4/3})+O(Ay^{-2/3})+O(\tilde{\Lambda})+(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1}[O(A^2)+O(\tilde{\Lambda}^2y^{4/3})]\} . \quad (2.12)$$

This means that, unless $k=\Lambda=0$, quantum effects persist, even at late times. The reason is that the highest-order derivatives in Eqs. (2.4) and (2.6) come from $\langle 0|T_{00}^{\text{QM}}|0\rangle$. If the classical solution is to be approached exactly at late times, then when the classical solution is substituted into Eqs. (2.4) and (2.6), the terms due to $\langle 0|T_{00}^{\text{QM}}|0\rangle$ should cancel, at least to the same order in the scale factor as the classical solution. This does not happen exactly if $k\neq 0$ and/or $\Lambda\neq 0$, which is why there are extra terms in Eq. (2.12).

The most significant of these occurs if $\Lambda>0$, and the universe is open. Then the universe expands at very late times as though it had a slightly different cosmological constant than it has. This implies that if, for some period, the universe expands like a de Sitter universe (which is what it does when a positive cosmological constant dominates the expansion), then $\langle 0|T_{00}^{\text{QM}}|0\rangle$ contributes a term which acts like an effective cosmological constant. Starobinsky[25] has shown that if the universe contains no matter or radiation, then there is an exact solution to Eq. (2.6) with $k=\Lambda=0$, $\alpha<0$, and $\beta>0$, which is a de Sitter universe with an effective cosmological constant equal to $6l^{-2}\beta^{-1}$.

One might be worried that some quantum effects persist even when the scale factor is large. However, all of these effects are either related to the spatial curvature or the spacetime curvature (since the curvature of de Sitter space depends on the cosmological constant) and they do not become large unless the curvature becomes large. If the curvature is large, then the effects of quantum field theory in curved spacetime should be important, so there is no inconsistency.

III. ASYMPTOTICALLY CLASSICAL SOLUTIONS AND THEIR PHYSICAL PROPERTIES

In this section the asymptotically classical solutions of Eqs. (2.4) and (2.6) are found for spacetimes with $k\neq 0$ and/or $\Lambda\neq 0$ and some of their physical properties are discussed. The early-time behaviors of these ACS are different for different values of α and β. Therefore the discussion of the ACS is broken up into the cases $\alpha>0$, $\alpha=0$

and $\alpha < 0$, as well as several subcases depending on the values of (β/α) and k. It is assumed throughout that the dimensionless constants E, A, B, and $\widetilde{\Lambda}$, which together determine the values of k, Λ, α, and β, are very small compared to unity, in keeping with their values for our universe [see Eqs. (2.7)–(2.10)]. The results for the various cases are summarized in Table I.

It is not hard to see that the terms containing A and $\widetilde{\Lambda}$ make negligible contributions to Eq. (2.6) unless $y \gtrsim |A|^{-3/2}$, $|\widetilde{\Lambda}|^{-3/4}$. For such large values of y, the classical solution (2.11) is dominated by either the term with A or the term with $\widetilde{\Lambda}$, which means that the expansion is dominated either by the spatial curvature or the cosmological constant. This is not yet true for our universe, which is why the ACS are required to approach the classical solution (2.11) at values of y which may be large compared to unity, but which are smaller than $|A|^{-3/2}$ or $|\widetilde{\Lambda}|^{-3/4}$.

The term containing B in (2.6) is important only for $y \lesssim |B|^{3/2}$. Even then, as shall be shown, this term only significantly affects the behavior of solutions if $\beta \simeq 3\alpha$. Thus for most values of α and β, the early-time behaviors of the ACS are qualitatively the same as those for universes with $k = \Lambda = 0$. That is, features such as the number of ACS beginning with $b = 0$ or $b = \infty$, beginning at $\chi = -\infty$ or at a finite value of χ, having a power-law expansion at early times, undergoing a single bounce or multiple bounces, etc., remain unchanged. For example, if there is a family of ACS for $k = \Lambda = 0$ which begin with $b = \infty$, bounce once and have no singularities, there is also such a family for $k \neq 0$ and/or $\Lambda \neq 0$.

As in paper I, only for $\alpha = 0$ were general analytic solutions to Eq. (2.6) found. For each value of β, there is one ACS. The solutions are displayed and their physical properties are discussed in Sec. III B.

For $\alpha > 0$, there is a one-parameter family of ACS for each value of α and β. In Sec. III A the behaviors of these ACS are examined in detail and their physical properties are discussed. Figures 1–3 show the results of numerical integrations of (2.4) for various values of k, Λ, α, and β.

For $\alpha < 0$, there is one ACS for each value of α and β. In Sec. III C, the behaviors of these ACS are examined in detail and their physical properties are discussed.

A more thorough discussion of the ACS and their early-time behaviors follows, beginning with the case $\alpha > 0$.

A. $\alpha > 0$.

In Sec. II A of paper I, this case was investigated for universes with $k = \Lambda = 0$. It was possible in that investigation to display the late-time behavior of the ACS and to prove theorems showing that ACS with certain initial behaviors exist. For universes with $k \neq 0$ and/or $\Lambda \neq 0$ the extra terms in (2.4) and (2.6) make it much more difficult to accomplish these tasks. Therefore a less rigorous approach is taken here and we replace the proofs with two assertions about the ACS. These assertions, along with some analytical and numerical work, allow nearly as much to be determined about the ACS for $k \neq 0$ and/or $\Lambda \neq 0$ as was determined about them for $k = \Lambda = 0$. The structure of the rest of this section is as follows. First the assertions are stated, then evidence for them is given and finally the early-time behaviors of the ACS are analyzed using them.

The two assertions are as follows:

(1) A one-parameter family of ACS exist for all values of α and β if $\alpha > 0$.

(2) The behavior of solutions to (2.6) in the range $|B|^{3/2} \ll y \ll |A|^{-3/2}$, $|\widetilde{\Lambda}|^{-3/4}$ is qualitatively the same as that for universes with $k = \Lambda = 0$.

There are two pieces of evidence supporting assertion (1). The first is that an explicit expression for the late-time behavior of one of the ACS can be obtained by as-

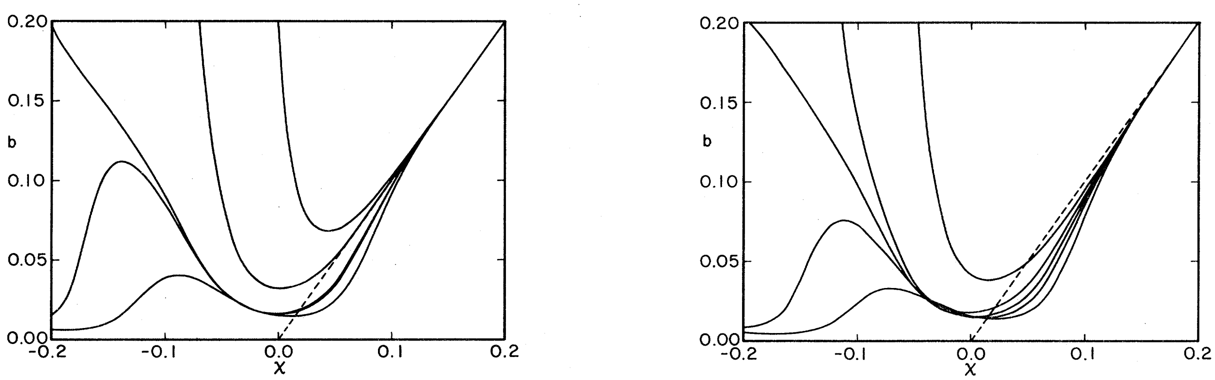

FIG. 1. This figure shows selected ACS for $\beta = 6\alpha = 6(2880\pi^2)^{-1}$, $\widetilde{\rho}_r = 1$, $\Lambda = 0$. The plot on the left is for $k = 1$ and that on the right is for $k = -1$. The dashed lines are the classical solutions. The solid lines were obtained by numerically integrating (2.4) backwards in time, using (3.7), with $c_2 = 0$, to obtain starting values for b' and b''. From top to bottom the curves in the left-hand plot correspond to solutions with $c_1 = 5$, 10, 18.9880, 20, and 35, while those for the right-hand plot correspond to solutions with $c_1 = 5$, 10, 14.3097, 20, and 35. Because of the large value of $\widetilde{\rho}_r^{\,-1}$, the spatial curvature has a visible effect on solutions. Even so, the qualitative behavior of solutions is the same. That is, for both values of k, the upper solutions begin with $b = \infty$ at a finite value of χ and behave initially like collapsing de Sitter universes with effective cosmological constants given by (3.13). They bounce once and approach the classical solution. In both plots the lower solutions are multiple-bounce solutions and begin with $b = 0$ at $\chi = -\infty$. Also in both cases a time-symmetric bounce solution exists, although it occurs for $c \approx 18.9880$ if $k = 1$ and $c \approx 14.3097$ if $k = -1$.

suming a solution to (2.6) of the form

$$f=f_0+\alpha f_1+\alpha^2 f_2+\cdots\ , \tag{3.1}$$

where f_0 is the classical solution (2.11). Here α is considered to be a small number and in what follows β is assumed to be of the same order of magnitude as α. The motivation for this expansion is that in the limit $\alpha,\ \beta\to 0$ quantum effects vanish and Eqs. (1.2) and hence (2.4) reduce to Einstein's equations.

If (3.1) is substituted into (2.6) and the dependence of y on α is taken into account [see Eq. (2.5)], then keeping terms of order α results in an algebraic equation for f_1. Substituting the solution for f_1 back in (2.6) and keeping terms of order α^2 results in an algebraic equation for f_2, etc. The result to order α is

$$\begin{aligned} f=&(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}+\tfrac{1}{16}\beta\alpha^{-1}y^{-4/3}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{7/4}\\ &+\tfrac{1}{4}[(3B-\tfrac{1}{2}A)y^{-2/3}+\tilde{\Lambda}](1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}\\ &-\tfrac{1}{4}(\tfrac{1}{4}A^2-3E+A\tilde{\Lambda}y^{2/3}+\tilde{\Lambda}^2y^{4/3})(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1/4}+O(\alpha^2)\ . \end{aligned} \tag{3.2}$$

For the solution in (3.2) to be an ACS, it must satisfy the two criteria given in Sec. II; that is, it must approach the classical solution for values of y such that $y\ll|A|^{-3/2},\ |\tilde{\Lambda}|^{-3/4}$ and it must continue to approach the classical solution for all larger values of y.

For open universes (ones which expand forever), (3.2) is valid for all $y\gg 1$. The classical solution, (2.11), is approached to within $O(y^{-4/3})$ for $1\ll y\ll|A|^{-3/2}$, $|\tilde{\Lambda}|^{-3/4}$ and to within $O(|A|y^{-2/3}+|\tilde{\Lambda}|)$ for $y>|A|^{-3/2},\ |\tilde{\Lambda}|^{-3/4}$. Thus (3.2) is an ACS. In the limit $y\to\infty$, this solution approaches

$$f=\tilde{\Lambda}^{3/4}(1+\tfrac{1}{16}\alpha^{-1}\beta\tilde{\Lambda})y,\quad \Lambda>0\ , \tag{3.3a}$$

$$f=A^{3/4}y^{1/2},\quad \Lambda=0\ . \tag{3.3b}$$

For closed universes (ones which expand to a maximum size and then contract), (3.2) is valid in the region $1\ll y\ll|A|^{-3/2},\ |\tilde{\Lambda}|^{-3/4}$ so the first criterion for an ACS is satisfied. However, as y approaches the value at which the classical maximum occurs, the quantity $(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})$ approaches zero [because an extremum in the $(\mathcal{X},b)$ plane corresponds to $f=0$ in the (y,f) plane] and the fourth term on the right in (3.2) diverges. This means that the expansion (3.1) breaks down in this limit and other techniques must be used to determine the behavior of the solution in (3.2). In the Appendix it is shown that a solution to (2.6) exists for which $f=0$ at a value of y_0 which is within $O(k^2\alpha^{1/4}\tilde{\rho}_r^{-1/4}+|\tilde{\Lambda}|^{1/4})$ of the value of y at the classical maximum. Near $y=y_0$, this solution matches up with the one in (3.2) to $O(\alpha(y_0-y)^{3/4})$. Thus, the solution in (3.2) also satisfies the second criterion for an ACS and is, therefore, an ACS.

The second piece of evidence for assertion (1) has to do with solutions to the equation which results from linearizing (2.6) about the classical solution (2.11). These solutions also provide evidence for assertion (2).

To linearize (2.6) about the classical solution (2.11), one first writes $f=f_0+f_l$, where f_0 is again the classical solution and f_l is assumed to be small compared to f_0. Then to first order in f_l,

$$\begin{aligned} \frac{d^2f_l}{dy^2}=&\left[-\frac{\beta}{12\alpha}y^{-2}+\tfrac{1}{3}y^{-2/3}(4+By^{-2/3})(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1}+\tfrac{5}{3}Ey^{-2/3}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-2}\right]f_l\\ &-\tfrac{1}{12}\alpha^{-1}\beta y^{-2}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}+[(\tfrac{1}{6}A-B)y^{-4/3}-\tfrac{1}{3}\tilde{\Lambda}y^{-2/3}](1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1/4}\\ &+[(\tfrac{1}{12}A^2-E)y^{-2/3}+\tfrac{1}{3}A\tilde{\Lambda}+\tfrac{1}{3}\tilde{\Lambda}^2y^{2/3}](1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-5/4}\ . \end{aligned} \tag{3.4}$$

A particular solution may be obtained by writing $f_l=\alpha f_{l1}+\alpha^2 f_{l2}+\cdots$ and substituting this expression into (3.4). If powers of α are counted and the α dependence of y is taken into account, then keeping terms of order α results in an algebraic equation for f_{l1}. Substituting this back into (3.4) and keeping terms of order α^2 gives an algebraic equation for f_{l2}, etc. The resulting particular solution matches (3.2) to order α, as one would expect.

The homogeneous equation can be reduced to a first-order differential equation with the change of variable $f_l=\exp[\int K(y)dy]$. The resulting equation for K is

$$\begin{aligned} \frac{dK}{dy}=&-K^2-\tfrac{1}{12}\alpha^{-1}\beta y^{-2}\\ &+\tfrac{1}{3}y^{-2/3}(4+By^{-2/3})(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1}\\ &+\tfrac{5}{3}Ey^{-2/3}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-2}\ . \end{aligned} \tag{3.5}$$

Since (3.4) is a linear equation, only two solutions of (3.5) are needed, so long as they correspond to linearly independent solutions of (3.4). For $1\ll y\ll|A|^{-3/2}$, $|\tilde{\Lambda}|^{-3/4}$, two such solutions may be obtained by the following method. Write $K=K_1+K_2+K_3+\cdots$ and substitute into (3.5). Assume that $|K_i| \gg |K_{i+1}|$ and that $|dK_i/dy| \ll |K_i|$ for all i. Then, keeping only lowest-order terms, solve for K_1. Substitute the expression for K_1 back into (3.5) and keeping only lowest-order terms solve for K_2, etc. The result is that for $1 \ll y \ll |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$, the above conditions on the K_i are satisfied and

$$K_1=\pm[-\tfrac{1}{12}\alpha^{-1}\beta y^{-2} +\tfrac{1}{3}y^{-2/3}(4+By^{-2/3})(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1} +\tfrac{5}{3}Ey^{-2/3}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-2}]^{1/2} . \quad (3.6)$$

Then the general solution to (3.4) in the region $1 \ll y \ll |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$ is

$$f_l=1+\beta(16\alpha)^{-1}y^{-4/3}+\tfrac{3}{4}Ay^{2/3}+\tfrac{3}{4}\tilde{\Lambda}y^{4/3}+O(A^2y^{4/3})+O(\tilde{\Lambda}^2y^{8/3})+O(E)+O(\alpha^2) +c_1y^{1/6}\exp\{-3^{1/2}y^{2/3}[1+O(E)+O(y^{-4/3})+O(Ay^{2/3})+O(\tilde{\Lambda}y^{4/3})]\} +c_2y^{1/6}\exp\{3^{1/2}y^{2/3}[1+O(E)+O(y^{-4/3})+O(Ay^{2/3})+O(\tilde{\Lambda}y^{4/3})]\} , \quad (3.7)$$

where c_1 and c_2 are arbitrary constants.

The solutions with $c_2=0$ form a one-parameter family which approach the classical solution. For open universes, it is not hard to show, using (3.3) and (3.4), that the classical solution is approached by these solutions for all $y \gg 1$. Thus it is very likely that the exact solutions of (2.6) which they correspond to are ACS. For closed universes, the perturbation expansion leading to (3.4) breaks down for $y \sim |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$. However, it seems likely that the solutions of (2.6) to which the solutions in (3.7) with $c_2=0$ correspond, continue to approach the classical solution and have maximum values of y close to that of the classical solution, just as the solution in (3.2) does. This implies that they are ACS and therefore that assertion (1) is correct, for both open and closed universes.

Given then that the solutions with $c_2=0$ in (3.7) correspond to ACS, one sees that assertion (2) is satisfied by these ACS in the region $1 \ll y \ll |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$, $f \sim 1$. There are three other pieces of evidence which support assertion (2). They consist of an analysis of the behavior of solutions in the limit $f \to 0$ which covers the region $|B|^{3/2} \ll y \ll |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$, the results of some numerical integrations which cover the region $10^{-14} < y < 10^4$, $0 < f < 10^2$, and an analysis of solutions which approach $f=\infty$, $y=\infty$ with $f > y$, which covers the region $f > y \gg 1$. Thus, for the most part, these pieces of evidence do not cover overlapping regions, but taken together, they do cover much of the region for which assertion (2) is expected to hold.

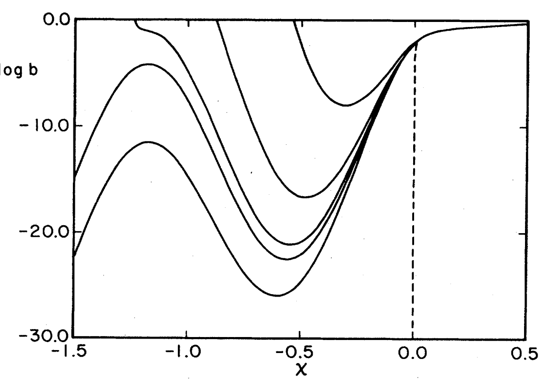

FIG. 2. This figure shows selected ACS for $\beta=3\alpha=(2880\pi^2)^{-1}$, $k=1$, $\tilde{\rho}_r=1$, $\Lambda=0$. The dashed line is the classical solution. There are two types of ACS shown. The upper curves are a family of solutions which begin with $b=\infty$ at a finite value of χ. Initially, they behave like contracting de Sitter universes with effective cosmological constants given by (3.13). They bounce once and approach the classical solution. The lower curves are multiple-bounce solutions. They begin with $b=0$ at $\chi=-\infty$ and bounce an infinite number of times before approaching the classical solution. A time-symmetric bounce solution is hinted at and almost certainly exists. For $\chi > -1$, it has approximately the same behavior as the central curve.

Because $f=|b'|^{3/2}$, solutions for which $f \to 0$ at a nonzero value of y have an extremum in the (χ,b) plane. If $f=0$ at $y=y_0$, then near $y=y_0$, Eq. (2.6) can be solved and the result is

$$f=(\tfrac{16}{3})^{3/8}y_0^{-1/4}(1+E+Ay_0^{2/3}+\tilde{\Lambda}y_0^{4/3})^{3/8}|y-y_0|^{3/4} +\mathrm{sgn}(\chi-\chi_0)D|y-y_0|^{5/4}+\cdots , \qquad y_0^{-1}|y-y_0| \ll 1 , \quad (3.8)$$

where the extremum occurs at $\chi=\chi_0$ and D is an arbitrary constant. For $y_0 \ll |A|^{-3/2}, |\tilde{\Lambda}|^{-3/4}$, the solutions clearly are only small perturbations of the solutions for $k=\Lambda=0$.

If $D=0$, then, in the (χ,b) plane, the solution is time symmetric about the extremum. Some of the numerical integrations mentioned above were performed for time-symmetric bounce solutions. Equation (2.6) was linearized about its solutions for $k=\Lambda=0$ and this linearized equation was numerically integrated for the cases $k\neq 0$, $\Lambda=0$, and $k=0$, $\Lambda\neq 0$. The solutions of this linearized equation give the deviations of solutions with $k\neq 0$ and/or $\Lambda\neq 0$ from solutions with $k=\Lambda=0$. In all cases, negligible deviations were found for values of A and $\tilde{\Lambda}$ in accord with (2.7) and (2.10). Most of the region $10^{-14} < y < 10^4$, $10^{-6} < f < 10^2$ was covered by these numerical integrations.

The rest of the numerical integrations were done for Eq. (2.4). The values used for $\tilde{\rho}_r$ were 1, 0.1, and 0.01, those for k were ± 1, 0 and those for $|l^2\Lambda|$ were 1, 10, and 100. In each case, starting values for b'' and b' were obtained from the solutions in (3.7) (to the linearized equation) with $c_2=0$.

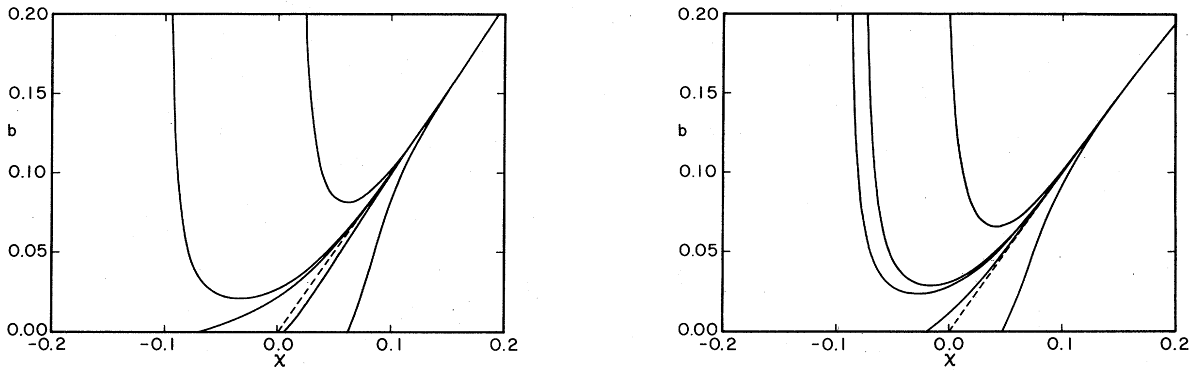

FIG. 3. This figure shows selected ACS for $\beta=\alpha=(2880^2)^{-1}$, $k=0$. The plot on the left is for $\Lambda=100l^{-2}$, while that on the right is for $\Lambda=-100l^{-2}$. The dashed lines are the classical solutions. The solid lines were obtained by numerically integrating (2.4) backwards in time, using (3.7), with $c_2=0$, to obtain starting values for b' and b''. From top to bottom the curves in each plot correspond to solutions with $c_1=-2$, 0, 0.1, 1, and 20. Because of the large values of Λ, the cosmological constant has a visible effect on solutions. Even so, the qualitative behavior of solutions is the same. That is, for both positive and negative Λ, the upper solutions begin with $b=\infty$ at a finite value of χ and behave initially like collapsing de Sitter universes with effective cosmological constants given by (3.13). They bounce once and approach the classical solution. The lower solutions begin with $b=0$ at a finite value of χ and do not bounce. A third solution is hinted at in each plot and almost certainly exists. It begins with $b=0$ at $\chi=-\infty$ and does not bounce.

The values of $\widetilde{\rho}_r^{\,-1}$ and $|l^2\Lambda|$ were large enough so that differences between universes with different values of k and Λ could be observed. However, even for such unrealistically large values of $\widetilde{\rho}_r^{\,-1}$ and $l^2\Lambda$, the qualitative behavior of solutions in the region $|B|^{3/2}<y<|A|^{-3/2}$, $|\widetilde{\Lambda}|^{-3/4}$ was the same as that found in paper I for $k=\Lambda=0$. Some examples are shown in Figs. 1—3.

The large-f, large-y behavior of solutions to (2.6) can be obtained by noting that for $f>y\gg 1$, the terms with B, $(1+E)$, and A can be ignored. Then, with the change of variables $f=yr$, $y=e^w$, $s=dr/dw$, one finds

$$\frac{ds}{dr}=s^{-1}[-(12\alpha)^{-1}\beta r+r^{-1/3}-\widetilde{\Lambda}r^{-5/3}]-1\ . \quad (3.9)$$

A phase-plane analysis shows that, if $\beta>0$, solutions spiral into the point

$$s=0,\ r=(6\alpha\beta^{-1})^{3/4}[1+(1-\tfrac{1}{3}\alpha^{-1}\beta\widetilde{\Lambda})^{1/2}]^{3/4}\ .$$

Since $\widetilde{\Lambda}\ll 1$, this corresponds to the behavior

$$f=(12\alpha/\beta)^{3/4}[1-(16\alpha)^{-1}\beta\widetilde{\Lambda}+\cdots]y,\ \ y\to\infty\ , \quad (3.10a)$$

$$b=(\beta/12)^{1/2}[1+(24\alpha)^{-1}\beta\widetilde{\Lambda}+\cdots](\chi-\chi_0)^{-1}\ ,\quad \chi\to\chi_0\ , \quad (3.10b)$$

where χ_0 is an arbitrary constant.

If $\beta=0$, a phase-plane analysis shows that solutions approach the curve $s=r^{-1/3}$ at large r. This corresponds to the behavior[26]

$$f=(\tfrac{4}{3})^{3/4}y(\ln y)^{3/4},\ \ y\to\infty\ , \quad (3.11a)$$

$$b(\ln b)^{1/2}=\tfrac{1}{2}\alpha^{1/2}(\chi-\chi_0)^{-1},\ \ \chi\to\chi_0\ . \quad (3.11b)$$

If $\beta<0$, a phase-plane analysis shows that solutions approach $s=(-\tfrac{1}{2}+\sigma)r$ at large r, where $\sigma\equiv\tfrac{1}{2}(1-\beta/3\alpha)^{1/2}$. This corresponds to the behavior

$$f=c_1y^{1/2+\sigma},\ \ y\to\infty\ , \quad (3.12a)$$

$$b=\text{const}\times(\chi-\chi_0)^{-1/2\sigma},\ \ \chi\to\chi_0\ , \quad (3.12b)$$

where c_1 is a positive constant. Thus, for $f>y\gg 1$, solutions for $k\neq 0$ and/or $\Lambda\neq 0$ are only small perturbations of the corresponding solutions for $k=\Lambda=0$.

Having provided evidence for assertions (1) and (2), and, in the process, discussed the late-time behavior of the ACS, we now turn to their early-time behaviors. These fall into two categories: those which begin with $b=0$ and those which do not. Because much of the groundwork has been laid for the latter category, it is convenient to begin there.

1. Asymptotically classical solutions which do not begin with $b=0$

In paper I, for each value of α and β, a one-parameter family of ACS which begin with $b=\infty$ at a finite value of χ were found. These bounce once and approach the classical solution. Assertion (2) and the numerical work mentioned above and shown in Figs. 1—3 indicate that, for $k\neq 0$ and/or $\Lambda\neq 0$, there is again a one-parameter family of ACS. Their early-time behavior is given by Eqs. (3.10)—(3.12) for $\beta>0$, $\beta=0$, and $\beta<0$, respectively.

These solutions have particle horizons, but if $\beta>0$, they do not have singularities. Instead they initially appear like collapsing de Sitter universes with effective cosmological constants:

$$\Lambda_{\text{effective}}=6l^{-2}\beta^{-1}[1+\tfrac{1}{16}\alpha^{-1}\beta\widetilde{\Lambda}+O(\widetilde{\Lambda}^2)]\ . \quad (3.13)$$

If $\beta \leq 0$, the solutions begin with initial singularities. This can be seen by substituting (3.11) and (3.12) into the formula for the scalar curvature,

$$R = l^{-2} b'' b^{-3} + 6kl^{-2} \tilde{\rho}_r^{-1/2} b^{-2}$$
$$= 2l^{-2} |\alpha|^{-1} y^{-1/3} f^{1/3} \frac{df}{dy}$$
$$+ 6lk^{-2} \tilde{\rho}_r^{-1/2} |\alpha|^{-1/2} y^{-2/3} . \qquad (3.14)$$

In paper I, a time-symmetric bounce ACS was found for each value of α and β, if $\beta > 3\alpha$. The numerical work used as evidence for assertion (2) implies that a time-symmetric bounce ACS also occurs for $k \neq 0$ and/or $\Lambda \neq 0$ if $\beta > 3\alpha > 0$. This solution has no singularities. For open universes it begins with $b = \infty$. If $\Lambda > 0$, then it also begins at a finite value of χ, since when the scale factor is large, the cosmological constant dominates the classical equation (1.5), and the approximate solution is $b = \text{const} \times (\chi - \chi_0)^{-1}$. In this case there are particle horizons. If $\Lambda = 0$, then the solution begins at $\chi = -\infty$ and there are no particle horizons. For closed universes, the scale factor oscillates between minimum and maximum values so there are no particle horizons.

There is one other case for which a time-symmetric bounce ACS exists. That is the case $k = 1$, $\beta = 3\alpha$. The evidence for this is given in Sec. III A 2 b.

This ends the discussion of the ACS which do not begin with $b = 0$.

2. *Asymptotically classical solutions which begin with b = 0*

The remaining ACS begin with $b = 0$ and therefore with $y = 0$. For $\beta > 0$ it is not hard to show that all solutions to (2.6) which begin with $y = 0$ also begin with $f = 0$. In this limit, (2.6) reduces to

$$\frac{d^2 f}{dy^2} = \frac{-\beta}{12\alpha} \frac{f}{y^2} - (1+E) y^{-2/3} f^{-5/3} - B y^{-4/3} f^{-1/3} . \qquad (3.15)$$

With the change of variables $f = y^{1/2} |v|^{3/2}$, $y = e^w$, the resulting equation can be integrated once to give

$$\frac{dv}{dw} = \pm \tfrac{2}{3} v^{-1} [\sigma^2 v^4 - 3Bv^2 + cv + 3(1+E)]^{1/2} , \qquad (3.16)$$

where c is an arbitrary constant. Both these equations differ from their counterparts in Sec. II A of paper I by the terms with E and B. The contribution of the term with E is clearly always negligible. The contribution of the term with B is a bit more difficult to ascertain, but it is certainly important if $\sigma \simeq 0$. Recalling that $\sigma = \frac{1}{2}(1 - \beta/3\alpha)^{1/2}$, this occurs for $\beta \simeq 3\alpha$.

A closer examination of (3.16) shows that in the limit $w \to -\infty$ $(y \to 0)$, dv/dw is imaginary at large $|v|$ if $\sigma^2 < 0$ or if $\sigma^2 = 0$ and $B > 0$, while for other values of σ^2 and B, dv/dw is real at large $|v|$. Thus, it is useful to break the discussion of the early-time behaviors of the ACS into the subcases $\beta > 3\alpha$, $\beta = 3\alpha$, and $\beta < 3\alpha$ corresponding to $\sigma^2 < 0$, $\sigma^2 = 0$, and $\sigma^2 > 0$, respectively.

(a) $\beta > 3\alpha$. For this subcase, dv/dw becomes imaginary at large values of v and it has two real roots: one at a positive value of v and one at a negative value of v. This phase-plane structure is the same as that which occurs for $\beta > 3\alpha > 0$ if $k = \Lambda = 0$. Thus, as discussed in Sec. II A 1 of paper I, solutions spiral about the "w" axis. This implies the existence of a two-parameter family of multiple-bounce solutions, since every time the w axis is crossed an extremum occurs in the (χ, b) plane. Assertion (2) and numerical integrations of (2.4) such as those shown in Fig. 1 indicate that a one-parameter family of these are ACS. As shown in paper I, the multiple-bounce ACS begin with an initial singularity at a finite proper time in the past and they have no particle horizons.

If $k = -1$, then for

$$3\alpha < \beta < 3\alpha[1 + B^2(1+E)^{-1}]$$

there is a range of values of c for which dv/dw has four real roots—three of which have the same sign. This implies the existence of a two-parameter family of solutions to (3.16) which oscillate between two positive or two negative values of v. For two values of c, dv/dw has a double root which implies a one-parameter family of solutions with the behavior

$$f = |v_0|^{3/2} y^{1/2} (1 + c_1 y^p + \cdots), \quad y \to 0 \qquad (3.17a)$$

$$b = \text{const} \times \exp(\alpha^{-1/4} |v_0| \chi), \quad \chi \to -\infty , \qquad (3.17b)$$

where the double root occurs at $v = v_0$, c_1 is an arbitrary constant, and $p = (\frac{8}{3})^{1/2} |v_0|^{-1} [\frac{1}{4} B^2 + (1+E)\sigma^2]^{1/4}$.[27] For both the two-parameter family mentioned above and the one-parameter family in (3.17a), it is possible to show, using an argument similar to the proof in Sec. II A 2 of paper I, that all solutions to (2.6) with this initial behavior approach $f = \infty$, $y = \infty$ with $f > y$ and therefore that no ACS have this initial behavior for $\beta > 3\alpha > 0$.

(b) $\beta = 3\alpha$. In this subcase, $\sigma = 0$ and the behavior of solutions in the small-y limit depends on the sign of B. If $B > 0$, then $k = 1$ and dv/dw in (3.16) is imaginary at large values of $|v|$. This is the same situation as for $\beta > 3\alpha$, so a two-parameter family of multiple-bounce solutions exists. Assertion (2) and the numerical integrations shown in Fig. 2 indicate that a one-parameter family of these are ACS. Integration of (3.16) near $v = 0$ shows that solutions with $c = 0$ are time-symmetric bounce solutions. For $\beta > 3\alpha$, a time-symmetric bounce always provides a continuous transition between the multiple-bounce ACS and the ACS which initially appear like contracting de Sitter universes. Therefore, it is likely that in this subcase a time-symmetric bounce ACS exists and that it provides the above transition. Such an ACS is hinted at, although not explicitly shown, in Fig. 2.

If $B = 0$, then $k = 0$ and (3.16) reduces to Eq. (2.10) of paper I. Using assertion (2), we therefore expect that a one-parameter family of ACS have initial behaviors given by Eq. (2.19) of paper I, and that one ACS has its initial behavior given by Eq. (2.20) of paper I. All of these solutions begin with initial singularities and none of them have particle horizons.

If $B < 0$, then $k = -1$ and (3.16) can be integrated with the result that for $|c| \neq 6[-B(1+E)]^{1/2}$,

$$f=(-\tfrac{4}{3}B)^{3/4}y^{1/2}(-\ln y)^{3/2}, \quad y \to 0, \tag{3.18a}$$

$$b=\text{const}\times \exp\{-\exp[-12^{1/2}\alpha^{-1/4}|B|^{1/2} \times(\chi-\chi_1)]\}, \quad \chi \to -\infty, \tag{3.18b}$$

where χ_1 is an arbitrary constant. For $|c|=6[-B(1+E)]^{1/2}$, dv/dw has a double root at $v_0=[-(1+E)/B]^{1/2}$ so the initial behavior of these solutions is given by (3.17). Substitution of (3.18) and (3.17) into (3.14) shows that these solutions begin with an initial singularity. They do not have particle horizons.

Although the parameters are not displayed in the asymptotic expression (3.18a), there is a two-parameter family of solutions with this initial behavior. There is also a one-parameter family of solutions whose initial behavior is given by (3.17a). The ACS condition gives one condition between the two parameters in (3.18a) and the one parameter in (3.17a). Thus we expect a family of ACS to have their initial behaviors given by (3.18a) and one ACS to have its initial behavior given by (3.17a).

(c) $\beta<3\alpha$. For $\beta<3\alpha$, $\sigma^2>0$ and dv/dw is real for both large and small values of $|v|$. Inspection of (3.16) shows that, at most, dv/dw may have two real roots and they either both occur at positive values of v or at negative values of v. Thus solutions do not spiral around the v axis and multiple-bounce solutions do not occur. However, for $c=c_0$, where

$$c_0=6Bv_0-4v_0{}^3\sigma^2, \quad v_0=\pm\{\tfrac{1}{2}B\sigma^{-2}+[\tfrac{1}{4}B^2\sigma^{-4}+(1+E)\sigma^{-2}]^{1/2}\}^{1/2}, \tag{3.19}$$

a double root of dv/dw occurs at $v=v_0$. This implies that a one-parameter family of solutions, whose initial behavior is given by (3.17), exists. In paper I, it was shown that for $k=\Lambda=0$ one of these is an ACS. Thus, using assertion (2), we expect one of them to be an ACS for $k\neq 0$ and/or $\Lambda\neq 0$ as well. Such an ACS is hinted at, although not explicitly shown, in Fig. 3.

A two-parameter family of solutions to (2.6) which begin with $y=0$ also exists for this subcase. They have the initial behavior

$$f=y^{1/2-\sigma}[C+\tfrac{9}{8}C^{-1/3}\sigma^{-2}By^{4\sigma/3}+C'y^{2\sigma}+\cdots], \quad y\to 0, \tag{3.20a}$$

$$b=\text{const}\times(\chi-\chi_0)^{1/2\sigma}, \quad \chi\to\chi_0, \tag{3.20b}$$

where C, C', and χ_0 are arbitrary constants with the restriction $C>0$. Substitution of (3.20) into (3.14) shows that these solutions begin with an initial singularity; they have particle horizons. Assertion (2) and the numerical work shown in Fig. 3 imply that a one-parameter family of them are ACS. This ends the discussion of the case $\alpha>0$.

B. $\alpha=0$

For the case $\alpha=0$, it is useful to define the variables f and x so that $|b'|=f^{2/3}$ as before and $b=x^{1/3}$. Then (2.6) becomes

$$0=-\tfrac{1}{12}\beta x^{-2}f+x^{-2/3}f^{-1/3}(1-\bar{B}x^{-2/3}) -x^{-2/3}f^{-5/3}(1+\bar{E}+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3}), \tag{3.21}$$

where $\bar{E}=3k^2\beta\tilde{\rho}_r{}^{-1}$, $\bar{A}=-6k\tilde{\rho}_r{}^{-1/2}$, $\bar{B}=-\tfrac{1}{6}\bar{A}\beta$, and $\bar{\Lambda}=2l^2\Lambda$. For $\beta=0$, (3.21) reduces to the classical solution (2.11). For $\beta>0$, the ACS is

$$f=(6\beta^{-1})^{3/4}x\{1-\bar{B}x^{-2/3}-[1-2\bar{B}x^{-2/3}+\bar{B}^2x^{-4/3}-\tfrac{1}{3}\beta x^{-4/3}(1+\bar{E}+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})]^{1/2}\}^{3/4}. \tag{3.22}$$

Substitution of (3.22) into (3.14) shows that this solution begins with an initial singularity at

$$x=x_0=((\beta\bar{A}+6\bar{B})(6-2\beta\bar{\Lambda})^{-1}+\{(\beta\bar{A}+6\bar{B})^2(6-2\beta\bar{\Lambda})^{-2}+[\beta(1+\bar{E})-3\bar{B}^2](3-\beta\bar{\Lambda})^{-1}\}^{1/2})^{3/2}. \tag{3.23}$$

Near the singularity one finds

$$b\simeq x_0{}^{1/3}[1-(6\beta^{-1})^{1/2}x_0{}^{1/3}(\chi-\chi_0)]^{-1}, \quad \chi\to\chi_0, \tag{3.24}$$

where χ_0 is an arbitrary constant. Since it begins at $\chi=\chi_0$, this solution has particle horizons.

For $\beta<0$, the ACS is

$$f=(6|\beta|^{-1})^{3/4}x\{[1-2\bar{B}x^{-2/3}+\bar{B}^2x^{-4/3}+\tfrac{1}{3}|\beta|x^{-4/3}(1+\bar{E}+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})]^{1/2}-1+\bar{B}x^{-2/3}\}^{3/4}. \tag{3.25a}$$

Substitution of (3.25a) into (3.14) shows that these solutions begin with an initial singularity at $x=0$. Near $x=0$, one finds

$$b=\text{const}\times e^{\kappa\chi}, \quad \chi\to-\infty, \tag{3.25b}$$

where

$$\kappa=\left|\frac{6}{\beta}\right|^{1/2}\left[\left[\bar{B}^2+\frac{|\beta|}{3}(1+\bar{E})\right]^{1/2}+\bar{B}\right]^{1/2}.$$

Since it begins at $\chi=-\infty$, this solution has no particle horizons. This ends the discussion of the case $\alpha=0$.

C. $\alpha<0$

In this case, for each value of α and β, there is one ACS whose late-time behavior may be obtained by substituting the expansion (3.1), for f, into (2.6). Keeping terms of order α, one solves for f_1, then substituting this back into (2.6) and keeping terms of order α^2, one solves for f_2, etc. The result is

$$f=(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}-\tfrac{1}{16}\alpha^{-1}\beta y^{-4/3}(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{7/4}+[(\tfrac{1}{8}A+\tfrac{3}{4}B)y^{-2/3}-\tfrac{1}{4}\tilde{\Lambda}](1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{3/4}$$
$$+(\tfrac{1}{16}A^2+\tfrac{3}{4}E+\tfrac{1}{4}A\tilde{\Lambda}y^{2/3}+\tfrac{1}{4}\tilde{\Lambda}^2y^{4/3})(1+Ay^{2/3}+\tilde{\Lambda}y^{4/3})^{-1/4}+O(\alpha^2)\,. \quad (3.26)$$

For open universes this solution approaches the classical solution for all $y\gg1$ and is therefore an ACS. For closed universes, it is shown in the Appendix that a solution to (2.6) exists for which $f=0$ at a value of y_0 which is within $O(k^2|\alpha|^{1/4}\tilde{\rho}_r{}^{-1/4}+|\tilde{\Lambda}|^{1/4})$ of the value of y at the classical momentum. Near $y=y_0$, this solution matches the solution in (3.26) to $O(\alpha(y_0-y)^{3/4})$. Thus, the classical solution is approached for all $y\gg1$ and the solution (3.26) is an ACS for closed universes as well.

If Eq. (2.6) is linearized about the classical solution (2.11), then it is found, using arguments similar to those near the beginning of Sec. III A, that it is very likely there are no other ACS.

Examination of (2.4) with $b'=0$ shows that there are no extrema for open universes. For closed universes, there are extrema only for values of b which are on the order of or larger than the value of b at the classical maximum. Thus the ACS do not bounce if $\alpha<0$, which implies that all ACS begin with $b=y=0$.

It is not hard to show, that if $\beta<0$, then all solutions to (2.6) which begin with $y=0$, also begin with $f=0$. The change of variables $f=y^{1/2}|v|^{3/2}$, $y=e^w$ allows (2.6) to be integrated once in the limit $f\to0$. The result is

$$\frac{dv}{dw}=\pm\tfrac{2}{3}v^{-1}[\sigma^2v^4+3Bv^2+cv-3(1+E)]^{1/2}\,, \quad (3.27)$$

where c is an arbitrary constant. This equation differs from its counterpart in Sec. II C of paper I by the terms with B and E. The term with E is never important, but that with B is important if $\sigma^2\simeq0$, i.e., if $\beta\simeq3\alpha$.

A closer examination of (3.27) shows that dv/dw is always imaginary near $v=0$. For $\sigma^2<0$ or $\sigma^2=0$, $B<0$, dv/dw is imaginary at large $|v|$, while for other values of σ^2 and B, dv/dw is real at large $|v|$. Further, if $\sigma^2>0$, $B<0$, and $3\alpha<\beta\leq3\alpha[1-B^2(1+E)^{-1}]$, then there are values of c for which dv/dw has four positive real roots. If $\sigma^2>0$, then for other values of B, α, and β, dv/dw has only two real roots: one positive and one negative. Since all these factors influence the behavior of solutions, the discussion is broken up into several subcases whose dependence on α, β, and B can be inferred from the above analysis.

1. $\beta<3\alpha$

In this subcase, $\sigma^2<0$ and dv/dw is imaginary for large and small values of $|v|$. A phase-plane analysis shows that solutions oscillate between either the two positive or the two negative roots of dv/dw. This is exactly the same behavior as found for solutions in Sec. II C of paper I. Thus these solutions begin with an initial singularity at $\chi=-\infty$ and they do not have particle horizons. Because there is no other small-y behavior for solutions of (2.6) in this subcase, one of these solutions is an ACS.

2. $\beta=3\alpha$

In this subcase, $\sigma^2=0$ and the term containing B in (3.27) is important. If $B<0$, then $k=1$ and dv/dw is again imaginary at large and small values of $|v|$. In this case, dv/dw has either two positive or two negative roots and solutions have the same initial behavior as solutions in the previous subcase.

If $B=0$, then $k=0$ and (3.27) reduces to Eq. (2.29) of paper I. Thus the initial behavior of solutions to (2.6) is given by Eq. (2.31) of paper I. These solutions begin with an initial singularity at $\chi=-\infty$ and do not have particle horizons. There are no other small-y behaviors, so one of these solutions is an ACS.

If $B>0$, then $k=-1$ and a phase-plane analysis of (3.27) shows that $|v|\to\infty$ in the limit $w\to-\infty$. In this limit, (3.27) can be integrated with the result that

$$f=(\tfrac{4}{3}B)^{3/4}y^{1/2}(-\ln y)^{3/2},\quad y\to0\,, \quad (3.28a)$$

$$b=\text{const}\times\exp\{-\exp[-12^{1/2}|\alpha|^{-1/4}B^{1/2}(\chi-\chi_1)]\}\,,\quad \chi\to-\infty\,, \quad (3.28b)$$

where χ_1 is an arbitrary constant. Substitution of (3.28) into (3.14) shows that these solutions begin with an initial singularity; they do not have particle horizons. There are no other small-y behaviors so one of these solutions is an ACS.

3. $3\alpha<\beta\leq3\alpha[1-B^2(1+E)^{-1}]$, $k=1$

This is the only subcase for $\alpha<0$ in which there is more than one type of small-y behavior which solutions may have. It is therefore the only subcase for which the early-time behavior of the ACS is uncertain. However, it is possible to determine what types of initial behaviors solutions may have and this is done next.

A phase-plane analysis of (3.27) shows that for all values of c, in this subcase, there are solutions which approach $|v|=\infty$ in the limit $w\to-\infty$. In this limit, (3.27) can be integrated with the result that

$$f=y^{1/2-\sigma}(C-\tfrac{9}{8}C^{-1/3}\sigma^{-2}By^{4\sigma/3}+C'y^{2\sigma}+\cdots)\,,\quad y\to0 \quad (3.29a)$$

$$b=\text{const}\times(\chi-\chi_0)^{1/2\sigma},\quad \chi\to\chi_0\,. \quad (3.29b)$$

Substitution of (3.29) into (3.14) shows that these solutions begin with an initial singularity. They have particle horizons. For $|c|<|c_-|$ and $|c|>|c_+|$, where

$$c_\pm=-6Bv_\pm-4\sigma^2 v_\pm{}^3\ ,$$
$$|v_\pm|=\{-\tfrac{1}{2}\sigma^{-2}B\pm[\tfrac{1}{4}\sigma^{-4}B^2-(1+E)\sigma^{-2}]^{1/2}\}^{1/2}\ , \quad (3.30)$$

and dv/dw in (3.27) has two real roots: one positive and one negative. A phase-plane analysis shows that all solutions have initial behaviors given by (3.29).

For $c=c_\pm$, dv/dw has a double root at $v=v_\pm$. For $c=c_+$, Eq. (3.27) can be integrated near $v=v_+$ with the result that

$$f=|v_+|^{3/2}y^{1/2}(1+c_2y^p+\cdots),\ \ y\to 0 \quad (3.31a)$$

$$b=\text{const}\times\exp(|\alpha|^{-1/4}|v_+|\chi),\ \ \chi\to-\infty\ , \quad (3.31b)$$

where c_2 is an arbitrary constant and

$$p=(\tfrac{8}{3})^{1/2}|v_+|^{-1}[\tfrac{1}{4}B^2-(1+E)\sigma^2]^{1/4}\ .$$

There is a one-parameter family of these solutions and substitution of (3.31) into (3.14) shows that they begin with an initial singularity. They do not have particle horizons. For $c=c_-$, dv/dw is imaginary near $v=v_-$, so no solutions approach $v=v_-$.

For $|c_-|<|c|<|c_+|$, dv/dw has four real roots, three of which have the same sign. A phase-plane analysis shows that along with the solutions in (3.29) and (3.31), there is a two-parameter family of solutions which oscillate between two positive or two negative values of v. As in Sec. III C 1, they begin with an initial singularity at $\chi=-\infty$ and do not have particle horizons.

For the special case $\beta=3\alpha[1-B^2(1+E)^{-1}]$, $k=1$, dv/dw has a triple root at $v_0=\pm(-B/2\sigma^2)^{1/2}$ for $c=-4Bv_0$. Integrating (3.27) near $v=v_0$, one finds

$$f=(-B/2\sigma^2)^{3/4}y^{1/2}[1+\tfrac{27}{8}\sigma^{-2}(\ln y+c_2)^{-2}],\ \ y\to 0 \quad (3.32a)$$

$$b=\text{const}\times\exp[|\alpha|^{-1/4}(-B/2\sigma^2)^{1/2}\chi],\ \ \chi\to-\infty\ , \quad (3.32b)$$

where c_2 is an arbitrary constant. Substitution of (3.32) into (3.14) shows that these solutions begin with an initial singularity. They do not have particle horizons. There are no other initial behaviors which solutions to (2.6) may have for this subcase.

4. $\beta>3\alpha$, $k=0,-1$ or $\beta>3\alpha[1-B^2(1+E)^{-1}]$, $k=1$

In this subcase, dv/dw in (3.27) has two real roots, one positive and one negative for all values of c. A phase-plane analysis shows that all solutions including the ACS have initial behaviors given by Eq. (3.29). Thus the ACS in this subcase has both particle horizons and an initial singularity. This ends the discussion of the case $\alpha<0$.

We have seen for general homogeneous and isotropic spacetimes containing classical radiation that quantum effects due to conformally invariant free fields can radically alter the behavior of the early universe. If $\alpha>0$ and $\beta>0$, the initial singularity predicted by classical general relativity can be removed, and if $\alpha>0$, $\alpha=0$ and $\beta<0$, or $\beta<3\alpha<0$ particle horizons can be removed. Further, if $\beta\geq 3\alpha>0$, both the initial singularity and particle horizons can be simultaneously removed. It is not yet clear whether they are removed for our universe because it is not known which ACS (if any) is the correct one for our universe. Nevertheless, it is clear that the universe may have had a very different beginning from that predicted by classical general relativity.

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APPENDIX

In this appendix, the solutions (3.2) and (3.26) for closed universes are matched to an extremum solution and it is shown that this solution obeys the second criterion for an ACS.

In terms of the variables x, $\bar{A}$, $\bar{B}$, and $\bar{\Lambda}$ defined in Sec. III B, Eqs. (3.2) and (3.28) are given by the single equation

$$\begin{aligned}f=&(1+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})^{3/4}+\tfrac{1}{16}\beta x^{-4/3}(1+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})^{7/4}\\&+\alpha[\tfrac{1}{4}\bar{\Lambda}+(\tfrac{3}{4}\alpha^{-1}\bar{B}-\tfrac{1}{8}\bar{A})x^{-2/3}](1+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})^{3/4}\\&-\alpha(\tfrac{1}{16}\bar{A}^2-\tfrac{3}{4}\alpha^{-1}E+\tfrac{1}{4}\bar{A}\,\bar{\Lambda}x^{2/3}+\tfrac{1}{4}\bar{\Lambda}^2x^{4/3})(1+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})^{-1/4}+O(\alpha^2)\ .\end{aligned} \quad (A1)$$

The classical solution $f=(1+\bar{A}x^{2/3}+\bar{\Lambda}x^{4/3})^{3/4}$ has a maximum in the (χ,b) plane, for closed universes, at $f=0$. This maximum occurs for a certain value of x, say $x=x_M$. Expanding (A1) about $x=x_M$ one finds

$$\begin{aligned}f=&(-\tfrac{2}{3}\bar{A}x_M{}^{-1/3}-\tfrac{4}{3}\bar{\Lambda}x_M{}^{1/3})^{3/4}(x_M-x)^{3/4}+O(\alpha(x_M-x)^{3/4})\\&-\alpha(-\tfrac{2}{3}\bar{A}x_M{}^{-1/3}-\tfrac{4}{3}\bar{\Lambda}x_M{}^{1/3})^{-1/4}(\tfrac{1}{16}\bar{A}^2-\tfrac{3}{4}\alpha^{-1}E+\tfrac{1}{4}\bar{A}\,\bar{\Lambda}x_M{}^{2/3}+\tfrac{1}{4}\bar{\Lambda}^2x_M{}^{4/3})(x_M-x)^{-1/4}+O(\alpha^2)\ .\end{aligned} \quad (A2)$$

Note that the expansion in powers of α breaks down for $(x_M - x) < |\alpha|$.

The goal is to find the extremum solution of (2.6) which corresponds to the solution (A1). From (3.8) it is seen that for a maximum an extremum solution has the form

$$f = (\tfrac{16}{3})^{3/8} x_0^{-1/4} [\,|\alpha|^{-1}(1+E+\bar{A}x_0^{2/3}+\bar{\Lambda}x_0^{4/3})]^{3/8}(x_0-x)^{3/4} + \mathrm{sgn}(\chi-\chi_0)D(x_0-x)^{5/4} + O((x_0-x)^{7/4}) , \qquad \text{(A3)}$$

where x_0 is the maximum value of x for this solution. Both (A2) and (A3) are asymptotic series. Therefore, the best one can hope to do is to find some x_0 such that the first few terms of (A2) and (A3) agree in some region where they are both valid approximations.

To find such an x_0, one can write $x_0 = x_M + \alpha x_1 + \alpha^2 x_2 + \cdots$, substitute this into (A3), and expand in powers of α. There are then two small quantities in (A2) and (A3): α and $(x_M - x)$. Equating (A2) and (A3) to $O[\alpha(x_M-x)^{3/4}]$, one finds

$$x_1 = -E\alpha^{-1}(\tfrac{2}{3}\bar{A}x_M^{-1/3} + \tfrac{4}{3}\bar{\Lambda}x_M^{1/3})^{-1} + \frac{\alpha}{|\alpha|}\frac{3}{16}x_M^{2/3}(\tfrac{2}{3}\bar{A}x_M^{-1/3} + \tfrac{4}{3}\bar{\Lambda}x_M^{1/3}) . \qquad \text{(A4)}$$

In a similar way, the quantities $x_2, x_3 \ldots$ may be chosen so that even better agreement between (A2) and (A3) is obtained.

One expects (A1) and (A2) to be valid for $|\alpha| \ll 1$ and $(x_M - x) \gg |\alpha|$. One also expects (A3) to be valid for $x_0^{-1}(x_0 - x) \ll 1$, so if $x_0 = x_M + \alpha x_1 + \cdots$, then both (A2) and (A3) should be valid in the region $|\alpha| \ll (x_M - x) \ll x_M$. This is the region in which they match up to $O(\alpha(x_M - x)^{3/4})$, so it is likely that they correspond to the same solution. Since $x_1 \sim (k^2\tilde{\rho}_r^{-1/4} + |\bar{\Lambda}|^{1/4}) \ll 1$, the maximum value of x for this solution is very close to that for the classical solution, so the second criterion for an ACS is satisfied.

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