

Quark-gluon-plasma evolution in scaling hydrodynamics

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The relation between the rapidity density of produced particles in ultrarelativistic nuclear collisions and the maximum proper energy density ϵ_0 is derived. The new scaling hydrodynamic equations of Bjorken, Kajantie, and McLerran are employed. The results exceed earlier estimates obtained with inside-outside cascade models and provide an independent estimate of ϵ_0 from collision data. We also derive a lower bound on ϵ_0 incorporating viscous heating and the first-order phase transition between the quark and hadronic phases. We infer that $\epsilon_0 > 2 \text{ GeV}/\text{fm}^3$ can indeed be reached in the collision of heavy nuclei at cosmic-ray energies.

I. INTRODUCTION

Hydrodynamic calculations¹⁻⁴ of hadronic processes have been reexamined⁵⁻⁸ recently in the light of the longitudinal growth of the reaction zone at high energies. In earlier work by Landau and others it was assumed that when two hadrons scatter a Lorentz-contracted fireball is formed in the c.m. frame. The energy density is therefore assumed to be enormous during the initial phase of hydrodynamic expansion,

$$\epsilon \sim \epsilon_L \gtrsim 2\gamma_{c.m.}^2 \epsilon_H, \quad (1)$$

where $\gamma_{c.m.} = \text{cosh } y_{c.m.}$ is the Lorentz factor and $\epsilon_H \sim 0.5 \text{ GeV}/\text{fm}^3$ is the proper energy density in a typical hadron. For nuclear collisions ϵ_H is replaced by the energy density in nuclei, $\epsilon_{\text{nuc}} \sim m_N \rho_0 \sim 0.15 \text{ GeV}/\text{fm}^3$. The right-hand side in Eq. (1) follows from assuming complete stopping⁹ in a Lorentz-contracted volume. If shock waves are produced,¹⁰ then ϵ_L could be ~ 2 times as large. To appreciate the scale, note that $\epsilon_L \sim 500 \text{ GeV}/\text{fm}^3$ for collisions involving laboratory energies $\sim 1 \text{ TeV}$.

Such enormous energy densities are well above all estimates^{11,12} for the energy density necessary to produce a quark-gluon plasma. Such estimates indicate that already for $\epsilon > \epsilon_{\text{SB}} \sim 2 \text{ GeV}/\text{fm}^3$, hadronic matter dissolves into an ideal Stefan-Boltzmann gas of quarks and gluons. In fact, deconfinement of hadrons (over the reaction volume) may occur when ϵ is as low as $\epsilon_H \sim 0.5 \text{ GeV}/\text{fm}^3$.

To reach ϵ_L the incident hadron must be able to lose all of its longitudinal momentum over an ever *decreasing* linear dimension. However, Landau and Pomeranchuk noted that the reaction time for hadronic processes must *increase* with increasing energy due to time dilation.¹³ This phenomenon is known as longitudinal growth, and experimental evidence for this effect has been seen in hadron-nucleus data.¹³ In simplest terms, to form a secondary hadron of dimensions $\tau_0 \sim 1 \text{ fm}$ requires a proper time τ_0 . In a frame where that secondary has rapidity y the formation time grows as

$$t(y) = \tau_0 \text{cosh } y. \quad (2)$$

In the parton picture τ_0 is replaced by the transverse

Compton wavelength $\tau_0 \sim 2/m_1$. Equation (2) means that the fastest secondaries can be produced only very far downstream from the reaction zone. This is the basis for the inside-outside cascade picture of hadronic processes.^{5,6,14}

We therefore see that (Landau's) longitudinal growth contradicts the assumptions of Landau hydrodynamics. To remove this inconsistency, Bjorken, Kajantie, and McLerran have proposed⁵⁻⁸ a new hydrodynamic picture that incorporates Eq. (2). Remarkably, the final numerical results turn out to be rather insensitive to the actual initial conditions.⁸ This is largely due to the fact that the breakup condition is always expressed in a Lorentz-covariant form; namely, when the proper energy density falls below a critical value $\epsilon_f \sim m_\pi^4$, the hydrodynamic expansion is terminated and the distribution of momenta is frozen out. Therefore, the original qualitative successes² of the Landau hydrodynamics are not altered by incorporating Eq. (2).

What is altered significantly, though, is the space-time picture of the reaction. In particular, the proper energy density in scaling hydrodynamics is always much smaller than ϵ_L as we show in the next section. Nevertheless, we find that the initial proper energy density can still exceed the critical values required to reach the quark-plasma phase. The nonlinear relation connecting the observed rapidity density dN/dy to ϵ_0 is derived in Sec. II. In addition to the ideal (Stefan-Boltzmann) equation of state, we consider a broader class of (Shuryak) equations of state. We then note some novel scaling of dN/dy with atomic number A that follow from ideal hydrodynamic expansion. In Sec. III, we consider entropy production due to viscous effects. A lower bound on ϵ_0 is derived by following the path of maximum entropy generation. We show that the earlier estimates of ϵ_0 by Bjorken⁵ coincide with the lower bounds appropriate for maximum entropy expansion. An upper bound on ϵ_0 is derived in Sec. IV, which incorporates a first-order phase transition from the plasma phase to the hadronic phase. The bag model is used to obtain qualitative insight. In Sec. V, numerical examples show that dN/dy depends only very weakly on the details of the phase transition and that viscosity and

phase transitions can lower the ideal-hydrodynamics estimate of ϵ_0 for fixed dN/dy by at most a factor ~ 2 .

II. SCALING HYDRODYNAMICS

The basic equations of scaling hydrodynamics⁶⁻⁸ are

$$\partial_\mu T^{\mu\nu} = \Sigma^\nu, \quad (3)$$

where

$$\Sigma^\nu = \epsilon_0(y) u^\nu \delta(\tau - \tau_0) \quad (4)$$

is the source function in terms of the variables $\tau^2 = t^2 - z^2$, $y = \frac{1}{2} \ln(t + z/t - z)$, and $u^\mu = x^\mu/\tau$ is the collective flow velocity. The form of Σ^ν is chosen to incorporate longitudinal growth in a natural way. The partons come on-shell at proper time $\tau_0 \sim 1$ fm/c and are assumed to evolve thereafter according to ideal hydrodynamics. Integrating Eq. (3) around $\tau = \tau_0$ noting that $T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - p g_{\mu\nu}$, we see that

$$\epsilon(\tau_0, y) = \epsilon_0(y) \quad (5)$$

is the initial proper energy density of the fluid element with collective flow rapidity y . Using the inside-outside cascade picture, Bjorken⁵ estimated that

$$\epsilon_0(y) = \langle m_\perp \rangle \frac{1}{\tau_0 A_\perp} \frac{dN}{dy}, \quad (6)$$

where A_\perp is the transverse area of the reaction, $\langle m_\perp \rangle \sim 0.3-0.4$ GeV is the typical transverse mass of produced particles, and dN/dy is the rapidity density of produced particles. For pp collisions at collider energies, Eq. (6) predicts that $\epsilon_0 < \epsilon_H \ll \epsilon_L$.

The point of this section is to note that if hydrodynamics, Eq. (3), is valid for the final expansion phase, then $\epsilon_0(y)$ is related to the final observed dN/dy and the equation of state as

$$\epsilon_0(y) = \left[\frac{k}{\tau_0 A_\perp} \frac{dN}{dy} \right]^{1+c_0^2}, \quad (7)$$

where k is a constant that we determine below and c_0 is the speed of sound. The difference between Eqs. (6) and (7) comes about by taking into account the work done by the fluid during expansion.

To derive Eq. (7), we recall that Eq. (3) implies that the entropy current $s_\mu = \sigma u_\mu$ obeys^{7,8}

$$T \partial_\mu s^\mu = u^\mu \Sigma_\mu, \quad (8)$$

where T is the local temperature related to σ via $T\sigma = \epsilon + p$. Neglecting transverse flow⁸ $u^\mu \partial_\mu \sigma = \partial\sigma/\partial\tau$, $\partial_\mu u^\mu = 1/\tau$, and $u^\mu u_\mu = 1$. Therefore, in this case Eq. (8) reduces to ($\sigma = dp/dT$)

$$\begin{aligned} \frac{T}{\tau} \frac{d}{d\tau} (\tau\sigma) &= \frac{d}{d\tau} (T\sigma - p) + \frac{T\sigma}{\tau} \\ &= \epsilon_0(y) \delta(\tau - \tau_0). \end{aligned} \quad (9)$$

The solution of Eq. (9) is clearly⁵⁻⁸

$$\sigma(\tau, y) = \sigma_0(y) \tau_0/\tau, \quad (10)$$

independent of the equation of state. The initial entropy density is

$$\sigma_0(y) = [\epsilon_0(y) + p(\tau_0, y)]/T(\tau_0, y). \quad (11)$$

We now recall the relation between the proper entropy density σ and the proper density of quanta n for an ultrarelativistic ($T \gg m$) gas with zero chemical potential

$$\sigma = \xi n, \quad (12)$$

where $\xi = 3.6, 4.0,$ and 4.2 for Bose, Boltzmann, and Fermi gases, respectively. For an SU(3) up-down glue plasma, $\epsilon = 12.2 T^4$, $p = \epsilon/3$, $n = 4.14 T^3$, and $\sigma = 16.2 T^3$, so that $\xi = 3.9$. For illustration, note that at $T = 200$ MeV, $\epsilon = 2.6$ GeV/fm³, and there are $n = 4.3$ quanta per fm³.

From Eqs. (10) and (12) we see that σ and hence n decrease⁵ as $1/\tau$. The hydrodynamic expansion continues until the energy density falls below a critical value $\epsilon_f \sim m_\pi^4$. The breakup surface on which $\epsilon = \epsilon_f$ follows from the solution of Eq. (3). In general, that surface must be solved for numerically. However, in the scaling regime, where $d\epsilon/dy$ can be neglected and the collective velocity can be well approximated by $u_\mu = x_\mu/\tau$, Eq. (3) simplifies⁵ to

$$\tau \frac{\partial \epsilon}{\partial \tau} + (\epsilon + p) = \tau_0 \epsilon_0 \delta(\tau - \tau_0). \quad (13)$$

For $p = c_0^2 \epsilon$,

$$\epsilon(\tau) = \epsilon_0 (\tau_0/\tau)^{1+c_0^2}. \quad (14)$$

From Eq. (14) we see that the breakup surface is simply a fixed proper time $\tau = \tau_f$. The volume element on that surface is just $\tau_f dy d^2x_\perp$, so that

$$\frac{dN}{dy} = \int d^2x_\perp \tau_f n(\tau_f, y, x_\perp). \quad (15)$$

Combining Eqs. (10), (12), and (15), we obtain

$$\frac{dN}{dy} = \xi^{-1} A_\perp \tau_0 \sigma(\tau_0, y), \quad (16)$$

which depends on the *initial* entropy density. For a Stefan-Boltzmann gas, $\sigma = 4\epsilon/3T$, Eq. (16) leads to

$$\epsilon_0 = \frac{3}{4} \xi T_0 \frac{1}{A_\perp \tau_0} \frac{dN}{dy}. \quad (17)$$

This shows explicitly that the transverse mass in Eq. (6) is replaced in hydrodynamics by $m_{\text{eff}} \sim 3T_0$. Since T_0 depends on ϵ_0 via $\epsilon = K_{\text{SB}} T^4$, we see that Eq. (17) is a special case of Eq. (7) with $k = 3\xi/(4K_{\text{SB}}^{1/4})$ and $c_0^2 = \frac{1}{3}$.

For a more general equation of state, we consider a Shuryak resonance gas characterized by a mass spectrum $\rho(m) \propto m^a$. The thermodynamic relations are²

$$\begin{aligned}
p_s &= c_0^2 \epsilon_s, \\
\epsilon_s &= \lambda m_\pi^4 \left[\frac{T}{m_\pi} \right]^{1+c_0^{-2}}, \\
\sigma_s &= \lambda m_\pi^3 (1+c_0^2) \left[\frac{\epsilon_s}{\lambda m_\pi^4} \right]^{1/(1+c_0^2)}, \\
\sigma_s &= \xi(c_0^2) n_s, \\
\xi(c_0^2) &= (1+c_0^{-2}) \frac{\zeta(1+c_0^{-2})}{\zeta(c_0^{-2})},
\end{aligned} \tag{18}$$

where $\zeta(z) = \sum n^{-z}$ for bosons, c_0 is the velocity of sound related to a by $c_0^{-2} = a + 4$, and λ is an arbitrary constant. Note that Eq. (18) reduces to the Stefan-Boltzmann form for $c_0^2 = \frac{1}{3}$ and $\lambda = K_{SB}$. When dealing with a resonance gas it is important to note that dN/dy in Eqs. (15) and (16) is not equal to the final pion multiplicity density. It is the total number of pions plus heavy resonances per unit rapidity at the breakup time. To convert dN/dy into dN_π/dy we must estimate how many pions will emerge after all resonances have decayed. The average energy per resonance at breakup ($T_f \sim m_\pi$) is

$$\begin{aligned}
\bar{E} &= \epsilon_s(m_\pi) / n_s(m_\pi) = m_\pi \xi(c_0^2) / (1+c_0^2) \\
&\approx m_\pi / c_0^2.
\end{aligned}$$

Estimating the average kinetic energy per resonance to be

$$\left(\frac{3}{2} - 3\right) T_f \sim 2m_\pi$$

at breakup, we see that the average resonance mass

$$\bar{m} \approx \bar{E} - 2m_\pi \approx m_\pi (c_0^{-2} - 2).$$

The average number of pions with kinetic energies $\sim (1-2)m_\pi$ resulting from the decay of these resonances is

$$\bar{n}_\pi \sim \bar{m} / 2m_\pi \sim 1/2c_0^2$$

for $c_0^2 \ll 1$. Consequently, $dN_\pi/dy \approx (1/2c_0^2) dN/dy$ with uncertainties on the order of a factor of 2. Since Eqs. (16) and (18) show that

$$dN/dy \propto \xi^{-1}(c_0^2) \propto c_0^2,$$

we find that for $c_0^2 \ll 1$

$$\frac{dN_\pi}{dy} = \xi_\pi^{-1} A_1 \tau_0 \sigma(\tau_0, y), \tag{19}$$

where ξ_π is independent of c_0^2 to lowest order in c_0^2 . To fix ξ_π , note that for an ideal pion gas ($c_0^2 = \frac{1}{3}$) $\xi_\pi \approx 4$. Therefore, the only role of the equation of state is to establish the relationship between $\sigma(\tau_0)$ and $\epsilon(\tau_0)$.

For the Shuryak gas we obtain in this way

$$\epsilon_0 = \epsilon_\pi \left[\frac{1}{\epsilon_\pi} \frac{3m_\pi}{\tau_0 A_1} \frac{dN_\pi}{dy} \right]^{1+c_0^2}, \tag{20}$$

where we used $\xi_\pi^{-1}(1+c_0^2) \approx 3$ to lowest order in $c_0^2 - \frac{1}{3}$ and defined $\epsilon_\pi \equiv \epsilon_s(T = m_\pi) = \lambda m_\pi^4$. This completes the derivation of Eq. (7).

An interesting consequence of Eq. (20) is the dependence of dN/dy on the atomic number A in nuclear col-

lisions. If the initial energy density depends on A as $\epsilon \propto A^\delta$, then

$$\frac{dN}{dy} \propto A^{[2/3 + \delta/(1+c_0^2)]}. \tag{21}$$

The power δ depends, of course, on the plasma-production mechanism. In many models, such as the additive quark model¹² or Low-Nussinov model,¹⁵ $\delta = \frac{1}{3}$ and consequently the power in Eq. (21) varies between $\frac{5}{6}$ and 1. Hydrodynamic flow thus tends to lower the A dependence of dN/dy because the entropy density rather than the energy density determines the multiplicity.

Up to now we have treated τ_0 as an independent parameter $\tau_0 \sim 1$ fm/c. However, for production of structureless partons longitudinal growth is controlled by the transverse Compton wavelength $\tau_0 \sim 2/p_\perp$. For a given initial temperature T_0 , $\langle p_\perp \rangle \sim (2-4)T_0$, and we could expect that $\tau_0 \sim T_0^{-1}$. Such a dependence of τ_0 on T_0 would lead via Eq. (17) to a highly nonlinear relation

$$\epsilon_0 \propto \left[\frac{dN}{dy} \right]^{(1+c_0^2)/(1-c_0^2)}.$$

(For $c_0^2 = \frac{1}{3}$ the relation would be quadratic.)

However, the use of $\tau_0 \sim T_0^{-1}$ in Eq. (17) would not be correct, because in colliding nuclei the relevant τ_0 cannot be smaller than the thickness of the parton cloud around the nuclei. Due to the wee ($1/x$) partons the limiting thickness¹⁴ of any hadronic system at very high energies is ~ 1 fm/c. Hence, it takes $\Delta t > 1$ fm/c for the colliding nuclei to pass through each other in any frame. Even though higher- p_\perp partons can be created on a faster time scale, their production times are distributed over a finite time interval Δt . In thermodynamic terms, entropy continues to be produced at least up to $\tau \sim \Delta t$. Since in Eq. (16) we used entropy conservation to get from τ_f to τ_0 , we see that $\tau_0 > \Delta t \sim 1$ fm/c in that relation.

III. MAXIMUM ENTROPY PRODUCTION IN VISCOUS EXPANSION

In Sec. II, we solved for the ideal fluid dynamics relation between ϵ_0 and dN/dy . The central assumption was that the total entropy per unit rapidity $dS/dy = A_\perp \tau \sigma(\tau, y)$ is a constant of motion. However, there are two obvious sources of additional entropy: (a) viscous heating and (b) the phase transition from the quark to the hadronic phase. Therefore, we rewrite Eq. (19) as

$$\xi_\pi \frac{dN_\pi}{dy} = \frac{dS_0}{dy} + \frac{dS_\eta}{dy} + \frac{dS_{tr}}{dy}, \tag{22}$$

where S_η is the total entropy produced via viscous heating and S_{tr} is the total entropy produced in the phase transition. In this section, we derive an upper bound on S .

Suppose at time τ_0 we start with a plasma drop in equilibrium with total energy E in a volume V_0 ($\epsilon_0 = E/V_0$). As the system expands, the state of maximum entropy is obtained in a volume V if global thermal equilibrium is maintained, and no energy is lost from the system. The maximum increase of entropy on expansion is therefore

$$\Delta S \leq \sigma V - \sigma_0 V_0 = S_0 \left[\frac{\sigma V}{\sigma_0 V_0} - 1 \right], \quad (23)$$

where $S_0 = \sigma_0 V_0$ is the initial entropy. By energy conservation $V/V_0 = \epsilon_0/\epsilon$, and, hence, Eq. (23) reduces to

$$S_0 \leq S \leq \sigma \frac{\epsilon_0}{\epsilon} V_0. \quad (24)$$

The system expands until the mean free paths of hadrons become comparable to the size of the system. At that point $\epsilon = \epsilon_f$, $T = T_f \sim 140$ MeV, and the system disintegrates.^{1,2} For a Shuryak gas, $\sigma_f/\epsilon_f = (1+c_0^2)/T_f$, and, therefore, Eqs. (22) and (24) give the bound

$$\epsilon_{\text{hyd}} \geq \epsilon_0 \geq \frac{m_{\text{eff}}}{\tau_0 A} \frac{dN_\pi}{dy}, \quad (25)$$

where

$$m_{\text{eff}} = \frac{\xi_\pi T_f}{1+c_0^2} \sim 3m_\pi, \quad (26)$$

and ϵ_{hyd} is given by Eq. (20). We have thus shown that Bjorken's estimate, Eq. (6), is a lower bound on ϵ_0 and follows in thermodynamics if the system follows the path of maximum entropy expansion. That path requires maintenance of global equilibrium at constant total energy. Of course, Eq. (6) also follows in the extreme nonequilibrium limit appropriate for the expansion of a noninteracting gas satisfying the scaling hypothesis.⁵

To understand better the bounds in Eq. (25) we contrast hydrodynamic expansion to maximum entropy expansion. For the scaling initial conditions, Eq. (3), matter is formed along the hyperbola $\tau = \tau_0$. Hydrodynamics assumes that this matter behaves as a continuous fluid capable of maintaining local thermal equilibrium between fluid cells with a large rapidity gradient. Consider now a fluid element with rapidity between y and $y + \delta y$. In the rest frame of that element, the initial volume is $\delta V_0 = A_1 \tau_0 \delta y$. In the scaling regime the volume expands to $\delta V = A_1 t \delta y$ by time t , in that frame. After a time increment Δt , the volume expands by $\Delta V = A_1 \Delta t \delta y$. The amount of work done in the expansion is $p \Delta V$ to first order in Δt . Therefore, energy conservation implies that

$$\epsilon \delta V = (\epsilon + \Delta \epsilon)(\delta V + \Delta V) + p \Delta V + O(\Delta t^2), \quad (27)$$

which leads to the hydrodynamic equation (13)

$$t \frac{\Delta \epsilon}{\Delta t} + (\epsilon + p) = 0. \quad (28)$$

Equation (28) shows that ϵ decreases as $(\tau_0/\tau)^{1+c_0^2}$. The energy density decreases faster than $1/\tau$, as appropriate for a noninteracting gas, because of work required to push neighboring fluid cells aside.

Now suppose that instead of a continuous fluid a series of fireballs are formed along $\tau = \tau_0$. This could arise due to unusual formation mechanisms or the inability of adjacent fluid elements to remain in thermal contact. Since these fireballs are all receding from one another in any frame, each expands independent of the other. No work is performed so that energy is conserved in each fireball. It

is of course possible for each fireball to expand hydrodynamically. However, the maximum entropy is generated if no collective flow velocities develop. This is the scenario that leads to the lower bound in Eq. (25).

It is important to note that if the path of maximum entropy is followed, then the final entropy $S_f = \sigma_f V_f = (\sigma_f/\epsilon_f)(\epsilon_0 V_0)$, is *independent* of any phase transitions during expansion. It depends only on the total energy of the fireball and the final freeze-out temperature. Therefore, the lower bound in Eq. (25) is very general in the scaling limit.

Finally, we note that if the scaling hypothesis is removed, then the initial state could be drastically different as, for example, Landau's fireball in Eq. (1). The simple connection of ϵ_0 with dN/dy would be lost. It is the assumed frame invariance of the production and expansion process that leads to the simple bounds in Eq. (25). This is a fundamental difference between Bjorken and Landau hydrodynamics.

IV. ISENTROPIC PATH THROUGH THE PHASE TRANSITIONS

During the expansion phase there may be a first-order transition between the quark and hadron phases.^{11,12} This could lead to entropy production. However, as noted in Sec. III, there is an upper bound on the maximum entropy that can be produced. This gives a lower bound Eq. (25) on ϵ_0 for a fixed dN/dy . In this section we derive an upper bound on ϵ_0 , which takes into account a possible first-order transition.

First we show that there exists an isentropic path through the transition. For illustration we consider the bag model^{16,17} equation of state. In the quark phase, we assume that

$$\begin{aligned} \epsilon_q &= KT^4 + B, \\ p_q &= \frac{1}{3}KT^4 - B, \\ \sigma_q &= \frac{4}{3}KT^3. \end{aligned} \quad (29)$$

For the hadronic phase we use Eq. (18). The critical temperature T_c is a solution of $p_q = p_s$. For example, for $c_0^2 = \frac{1}{3}$, $T_c = [3B/(K-\lambda)]^{1/4}$. Define the critical parameters $\epsilon_H = \epsilon_s(T_c)$, $\sigma_H = \sigma_s(T_c)$, $\epsilon_Q = \epsilon_q(T_c)$, $\sigma_Q = \sigma_q(T_c)$. Note that $\epsilon_Q - \epsilon_H$ is the latent heat per unit volume. For $\epsilon_H \leq \epsilon \leq \epsilon_Q$ the pressure and temperature are independent of ϵ .

Consider again a fluid element in the rapidity interval y to $y + \delta y$ with an initial proper energy density $\epsilon_0 > \epsilon_Q$. In the rest frame of that element the fluid expands from the initial volume $\delta V_0 = A_1 \tau_0 \delta y$ until the energy density is reduced to ϵ_Q . At that point the volume has increased to δV_Q . The total entropy in that element at that point is then $\delta S_Q = \sigma_Q \delta V_Q = \sigma_0 \delta V_0$. As the system expands further, ϵ decreases while the temperature and pressure remain constant at $T = T_c, p = p_c$. The expansion continues until ϵ is reduced to ϵ_H and $\delta V = \delta V_H$. During this expansion an amount of work $\delta W = p_c(\delta V_H - \delta V_Q)$ must be performed to counteract the pressure exerted by neighboring fluid elements. Therefore, energy conservation im-

plies that $\epsilon_Q \delta V_Q = \epsilon_H \delta V_H + \delta W$ and therefore

$$(\epsilon_Q + p_c) \delta V_Q = (\epsilon_H + p_c) \delta V_H. \quad (30)$$

The thermodynamic relation $\sigma_Q T_c = \epsilon_Q + p_c$ and $\sigma_H T_c = \epsilon_H + p_c$ then shows that the total entropy of the cell remains constant $\delta S_Q = \delta S_H$ through the transition. Clearly this result is independent of any particular functional form of the equation of state.

Therefore, if there are no viscous effects before or after the transition, then entropy can be conserved at all times, i.e., there exists an isentropic path through the phase transition. For this scenario to hold the characteristic time τ_c for the transition must be short in comparison to the time required to expand from δV_Q to δV_H . If τ_c is long, then supercooling of the plasma should occur. Such supercooling could be followed by explosive growth of hadronic bubbles leading to additional entropy.¹⁸

Following the isentropic path starting at $\epsilon_0 > \epsilon_Q$ leads to Eq. (19) with

$$\sigma_0 = \frac{4}{3} K^{1/4} (\epsilon_0 - B)^{3/4}. \quad (31)$$

For an initial energy density in the mixed phase ($\epsilon_H \leq \epsilon \leq \epsilon_Q$)

$$\sigma_0 = (\epsilon_0 + c_0^2 \epsilon_H) / T_c, \quad (32)$$

where we used that the pressure $p = c_0^2 \epsilon_H$ and $T = T_c$ are constant in the mixed phase. In the hadronic phase $\epsilon < \epsilon_H$, Eq. (18) applies.

Inserting these forms of the entropy density into Eq. (19) gives

$$\epsilon_0 = B + \left[\frac{k}{\tau_0 A_{\perp}} \frac{dN_{\pi}}{dy} \right]^{4/3} \quad \text{for } \epsilon_0 \geq \epsilon_Q, \quad (33)$$

where $k = 3\xi_{\pi} / (4K^{1/4})$ and $B = (\epsilon_Q - 3c_0^2 \epsilon_H) / 4$ as follows from $p_s = p_Q$. For the mixed phase we obtain

$$\epsilon_0 = \frac{m_c}{\tau_0 A_{\perp}} \frac{dN_{\pi}}{dy} - c_0^2 \epsilon_H \quad \text{for } \epsilon_H \leq \epsilon_0 \leq \epsilon_Q, \quad (34)$$

with a critical mass parameter

$$m_c = \xi_{\pi} T_c, \quad (35)$$

where $T_c = [3(\epsilon_Q + c_0^2 \epsilon_H) / 4K]^{1/4}$. Note that in the mixed phase, we have again a linear relationship between ϵ_0 and dN/dy . Finally, for the hadronic phase we must use Eq. (20). Equations (20), (33), and (34) give the upper bound on ϵ_0 as obtained by following the isentropic path through the transition.

We can also invert Eq. (34) to find the minimum observed pion rapidity density corresponding to $\epsilon = \epsilon_H$ or $\epsilon = \epsilon_Q$. For central $A+B$ collisions with $A < B$, $A_{\perp} \approx \pi r_0^2 A^{2/3}$, $r_0 \approx 1.2$ fm. Therefore, the minimum multiplicity density that corresponds to $\epsilon_0 > \epsilon_H$ is

$$\left[\frac{dN}{dy} \right]_H = A^{2/3} \frac{(1+c_0^2) \epsilon_H}{\xi_{\pi} \epsilon_c}, \quad (36)$$

where $\epsilon_c = T_c / (\pi \tau_0 r_0^2) \sim 0.03$ GeV/fm³. Since we expect $\epsilon_H \sim 0.3-0.5$ GeV/fm³, $(dN/dy)_H \sim (3-6)A^{2/3}$. The minimum multiplicity density corresponding to $\epsilon_0 > \epsilon_Q$,

on the other hand, is

$$\left[\frac{dN}{dy} \right]_Q = A^{2/3} \frac{\epsilon_Q + c_0^2 \epsilon_H}{\xi_{\pi} \epsilon_c} \sim \frac{\epsilon_Q}{\epsilon_H} \left[\frac{dN}{dy} \right]_H. \quad (37)$$

Therefore $(dN/dy)_Q \sim (6-24)A^{2/3}$ may be required for events involving the pure plasma phase. Detailed numerical results are given in the next section.

V. RESULTS

To illustrate the insensitivity of the results to the details of the phase transition, we consider two forms of the equation of state as shown in Fig. 1 that span a physically reasonable range of possibilities.^{11,12} Curve 1 corresponds to a strong first-order transition at $T_c = 200$ MeV with latent heat $\Delta\epsilon \approx 2.6$ GeV/fm³ and parameters $K = 12.2$, $B = 0.74$ GeV/fm³, $c_0^2 = 0.16$, and $\epsilon_H, \epsilon_Q = 0.7, 3.3$ GeV/fm³. Curve 2 corresponds to a weak first-order transition at $T_c = 140$ MeV with $\Delta\epsilon \approx 0.2$ GeV/fm³ and $K = 12.2$, $B = 0.05$ GeV/fm³, $c_0^2 = \frac{1}{3}$, and $\epsilon_H, \epsilon_Q = 0.45, 0.67$ GeV/fm³. Notice that the first bag constant is close to Shuryak's estimate, while the second is close to the MIT value.

For these two equations of state curves 1 and 2 in Fig. 2 show the relation [Eqs. (20), (33), and (34)] between the initial energy density ϵ_0 and the reduced, total pion rapidity density $A^{-2/3} dN/d\eta$. We used the pseudorapidity variable $\eta = -\ln \tan \theta_{\text{lab}}/2$ instead of y to make closer contact with experiment. As emphasized before, the crucial parameter setting the absolute scale of ϵ_0 is the proper time τ_0 marking the onset of hydrodynamic flow. We set $\tau_0 = 1$ fm/c in this example. For $\tau_0 = \frac{1}{2}$ fm/c the slope of all curves would increase by a factor ~ 2 . For $\tau_0 = 2$ fm/c the slopes would decrease by a factor ~ 2 . This factor is the intrinsic theoretical uncertainty in the conversion between $dN/d\eta$ and ϵ_0 .

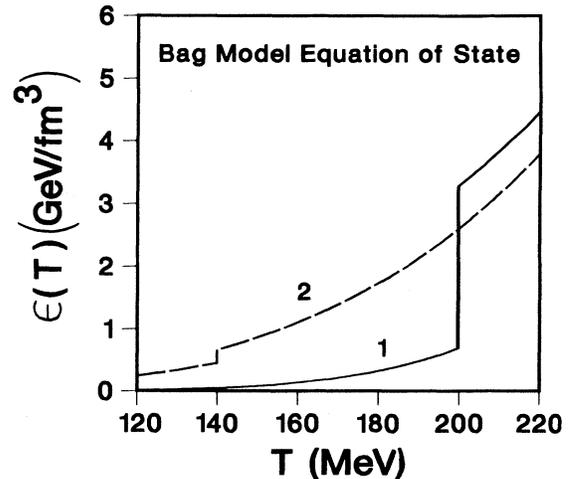


FIG. 1. Energy density versus temperature for two sets of bag-model parameters as described in text. Curves 1 and 2 correspond to strong and weak first-order transitions between the hadronic and quark-gluon plasma phases.

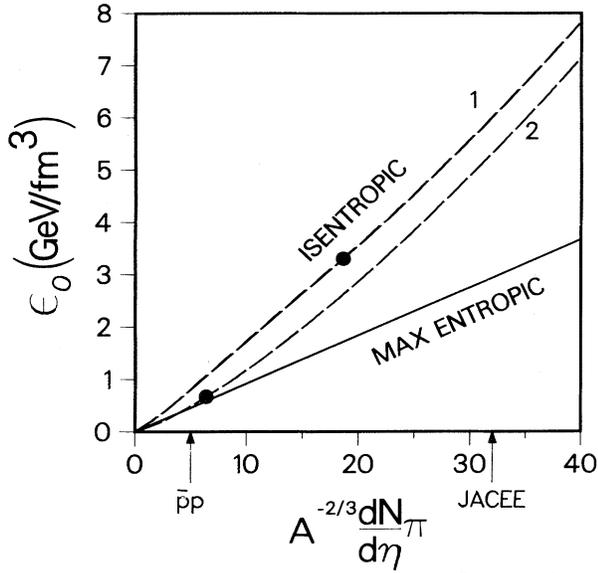


FIG. 2. The energy density ϵ_0 at the onset of scaling hydrodynamics versus total pion pseudorapidity density reduced by $A^{2/3}$, where A is the atomic number of the smaller nucleus in central nuclear collisions. Dashed curves 1 and 2 correspond to scaling hydrodynamic expansion with the equations of state shown in Fig. 1. The solid dots locate $\epsilon = \epsilon_0$ for each. The solid curve corresponds to expansion along the path of maximum entropy production. It also coincides with Bjorken's relation with $m_{\text{eff}} = 0.4$ GeV. The average reduced density in $\bar{p}p$ collider events (Ref. 19) and in the Si + Ag JACEE event (Ref. 21) are also shown.

In addition to τ_0 we must specify the transverse area A_{\perp} of the reaction zone. Assuming central collisions between nuclei A and B with $A < B$, we again take $A_{\perp} = \pi r_0^2 A^{2/3}$, $r_0 = 1.18$ fm. The reduced rapidity density $A^{-2/3} dN/d\eta$, therefore, removes the trivial geometric enhancement of $dN/d\eta$ for heavier nuclei and measures the rapidity density per unit area. For peripheral collisions the corresponding ϵ_0 value is a lower bound.

The most remarkable feature seen in Fig. 2 is that curves 1 and 2 are so close to each other. The solid dot marking the point $\epsilon = \epsilon_0$ for both curves shows that in spite of the large difference between the two equations of state the relation between ϵ_0 and $dN/d\eta$ is controlled mainly by plasma branch for $\epsilon > \epsilon_0$ given by Eq. (29). To a good approximation the curves are shifted vertically from one another by the difference of the bag constants for each.

Also shown in Fig. 2 is the curve corresponding to evolution along the path of maximum entropy production, Eq. (25). We took $m_{\text{eff}} = 0.4$ GeV in accord with Eq. (26). As noted before, with this effective mass that curve also coincides with Bjorken's estimate, Eq. (6). Viscous effects and entropy nonconserving processes associated with the phase transition can therefore lower the estimate of ϵ_0 for a fixed $dN/d\eta$ by at most a factor ~ 2 . Most likely, the entropy generation in the final state expansion will in fact be much smaller than the maximum value, and the true ϵ_0 would be closer to curves 1 and 2.

Finally, we have indicated in Fig. 2 the observed reduced rapidity density in typical $\bar{p}p$ reactions at collider energies.¹⁹ We assume that unobserved neutral pions account for $\sim \frac{1}{3}$ of the total rapidity density. We see from Fig. 2 that typical $\bar{p}p$ events lead to very low initial energy densities, $\epsilon_0 \sim 0.5$ GeV/fm³. However, in about 5% of the events,¹⁹ $dN/dy \sim 18$ is reached corresponding to $\epsilon_0 \sim 1.5\text{--}3$ GeV/fm³. For nuclear collisions, only limited cosmic-ray data are available.²⁰ The most spectacular reaction observed so far is the JACEE event²¹ Si + Ag at ~ 4 TeV/nucleon. The charged-particle multiplicity in the central region is $dN_{\text{ch}}/d\eta \approx 200$. Taking $A = 28$ for this reaction, the reduced total pion density is then ~ 32 . Simple model estimates²² indicate that such high multiplicities are to be expected in central Si + Ag collisions. Multiplicity densities in central U + U collisions are thus expected to reach reduced densities ~ 60 , assuming $A^{1/3}$ scaling. Reading off the conservative maximum-entropic curve, heavy nuclear collisions would then lead to initial energy densities $\epsilon_0 > 3$ GeV/fm³—well into the plasma phase. If scaling hydrodynamics applies, then energy densities as high as $\epsilon_0 \sim 10$ GeV/fm³ could be reached in central U + U collisions at JACEE energies. This value represents an optimistic upper bound achievable in nuclear collisions.

These estimates for the energy densities achievable in nuclear collisions are subject, however, to several caveats. First, they are only applicable to the central rapidity region where the baryon density is low. In the fragmentation regions, the relation between entropy density and pion multiplicity is more complicated because only a fraction of the total entropy is carried by mesonic degrees of freedom. That fraction is also a sensitive function of the temperature and varies during the expansion phase. Since each fragmentation region extends over a rapidity interval $\Delta y_F \approx \ln(4R/\tau_0) \lesssim 3$, the total rapidity gap necessary to form a central region is $2\Delta y_F \sim 6$. Therefore, Fig. 2 can be applied only for reactions above several hundred GeV per nucleon.

Second, we have assumed via Eq. (4) that the energy deposition in the central region occurs instantaneously at proper time τ_0 . However, as noted before, there exists a finite time interval^{5,14} $\Delta\tau \sim 1$ fm/c, asymptotically for the nuclei to pass through one another. This has the effect of replacing $\epsilon_0 \delta(\tau - \tau_0)$ in Eq. (4) by $\epsilon_0 / \Delta\tau \theta(\Delta\tau - |\tau - \tau_0|)$. The solution of Eq. (3) with such a source was found in Ref. 8 [see Eq. (53)]. The maximum energy density for $\Delta\tau \sim 1$ fm/c was found to be reduced by $\sim 30\%$ from what it would be if $\Delta\tau = 0$. This finite time dilution factor therefore has the effect of lowering the slopes of curves 1 and 2 by $\sim 30\%$. Nevertheless, the resulting isentropic curves still lie significantly above Bjorken's estimate. This shows, in particular, the importance of choosing the source function in hydrodynamical calculation self-consistently. Up to now, scaling hydrodynamic source terms were estimated^{6–8} using Bjorken's formula Eq. (6), relating ϵ_0 linearly with the final rapidity density. However, as stressed here, dN/dy is linearly related only to the entropy density.^{1,2} A higher initial value of ϵ_0 must be chosen, as shown in Fig. 2, to allow for work done on expansion. Such a self-consistent calculation would result,

according to our analysis, in approximately a factor of 2 higher ϵ_0 in the central region than computed in Ref. 8. Self-consistency in the fragmentation regions would also be important, especially since the scaling relationship, Eq. (2), between longitudinal coordinate and rapidity breaks down and even the linear relation between ρ and σ no longer holds.

A third caveat concerns the applicability of ideal (non-viscous) hydrodynamics. A necessary condition for the validity of such an approach is that gradients of field quantities such as $\epsilon(x)$ be small compared to characteristic collision rates. In the scaling regime the rapid longitudinal expansion causes ϵ to change significantly on a time scale $\delta t \sim (\partial \ln \epsilon / \partial \tau)^{-1} \propto \tau$. The collision rate in the plasma is, on the other hand, controlled by the local temperature¹⁷ $\gamma \propto \alpha^2 T(\tau)$, where α is an effective parton coupling constant. If local equilibrium is maintained, then³ $T(\tau) = T_0(\tau_0/\tau)^{1/3}$. Viscous effects can be neglected only if $\gamma \delta t \sim \alpha^2 T_0 \tau_0 > 1$. Therefore, ideal hydrodynamics can be applied only if the initial energy density $\epsilon_0 \propto T_0^4$ is sufficiently large. Any estimate of that critical energy density is, however, very uncertain at present and requires further development of a transport theory of quark-gluon plasmas.²³ It is likely to be on the order of $\epsilon_0 \sim 1 \text{ GeV}/\text{fm}^3$.

Finally, we note that we have neglected transverse

motion. In the central region, this is justified because the entropy per unit rapidity is still conserved, and we need only to interpret A_\perp as the initial transverse area of the plasma. The only effect of transverse flow is then to redistribute the transverse momenta of pions.²⁴ In the fragmentation regions, transverse expansion converts entropy from mesonic to baryon degrees of freedom, thereby lowering the pion multiplicity. For small A reactions, transverse expansion can have, on the other hand, an indirect influence even in the central regions. If the transverse dimension R becomes comparable to the parton mean free paths, then ideal hydrodynamic flow cannot be justified. Therefore, the isentropic curves in Fig. 2 are more relevant for heavy nuclear collisions than for hadron-hadron or hadron-nucleus collisions.

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