# Helicity selection rules in the Born approximation to Compton scattering of massive spin-1 particles on massless fermions

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The helicity amplitudes for the reaction $G+q\rightarrow\gamma+q$, where the gluon $G$ and photon have virtual masses, and where the quarks are massless, are studied. A remarkably simple picture is found to describe the reaction in the Born approximation. Despite the fact that there are three distinct polarization vectors, the transition matrix remains two-dimensional, provided the reaction takes place on a polarized quark. This simplification is achieved by introducing a nonorthogonal basis for the massive gluon and photon states. An interesting aspect is that one particular gluon polarization state cannot produce photons, while a certain massive photon polarization state cannot be produced. Our results represent a generalization to arbitrary-mass spin-1 particles of the usual helicity conservation known to hold for massless vector particles.

## I. INTRODUCTION

Recently interest has arisen in testing QCD through its predictions for processes involving polarized particles, such as the production of lepton pairs using a polarized beam or target.[1] The parton-model calculations of the corresponding subprocesses involving the fundamental constituents are usually carried out assuming massless quarks. It is a well-known fact that massless quarks acting via $\gamma^\mu$ or $\gamma^\mu\gamma^5$ coupling conserve helicity. An immediate consequence of this is that the number of amplitudes required is reduced by a factor of 2; a more subtle result follows from the theorem of Eberhard and Good.[2] To see how this works, we consider the reaction

$$a+q\rightarrow b+q\ ,$$

where we suppose that the initial and final quarks are joined by a continuous line involving only $\gamma^\mu$ and $\gamma^\mu\gamma^5$ couplings in any Feynman diagram. The theorem states that the rank (number of strictly positive eigenvalues) of the spin density matrix $\rho_b$ for particle $b$ is less than or equal to the product of the statistical weights of the other particles in the reaction. For an unpolarized particle of spin $s$, the statistical weight is $(2s+1)$, whereas for a polarized particle, it is the rank of its density matrix. If the initial quark is in a state of definite helicity, the final quark has the same helicity, and the product of the statistical weights for the quarks is unity. It then follows that

$$r_b\le r_a\ ,$$

where $r_b$ and $r_a$ denote the ranks of the density matrices for particles $a$ and $b$, respectively. In particular, if particle $a$ is in a pure state (i.e., $\rho_a$ is unitarily equivalent to a matrix with one eigenvalue of unity and the others zero), then particle $b$ is also in a pure state. This allows us to consider the transition amplitude as a mapping of pure states of particle $a$ onto pure states of particle $b$. One may then write the amplitude matrix for the reaction in the following form, using Dirac notation:

$$S=\sum_{i=1}^{(2S_a+1)} S_i\,|\beta_i\rangle\langle\tilde{\alpha}_i|\ , \tag{1}$$

where the $|\alpha_j\rangle$ span the space of states of particle $a$, and $\langle\tilde{\alpha}_i|$ denotes the dual state, i.e., those states which satisfy

$$\langle\tilde{\alpha}_i|\alpha_j\rangle=\delta_{ij}\ . \tag{2}$$

The $|\beta_i\rangle$ denote states for particle $b$, and the quantities $S_i$ are the reaction amplitudes corresponding to the transition in which state $|\alpha_i\rangle$ becomes $|\beta_i\rangle$. Thus a complete specification of the transition matrix is provided by the $(2S_a+1)$ states $|\alpha_i\rangle$ (or their duals), their images $|\beta_i\rangle$, and the corresponding amplitudes $S_i$.

In this paper we show that for the reaction

$$G+q\rightarrow\gamma+q$$

involving massive gluons and photons, the Born approximation to the transition amplitude on a positive-helicity quark may be written as

$$S=S_1\,|\gamma 1^s\rangle\langle G1^s|+S_{-1}\,|\gamma-1^u\rangle\langle G-1^u|\ , \tag{3}$$

where $|G1^s\rangle$ and $|\gamma 1^s\rangle$ denote $s$-channel helicity $+1$ states of massive gluon and massive photon, respectively, and where the states $|G-1^u\rangle$ and $|\gamma-1^u\rangle$ denote negative $u$-channel helicity states. (We recall that $s$-channel helicity for massive particles means that in the rest frame of the particle, the $z$ direction is antiparallel to the momentum of the partner of the particle in the $s$ channel. In the present case, this means the initial quark for particle $a$ and the final quark for particle $b$. The $u$-channel helicity axes correspond to choosing the $z$ axis antiparallel to the final-quark momentum in the particle-$a$ rest frame, and to the initial-quark momentum in the particle-$b$ rest frame.)

A remarkable aspect of this expression is that only $|1^s\rangle$ and $|-1^u\rangle$ states appear. It then follows that the gluon polarization state which is orthogonal to both of these states cannot produce a photon, nor can the corresponding photon state be produced. It should also be remarked that our expression (3), which looks as if it conserves helicity, does not. The Dirac bra vectors appearing are the dual vectors corresponding to the states $|G-1^s\rangle$, $|G1^u\rangle$ and the state which does not react. Thus, if an incident gluon is in the state $|1^u\rangle$, the final photon is necessarily in the state $|1^s\rangle$, whereas a gluon in a $|-1^s\rangle$ state produces a $|-1^u\rangle$ photon. The corresponding amplitudes are

$$\langle \gamma 1^s | S | G 1^u \rangle = S_1 \langle G 1^s | G 1^u \rangle \tag{4a}$$

and

$$\langle \gamma -1^u | S | G -1^s \rangle = S_{-1} \langle G -1^u | G -1^s \rangle \ . \tag{4b}$$

Note that

$$\langle G 1^s | G 1^u \rangle = \langle G -1^u | G -1^s \rangle = \frac{1+\cos\chi_G}{2} \ , \tag{5}$$

where, in the rest frame of the massive gluon, $\chi_G$ is the angle between the momenta of the initial and final quarks. In the limit of zero gluon mass, $\chi_G=0$, and there is no distinction between $s$-channel and $u$-channel helicity. In taking the limit $m_G \to 0$ we recover earlier results derived by Gottlieb and the present author in Ref. 3. If one also allows the photon mass $m_\gamma$ to approach zero, the usual helicity conservation rule is obtained.

In Sec. II we give a derivation of Eq. (3), as well as an extension to the annihilation process

$$\bar{q}+q \to \gamma + G \ ,$$

again with polarized quarks. We remark that our results apply also if the $\gamma$ is replaced by a $Z_0$ or $W$, and also if the gluon is replaced by a photon.

## II. DERIVATION OF RESULTS

### A. Compton process

The evaluation of the amplitudes for the Born approximation is straightforward, and results may be found in the literature. However, we believe that our derivation of Eq. (3) is sufficiently interesting in itself to merit a detailed presentation. Let us write the amplitude for the Born approximation to

$$G(p_1)+q(p_2) \to \gamma(p_3)+q(p_4)$$

in the standard form[4] (omitting color indices for simplicity)

$$M = eg\bar{u}(p_4) O u(p_2) \ , \tag{6}$$

where $O$ is defined by

$$O = \frac{\not\epsilon^{\gamma\dagger}(\not p_1+\not p_2)\not\epsilon^G}{s} + \frac{\not\epsilon^G(\not p_2-\not p_3)\not\epsilon^{\gamma\dagger}}{u} \ , \tag{7}$$

and where $\epsilon^G$ and $\epsilon^{\gamma\dagger}$ are the gluon and photon polarization vectors, $s=(p_1+p_2)^2$, and $u=(p_2-p_3)^2$. In the c.m. frame, the momenta may be written

$$p_1 = (s+m_G{}^2, 0, 0, s-m_G{}^2)/2s^{1/2} \ , \tag{8a}$$

$$p_2 = (s-m_G{}^2, 0, 0, -(s-m_G{}^2))/2s^{1/2} \ , \tag{8b}$$

$$p_3 = (s+m_\gamma{}^2, (s-m_\gamma{}^2)\sin\theta, 0, (s-m_\gamma{}^2)\cos\theta) \times (2s^{1/2})^{-1} \ , \tag{8c}$$

where $\theta$ is the production angle and where $p_1{}^2=m_G{}^2$ and $p_3{}^2=m_\gamma{}^2$ (although we shall assume, in the body of the paper, that $m_G{}^2$ and $m_\gamma{}^2$ are positive, our results may be extended to include spacelike momenta for the gluon or photon. The details are given in the Appendix). It is useful to introduce the vectors $n$ and $n^\dagger$, which in the c.m. frame have the form

$$n = (0,1,i,0)/\sqrt{2} \ , \tag{9a}$$

$$n^\dagger = (0,1,-i,0)/\sqrt{2} \ , \tag{9b}$$

and which satisfy $n_\mu n^\mu=0$, $n_\mu n^\dagger_\mu=-1$. They correspond to gluon polarization vectors of positive and negative $s$-channel helicity, respectively.

Using standard $\gamma$-matrix identities, the quantity $O$ of Eq. (7) may be written as

$$O = \frac{\not A(1+\gamma_5)}{2} + \frac{\not B(1-\gamma_5)}{2} \ , \tag{10}$$

and, if the nonorthogonal basis $p_2$, $p_4$, $n$, and $n^\dagger$ is introduced, $\not A$ may be written as

$$\not A = a\not p_2 + b\not p_4 + c\not n + d\not n^\dagger$$

with a similar expression for $\not B$. If the initial quark has positive helicity, $(1-\gamma_5)$ annihilates it, and so does $\not n^\dagger$, since it corresponds to $\sigma_x - i\sigma_y$ acting on a down-spinor. The quantities $\not p_2$ and $\not p_4$ have zero matrix elements, and one may write, for positive-helicity quarks,

$$M = eg\left[\bar{u}(p_4)\not n\left[\frac{1+\gamma_5}{2}\right]u(p_2)\right]c \ . \tag{11}$$

In order to obtain the coefficient $c$, we introduce $v^\dagger$, the vector dual to $n$ in the $p_2, p_4, n, n^\dagger$ basis and find

$$c = \tfrac{1}{4}\,\mathrm{Tr}[O\not v^\dagger(1-\gamma_5)] \ . \tag{12}$$

The dual vector may be written as

$$v^\dagger = \frac{p_4 \cdot n^\dagger}{p_4 \cdot p_2} p_2 - n^\dagger \ , \tag{13}$$

and satisfies

$$v^\dagger_\mu p_2^\mu = v^\dagger_\mu p_4^\mu = v^\dagger_\mu n^{\dagger\mu} = v^\dagger_\mu v^{\dagger\mu} = 0 \ , \quad v^\dagger_\mu v^\mu = -1 \ .$$

At this point one may simply replace $O$ by Eq. (7) and carry out the trace (12) to obtain $c$ as a function of $\epsilon^{\gamma\dagger}$ and $\epsilon^G$. We can avoid this effort, however, by writing Eq. (12) in another way, namely,

$$c = \epsilon^{\gamma\dagger\mu} C_{\mu\nu} \epsilon^{G\nu} \ . \tag{14}$$

One may then easily verify, using either Eq. (12) or (6), that the following identities hold:

$$C_{\mu\nu} p_1^\nu = 0 \ , \tag{15a}$$

$$C_{\mu\nu} v^{\dagger\nu} = 0 \ , \tag{15b}$$

$$p_3^\mu C_{\mu\nu} = 0 \ , \tag{15c}$$

$$v^{\dagger\mu}C_{\mu\nu}=0\ , \tag{15d}$$

$$p_2^{\mu}C_{\mu\nu}p_2^{\nu}=0\ , \tag{15e}$$

$$p_4^{\mu}C_{\mu\nu}p_4^{\nu}=0\ . \tag{15f}$$

Equations 15(a)–15(d) imply that $C_{\mu\nu}$ acts effectively on the right only in the subspace orthogonal to $p_1$ and $v^{\dagger}$, and on the left in the subspace orthogonal to $p_3$ and $v^{\dagger}$. A convenient set of basis vectors for these orthogonal subspaces may be introduced, namely,

$$\hat{s}_{\mu}=i\epsilon_{\mu\alpha\beta\gamma}p_1^{\alpha}v^{\dagger\beta}p_2^{\gamma}/p_1\cdot p_2\ , \tag{16a}$$

$$\hat{u}_{\mu}=i\epsilon_{\mu\alpha\beta\gamma}p_1^{\alpha}v^{\dagger\beta}p_4^{\gamma}/p_1\cdot p_4 \tag{16b}$$

for the right basis, and

$$\hat{s}'_{\mu}=i\epsilon_{\mu\alpha\beta\gamma}p_3^{\alpha}v^{\dagger\beta}p_4^{\gamma}/p_3\cdot p_4\ , \tag{16c}$$

$$\hat{u}'_{\mu}=i\epsilon_{\mu\alpha\beta\gamma}p_3^{\alpha}v^{\dagger\beta}p_2^{\gamma}/p_3\cdot p_2 \tag{16d}$$

for the left basis. The vector $s_{\mu}$ satisfies

$$\hat{s}_{\mu}\hat{s}^{\mu}=0\ ,\quad \hat{s}_{\mu}\hat{s}^{\mu\dagger}=-1$$

as do $\hat{u}$, $\hat{s}$, and $\hat{u}$. Using these vectors, one may write the most general expression for $C_{\mu\nu}$, such that all of the conditions (15) are satisfied, in the form

$$C_{\mu\nu}=C_1\hat{s}'_{\mu}\hat{s}_{\nu}+C_{-1}\hat{u}'_{\mu}\hat{u}_{\nu}\ . \tag{17}$$

The coefficients $C_1$ and $C_{-1}$ are readily evaluated by making special choices for $\epsilon^{G}$ and $\epsilon^{\gamma\dagger}$. For example, if $\epsilon^{G}=p_2$ and $\epsilon^{\gamma\dagger}=p_4$, one finds

$$C_{-1}=\frac{2(p_2\cdot p_4)(p_3\cdot v^{\dagger})}{(p_4\cdot\hat{u}')(p_2\cdot\hat{u})u}\ , \tag{18}$$

whereas if $\epsilon^{G}=p_4$ and $\epsilon^{\gamma\dagger}=p_2$, one obtains

$$C_1=\frac{-2(p_2\cdot p_4)(p_3\cdot v^{\dagger})}{(p_2\cdot\hat{s}')(p_4\cdot\hat{s})s}\ . \tag{19}$$

These expressions may be further simplified if one introduces the pseudovector $N_{\mu}$, via

$$N_{\mu}=\epsilon_{\mu\alpha\beta\gamma}p_1^{\alpha}p_2^{\beta}p_3^{\gamma}\ . \tag{20}$$

Evaluating the scalar products such as $p_2\cdot\hat{s}$, one finds

$$C_{-1}=\frac{4(p_2\cdot p_4)(p_3\cdot v^{\dagger})}{N_{\mu}N^{\mu}}\left[\frac{(p_2\cdot p_3)(p_1\cdot p_4)}{u}\right]\ , \tag{21a}$$

$$C_1=-\frac{4(p_2\cdot p_4)(p_3\cdot v^{\dagger})}{N_{\mu}N^{\mu}}\left[\frac{(p_3\cdot p_4)(p_1\cdot p_2)}{s}\right]\ . \tag{21b}$$

At this point we have essentially completed our demonstration. The key feature is that if either the gluon polarization $\epsilon^{G}=v^{\dagger}$ or if the photon polarization vector $\epsilon^{\gamma\dagger}=v^{\dagger}$, then the amplitude is zero. Noting that the vector $\hat{s}$ satisfies $\hat{s}\cdot p_1=\hat{s}\cdot p_2=\hat{s}\cdot\hat{s}=0$, along with $\hat{s}\cdot\hat{s}^{\dagger}=-1$, we can conclude that $\hat{s}$ corresponds to a state of definite $s$-channel helicity $+1$ or $-1$. After working out the details, taking into account the helicity convention of Jacob and Wick[5] [for $s$-channel helicity gluons, positive helicity implies $-(\hat{e}_x+i\hat{e}_y)/\sqrt{2}$, whereas negative helicity implies $(\hat{e}_x-i\hat{e}_y)/\sqrt{2}$] we obtain the following results.

Gluon polarization vectors:

$$\hat{s}=-\epsilon_s^{1\dagger}\ , \tag{22a}$$

$$\hat{u}=\epsilon_u^{-1\dagger}\ , \tag{22b}$$

where $s$ and $u$ refer to $s$- and $u$-channel helicity.

Photon polarization vectors:

$$\hat{s}'=-\epsilon_s^{1}\ , \tag{23a}$$

$$\hat{u}'=\epsilon_u^{-1}\ . \tag{23b}$$

The expression $\hat{s}'_{\mu}\hat{s}_{\nu}$ thus corresponds to the $|\gamma 1^{s}\rangle\langle G 1^{s}|$ expression of Eq. (3), while the $\hat{u}'_{\mu}u_{\nu}$ term corresponds to $|\gamma -1^{u}\rangle\langle G-1^{u}|$. Returning now to Eq. (11), we find, for positive-helicity quarks,

$$\bar{u}(p_4)\not{n}\left[\frac{1+\gamma_5}{2}\right]u(p_2)=-2(p_2\cdot p_4)^{1/2} \tag{24}$$

and we obtain, as our final result for positive-helicity quarks,

$$M=K\left[\frac{(s-m_{\gamma}^2)(s-m_G^2)}{s}|\gamma 1^{s}\rangle\langle G1^{s}| -\frac{(m_{\gamma}^2-u)(m_G^2-u)}{u}|\gamma-1^{u}\rangle\langle G-1^{u}|\right]\ , \tag{25}$$

where

$$K=-2eg(m_{\gamma}^2m_G^2-su)^{-1/2}\ . \tag{26}$$

If the reaction takes place on a quark with negative helicity, one obtains

$$M=K\left[\frac{(s-m_{\gamma}^2)(s-m_G^2)}{s}|\gamma-1^{s}\rangle\langle G-1^{s}| -\frac{(m_{\gamma}^2-u)(m_G^2-u)}{u}|\gamma 1^{u}\rangle\langle G1^{u}|\right]\ . \tag{27}$$

In this case the polarization vector which does not interact is $v$, rather than $v^{\dagger}$. In Eq. (17) the vectors $\hat{s}'$, $\hat{s}$, $\hat{u}'$, and $\hat{u}$ should be replaced by $\hat{s}'^{\dagger}$, $\hat{s}^{\dagger}$, $\hat{u}'^{\dagger}$, and $\hat{u}^{\dagger}$, if the quark has negative helicity.

If the gluon mass is set equal to zero, the results of Ref. 3 are recovered, in the sense that the $u$-channel and $s$-channel gluon helicity states become identical. And, if both photon and gluon masses are set equal to zero, the usual helicity-conserving amplitudes are recovered.

### B. Annihilation process

Results similar to those for the Compton process apply also in the annihilation process

$$\bar{q}(p_1)+q(p_2)\rightarrow\gamma(p_3)+G(p_4)\ .$$

In the c.m. frame the momenta are

$$p_1=\frac{\sqrt{s}}{2}(1,0,0,1)\ , \tag{28a}$$

$$p_2=\frac{\sqrt{s}}{2}(1,0,0,-1)\ , \tag{28b}$$

$$p_3=(s+m_\gamma{}^2,(s-m_\gamma{}^2)\sin\theta,0,(s-m_\gamma{}^2)\cos\theta)/2\sqrt{s}\ . \tag{28c}$$

The matrix element for the reaction on a positive-helicity quark (which requires a negative-helicity antiquark) is

$$M=eg\bar{v}(p_1)\not{n}\frac{(1+\gamma_5)}{2}u(p_2)\epsilon^{\gamma\dagger}_{\ \mu}\epsilon^{G\dagger}_{\ \nu}\bar{C}_{\mu\nu}\ . \tag{29}$$

If either $\epsilon_\gamma$ or $\epsilon^G$ is $n$, the amplitude is zero. We introduce the vectors $\hat{u}$, $\hat{u}'$, $\hat{t}$, $\hat{t}'$ (which should not be confused with our prior usage in the Compton process):

$$\hat{u}_\mu=i\epsilon_{\mu\alpha\beta\gamma}p_4^\alpha n^{\dagger\beta}p_1^\gamma/p_1\cdot p_4\ , \tag{30a}$$

$$\hat{t}_\mu=i\epsilon_{\mu\alpha\beta\gamma}p_4^\alpha n^{\dagger\beta}p_2^\gamma/p_2\cdot p_4\ , \tag{30b}$$

$$\hat{u}'_\mu=i\epsilon_{\mu\alpha\beta\gamma}p_3^\alpha n^{\dagger\beta}p_2^\gamma/p_2\cdot p_3\ , \tag{30c}$$

$$\hat{t}'_\mu=i\epsilon_{\mu\alpha\beta\gamma}p_3^\alpha n^{\dagger\beta}p_1^\gamma/p_1\cdot p_3\ . \tag{30d}$$

The tensor $\bar{C}_{\mu\nu}$ may be written as

$$\bar{C}_{\mu\nu}=K'(C_t\hat{t}'_\mu\hat{t}_\nu+C_u\hat{u}'_\mu\hat{u}_\nu)\ , \tag{31}$$

where

$$K'=\frac{4(p_1\cdot p_2)(n^\dagger\cdot p_3)}{N_\mu N^\mu}\ , \tag{32a}$$

$$C_t=(m_\gamma{}^2-t)(m_G{}^2-t)/4t\ , \tag{32b}$$

$$C_u=-(m_\gamma{}^2-u)(m_G{}^2-u)/4u\ . \tag{32c}$$

The transition matrix, instead of being three-dimensional, remains two-dimensional, just as for the massless spin-1 particles.

If the reaction occurs on a negative-helicity quark, similar results hold except that $\not{n}(1+\gamma_5)/2$ is replaced by $\not{n}^\dagger(1-\gamma_5)/2$, and $\hat{u}$, $\hat{t}$, $\hat{u}'$, $\hat{t}'$ should be replaced by $\hat{u}^\dagger$, $\hat{t}^\dagger$, $\hat{u}'^\dagger$, $\hat{t}'^\dagger$.

## III. CONCLUSIONS

The results derived here show that, at least in the Born approximation, there exists a rather simple way to express the amplitude for the scattering of massive spin-1 particles on polarized massless fermions. Although we have treated only the case of timelike spin-1 momentum, our results can be applied as well to spacelike momentum, provided the modifications indicated in the Appendix are taken into account. In our judgement, the most interesting aspect of our results is how the notion of helicity conservation is generalized to arbitrary spin-1 masses. If a negative $s$-channel gluon scatters on a positive-helicity quark, a negative $u$-channel photon is produced. If the gluon has positive $u$-channel helicity, the outgoing photon has positive $s$-channel helicity. A gluon whose polarization state is orthogonal to both a negative $s$-channel helicity state and a positive $u$-channel helicity state will not produce a photon, nor can a photon which has no overlap with either negative $u$-channel helicity or positive $s$-channel helicity be produced. Despite the simplicity of these results, it is not easy to find examples of reactions which could illustrate them. One (perhaps overly imaginative) reaction might be

$$e^-+e^+\to e^-+Z^0+e^+$$

using longitudinally polarized electron and positron beams, with a tag on the final $e^-$ to fix the virtual-photon momentum and polarization, the decay of the final $Z^0$ permitting a direct measurement of its polarization state. Other conceivable applications, such as the Drell-Yan process or deep-inelastic lepton scattering, must await the introduction of polarized beams and targets. Even then, unknown polarized parton distribution functions for polarized hadrons will intervene, making a direct test of our results problematic. Finally, the parity-conserving decay of a massive photon into $l^+l^-$, as in the Drell-Yan process, does not distinguish between a photon of pure positive helicity and one which is an arbitrary mixture of positive and negative helicity. Once again this tends to make direct tests extremely difficult.

An interesting open question is whether our result is true only in the Born approximation, or whether it is more generally true. As a first step in this direction, one may calculate the one-loop corrections to the Compton process. The formalism presented here is adequate for handling this problem. The essential question is whether Eqs. (15b) and (15d) continue to hold when the one-loop expression for the operator $O$ is used in place of Eq. (7). Another question is whether the results are valid for the Born approximation to massive gluon scattering, once the triple-gluon coupling is included. Work on these questions is in progress, and we hope to present results in a subsequent paper.

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## APPENDIX

The relations among the $s$-channel and $u$-channel helicity states for the massive gluons and photons are as follows. For the gluon (momentum $p_1$),

$$|\,Gm^u\rangle=\sum_{m'}d^1_{m'm}(\chi_G)\,|\,Gm'^s\rangle\ , \tag{A1}$$

where $0\le\chi_G\le\pi$ and

$$1-\cos\chi_G=\frac{(p_2\cdot p_4)p_1{}^2}{(p_1\cdot p_2)(p_1\cdot p_4)} \tag{A2}$$

and where the rotation functions are defined by

$$d^j_{mm'}(\theta)=\langle jm\,|\,e^{-i\theta J_y}\,|\,jm'\rangle\ . \tag{A3}$$

For the photon (momentum $p_3$),

$$|\gamma m^u\rangle=\sum_{m'} d^1_{m'm}(-\chi_\gamma)\,|\gamma m'^s\rangle\ , \tag{A4}$$

where $0\le\chi_\gamma\le\pi$, and

$$1-\cos\chi_\gamma=\frac{(p_2\cdot p_4)p_3{}^2}{(p_3\cdot p_2)(p_3\cdot p_4)}\ . \tag{A5}$$

If the mass of the gluon is set to zero, $\chi_G\to 0$, and the $s$- and $u$-channel axes coincide.

*Spacelike gluon momentum.* The results we have obtained, assuming explicitly that the gluon momentum $p_1$ and the photon momentum $p_3$ were timelike, may be extended to the case where either $p_1$ or $p_3$, or both, are spacelike. The essential point is that the tensor $C_{\mu\nu}$, introduced in Eq. (14), satisfies the identities (15a)—(15f), irrespective of the signs of $m_G{}^2$ or $m_\gamma{}^2$. From these identities follows our Eq. (17), which expresses $C_{\mu\nu}$ in terms of $\hat{s}'_\mu\hat{s}_\nu$, $\hat{u}'_\mu\hat{u}_\nu$, and the invariants $C_1$ and $C_{-1}$. These invariants may be expressed in terms of scalar products of four-vectors, as indicated in Eqs. (18) and (19). At this point there is no real distinction between timelike and spacelike momenta, since all the variables we have introduced remain well defined when $m_G{}^2$ or $m_\gamma{}^2$ are negative. What does change, of course, is the interpretation of $\hat{s}$ and $\hat{u}$ as corresponding to states of definite $+1$ or $-1$ spin projection along suitably chosen axes in the rest frame of the massive gluon. If $m_G{}^2<0$, there is no gluon rest frame, and $\hat{s}$ and $\hat{u}$ acquire a different characterization. It can be shown that in any Lorentz frame such that $p_1$ and $p_2$ lie in the $(t,z)$ plane, with $(p_2)^z<0$, $\hat{s}$ takes on the form $\hat{s}=(0,1/\sqrt{2},-i/\sqrt{2},0)$. Among these frames are all obtained from the c.m. frame by a $z$ boost. Similarly, in that class of Lorentz frame such that $p_1$ and $p_4$ span the $(t,z)$ plane, with $(p_4)^z<0$, one finds $\hat{u}=(0,1/\sqrt{2},i/\sqrt{2},0)$. In the standard treatment of deep-inelastic electron scattering, it is customary to use the terms of positive and negative helicity to describe the polarization state of the virtual photon. This standard usage is what we term $s$-channel helicity; our use of the term $u$-channel helicity to describe the virtual gluon, while not standard, is a natural generalization of the concept of $u$-channel helicity for timelike gluons. One cautionary remark concerning the crossing matrices is necessary. In the case of spacelike $p_1$, Eq. (A2) implies that $\chi_G$, the $s$-$u$ crossing angle, becomes complex, since one may easily show that

$$|\cos\chi_G|\ge 1 \tag{A6}$$

if $p_1{}^2$ is negative. Detailed examination of the kinematics written in Eqs. (8a), (8b), and (8c) leads to the following result:

For $0\le\theta<\theta_c$

$$\chi_G=i\cosh^{-1}\left[1-\frac{p_1{}^2(p_2\cdot p_4)}{(p_1\cdot p_2)(p_1\cdot p_4)}\right]\ . \tag{A7}$$

For $\theta_c<\theta\le\pi$

$$\chi_G=\pi+i\cosh^{-1}\left[\frac{p_1{}^2(p_2\cdot p_4)}{(p_1\cdot p_2)(p_1\cdot p_4)}-1\right]\ , \tag{A8}$$

where $\theta_c$ is the value of the c.m. scattering angle such that $p_1\cdot p_4=0$, namely,

$$\cos\theta_c=\left[\frac{s+p_1{}^2}{s-p_1{}^2}\right]\ . \tag{A9}$$

If these modifications are taken into account, the formulas presented for the case of timelike gluon momentum are valid as well as for spacelike momentum.

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