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Remarks on the chiral phase transition in chromodynamics

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The phase transition restoring chiral symmetry at finite temperatures is considered in a linear σ model. For three or more massless flavors, the perturbative ϵ expansion predicts the phase transition is of first order. At high temperatures, the $U_A(1)$ symmetry will also be effectively restored.

The phase transitions of chromodynamics are of theoretical interest in their own right, as well as for their possible relevance to hadronic matter under extreme conditions of high temperatures and densities. At present, the nature of the deconfining transition in purely gluonic theories appears to be clear, both conceptually[1] and, at least in part, numerically.[2] The significance of these results for theories with dynamical quarks is less certain. If the quark contribution to the free energy is large enough, the presence of the quarks can smooth out any singularity of a deconfining transition.[3] Whether or not this occurs for the first-order transition of SU(3) color can only be answered by further study.[4]

In this note, we do not directly consider the deconfining transition, but rather a phase transition which almost certainly occurs in hadronic matter at finite temperature—the restoration of the spontaneously broken chiral symmetry, associated with massless quarks, to the chromodynamic vacuum. We begin in Sec. I by analyzing the chiral phase transition in a linear σ model, for any number of colors and flavors. Known results from the ϵ expansion suggest that the phase transition is first order if there are three or more massless flavors. For less than three flavors, the order of the transition depends on the details of the dynamics.

We argue that besides the exact restoration of the flavor chiral symmetry, at high temperature there will also be, approximately, restoration of the flavor-singlet $U_A(1)$ symmetry. We suggest that the temperature scale at which the $U_A(1)$ symmetry is effectively restored need *not* be the same as that of the chiral phase transition. In Sec. II, we discuss some of the effects possible in hadronic matter if the $U_A(1)$ symmetry is effectively restored before the chiral transition.

I. ORDER OF THE CHIRAL PHASE TRANSITION

Consider N_f flavors of massless quarks which couple in the fundamental representation to a $SU(N_c)$ color gauge group. Classically, the quark part of the chromodynamic action is invariant under a global flavor symmetry of $G_f = U_A(1) \times SU(N_f) \times SU(N_f)$. The axial $U_A(1)$ symmetry, while valid classically, can be violated by quantum-mechanical effects. If $J^5_\mu = \bar{q}\gamma_\mu\gamma_5 q$ is the $U_A(1)$ current, where q denotes the quark fields, the conservation of J^5_μ is spoiled by the anomaly[5]

$$\partial_\mu J^5_\mu = \frac{g^2 N_f}{16\pi^2}\,\mathrm{tr}(\tilde{F}_{\mu\nu}F^{\mu\nu}) \quad . \tag{1}$$

The anomaly vanishes if there are no dynamical fermions, $N_f = 0$, or in the limit of infinite colors,[6] $N_c = \infty$; then the flavor symmetry is still G_f. Even with the anomaly present, there remains an axial $Z_A(N_f)$ symmetry,[7,8] so the flavor symmetry of the quantum theory becomes $G_f' = Z_A(N_f) \times SU(N_f) \times SU(N_f)$.

We henceforth assume that at zero temperature the chromodynamic vacuum spontaneously breaks G_f (or G_f') to $SU(N_f)$, and that there is a finite temperature T_{ch} at which the full chiral symmetry, be it G_f or G_f', is restored to the vacuum.

We further assume that the chiral-symmetry breaking is characterized by a quark bilinear which transforms as $(N_f, N_f^*) + (N_f^*, N_f)$ under G_f. As in the linear σ model of Gell-Mann and Levy,[9] we introduce a (color singlet) complex, N_f-by-N_f matrix $\underline{\Phi}$ to parametrize the symmetry breaking, $\Phi_{ij} \sim \langle \bar{q}_i(1+\gamma_5)q_j \rangle$. Under G_f, $\underline{\Phi}$ transforms as

$$\underline{\Phi} \to \exp(i\alpha)\, U_+ \underline{\Phi} U_- \quad , \tag{2}$$

where U_+ and U_- are arbitrary and independent $SU(N_f)$ matrices, and α generates $U_A(1)$ rotations. To discuss the low-energy excitations corresponding to slow space-time variations in $\underline{\Phi}$, we consider the most general renormalizable Lagrangian for $\underline{\Phi}$ consistent with the symmetries:

$$L_\Phi = \tfrac{1}{2}\mathrm{tr}(\partial_\mu \underline{\Phi}^\dagger)(\partial_\mu \underline{\Phi}) - \tfrac{1}{2} m_\Phi{}^2\, \mathrm{tr}\underline{\Phi}^\dagger\underline{\Phi} - \frac{\pi^2}{3} g_1 (\mathrm{tr}\underline{\Phi}^\dagger\underline{\Phi})^2 - \frac{\pi^2}{3} g_2\, \mathrm{tr}(\underline{\Phi}^\dagger\underline{\Phi})^2 \quad . \tag{3}$$

For stability at large $\underline{\Phi}$, $g_2 > 0$, $g_1 + g_2/N_f > 0$. If the anomaly is present, to L_Φ the term

$$L_\Phi' = c(\det\underline{\Phi} + \det\underline{\Phi}^\dagger) \tag{4}$$

must be added.[10,11] The $Z_A(N_f)$ symmetry is evident in terms of L_Φ'.

At zero temperature, $m_\Phi{}^2 < 0$, with a vacuum expectation value $\langle \underline{\Phi} \rangle = \Phi_0 \underline{1}$ chosen to be $SU(N_f)$ symmetric. If $c = 0$, this spontaneous symmetry breaking generates N^2 Goldstone bosons: a $SU(N_f)$ multiplet like pions, and a flavor singlet η'. When $c \neq 0$, since the broken $Z_A(N_f)$ symmetry is discrete, the η' is massive.

As the temperature increases, so does $m_\Phi{}^2$. In mean-field theory, $m_\Phi{}^2$ and Φ_0 vanish as $T \to T_{\mathrm{ch}}^-$. To improve on mean-field theory, we note that as fluctuations in the timelike direction are cut off by finite temperatures, the critical behavior will be described by an effective theory in three dimensions. Neglecting all excitations which are effectively massive about T_{ch}—such as any fermions, mesons not included in L_Φ, etc.—only L_Φ (and L_Φ') determine the critical properties.

We first consider the limit of infinite colors, where the anomaly vanishes and $c = 0$, and then discuss a small number of colors, where $c \neq 0$.

The critical behavior of models with G_f symmetry have not been studied directly in three dimensions, but they have in $4-\epsilon$ dimensions.[12-14] To leading order in ϵ, the β functions for g_1 and g_2, β_1 and β_2, are

$$\beta_1 = -\epsilon g_1 + \frac{(N_f^2+4)}{3} g_1^2 + \frac{4N_f}{3} g_1 g_2 + g_2^2 \ ,$$
$$\beta_2 = -\epsilon g_2 + 2 g_1 g_2 + \frac{2N_f}{3} g_2^2 \ . \tag{5}$$

A fixed point g^* is infrared stable if the stability matrix $\omega_{ij} = \partial\beta_i / \partial g_j$ has real and positive eigenvalues for $g = g^*$. From Eq. (5), it can be shown that for $\sqrt{2} > N_f \geq 0$, the infrared-stable fixed point has $g_2^* = 0$, with $O(2N_f)$ critical exponents. When $N_f > \sqrt{3}$, there is no infrared-stable fixed point for g_1, $g_2 \sim O(\epsilon)$. This is due critically to the presence of two coupling constants in L_Φ. For example, when $N_f > \sqrt{3}$, the $O(2N_f)$-like fixed point, with $g_2^* = 0$, becomes infrared unstable in the g_2 direction.

For a single flavor, since $G_f = U(1) = O(2)$ anyway, this instability does not arise. The phase transition, if second order,[15] has O(2) critical exponents.

The limit of zero flavors is measured by "quenched" fermions.[2] It is notable that when $N_f = 0$, the chiral phase transition, if second order,[15] has O(0) critical exponents. Since the anomaly always vanishes for $N_f = 0$, this result holds for any number of colors.

For two or more flavors, because there is no infrared-stable fixed point, the chiral phase transition should be first order, induced by fluctuations.[16,17] (Here and below we are assuming that there is not a nonperturbative infrared stable fixed point governing the transition. Strictly speaking, all that has been shown is that the usual Wilson-Fischer fixed point suggested by mean-field theory does not survive the critical fluctuations.) We emphasize that the transition being of first order is a striking prediction of the ϵ expansion, and is not expected from mean-field theory. While it is only possible to prove that the transition is first order for $\epsilon \ll 1$,[17] in many known examples[16] it appears that the results found at leading order in ϵ remain an excellent guide to three dimensions, when $\epsilon = 1$.

For a small number of colors, at zero temperature the effects of the anomaly should be large, with $c \sim g_1 \sim g_2 \sim O(1)$. For instance, this is necessary in hadronic matter in order that the η' be much more massive than the π, K, η octet. Assume for the moment that c were independent of temperature, $c(T) \approx c(0)$.

With flavor symmetry G_f', the phase transition remains first order if $N_f \geq 3$.[18] For three flavors, as L_Φ' is trilinear in the matrix elements of $\underline{\Phi}$, it alone drives the transition first order. When $N_f = 4$, L_Φ' is a relevant operator, but its presence does not generate an infrared-stable fixed point.[13] For $N_f > 4$, L_Φ' is an irrelevant operator, and so it does not affect the critical behavior.

For a single massless flavor, as L_Φ' acts like a background magnetic field for $\underline{\Phi}$, there is no phase transition to speak of.

The case of two massless flavors is special. If $c(T_{ch}) \approx c(0)$, the η' remains massive about T_{ch}. Since $G_f' = O(4)$, the phase transition can be second order,[15] with O(4) critical exponents.

We argue, however, that the effects of the anomaly may be strongly dependent on temperature. The anomaly removes the $U_A(1)$ symmetry solely because of nonperturbative effects, such as instantons, which carry topological charge.[5] In weak coupling, 't Hooft[5] showed that instantons generate a $2N_f$-point interaction between massless fermions with essentially the form of $\det\underline{\Phi}$ in L_Φ' (anti-instantons generate $\det\underline{\Phi}^\dagger$). Hence we take $c(T) \sim d_I(T)$, where $d_I(T)$ is the total temperature-dependent instanton density.[19]

By necessity, instantons require both electric and magnetic color fields. At high temperatures, fluctuations in the electric fields are suppressed because of Debye screening, and consequently, so are instantons. Calculation shows that as the temperature $T \to \infty$, $d_I(T) \to 0$.[20] Thus if $c(T) \sim d_I(T)$, at high temperatures there is inevitably the *effective restoration of the $U_A(1)$ symmetry.* The $U_A(1)$ symmetry restoration is only approximate, since for any finite temperature, $c(T) \sim d_I(T) \neq 0$: there are always some instantons present. Also, and unlike the restoration of the G_f' flavor symmetry, the variation of $d_I(T)$ with temperature is relatively smooth (see, e.g., Fig. 5 of Ref. 20). Even so, $d_I(T)$ decreases monotonically with temperature, decreasing to values $< 10\%$ that at zero temperature when $T \sim O(\Lambda)$, where Λ is the renormalization scale parameter of the chromodynamics.

For a single flavor, then, there is no chiral transition if $N_c < \infty$, but the (dynamically generated) quark mass decreases smoothly as the vacuum is heated, vanishing as $T \to \infty$.

With two massless flavors, there is the possibility of a first-order transition even for small N_c. This could happen if the density of instantons were sufficiently small below T_{ch}, so that the η' would be very light, and the critical behavior not like O(4), but O(2) $\times$ O(4).

For three or more flavors, even if $c(T)$ decreases significantly below T_{ch}, the chiral transition remains first order. Note that for three flavors, if $c(T_{ch}) \ll c(0)$, it is not the presence of L_Φ' which drives the phase transition first order, but rather the critical fluctuations of $U_A(1) \times SU(3) \times SU(3)$.

If instantons themselves are the primary chiral-symmetry-breaking mechanism,[21] then it is very difficult to imagine how $c(T_{ch})$ could be $\sim c(0)$. Our arguments only suggest but do not prove that $c(T_{ch}) \ll c(0)$. Numerical simulations are required finally to settle this question.[22]

In hadronic matter the quarks have nonzero bare masses. The effects of these bare masses can be represented by adding a background magnetic field $\sim \mathrm{tr}\underline{M}\underline{\Phi}$ to $L_\Phi + L_\Phi'$. As for any first-order transition, this background field will decrease the latent heat of the chiral phase transition. Granted the success of chiral SU(3) in describing the octet of pseudoscalar mesons, we suspect that the effect of the quark bare masses will only be to weaken the first-order chiral transition, but not to wash it out completely. This remains a prejudice which will be decided by detailed investigation.

II. EFFECTS OF APPROXIMATE $U_A(1)$ RESTORATION

We argued above that it is possible for the $U_A(1)$ symmetry to be effectively restored before the chiral phase transition. In this section, we discuss how this possibility will result in dramatic effects for the spectrum of pseudoscalar mesons in hadronic matter.

To first order in the quark masses, the elements of the

$(\text{mass})^2$ matrix for the π^0-η-η' system are

$$M^2_{\pi^0\pi^0} = (m_u + m_d)\frac{v}{f_\pi^2} ,$$

$$M^2_{\pi^0\eta} = \frac{(m_u - m_d)}{\sqrt{3}}\frac{v}{f_\pi^2} ,$$

$$M^2_{\eta\eta} = \frac{(m_u + m_d + 4m_s)}{3}\frac{v}{f_\pi^2} , \tag{6}$$

$$M^2_{\pi^0\eta'} = \sqrt{2/3}(m_u - m_d)\frac{v}{f_\pi f_{\eta'}} ,$$

$$M^2_{\eta\eta'} = \frac{\sqrt{2}}{3}(m_u + m_d - 2m_s)\frac{v}{f_\pi f_{\eta'}} ,$$

$$M^2_{\eta'\eta'} = \tfrac{2}{3}(m_u + m_d + m_s)\frac{v}{f_{\eta'}^2} + K .$$

The strength of the chiral-symmetry-breaking condensate is $-v = \langle \bar{u}u\rangle = \langle \bar{d}d\rangle = \langle \bar{s}s\rangle$. By SU(3) symmetry, we have taken $f_\eta = f_\pi$. SU(3) symmetry does not relate $f_{\eta'}$ and f_π, so we leave their ratio as a free parameter. To represent the effects of the anomaly, we add a term $\sim K$ to $M^2_{\eta'\eta'}$; K is proportional to the coupling c of L'_Φ.

For the purposes of discussion, we take the values of the quark masses to be $m_u = 6.3$ MeV, $m_d = 11$ MeV, and $m_s = 215$ MeV.[23] We neglect all electromagnetic mass splittings, so for m_{π^0} we take the average pion mass, $(2m_{\pi^\pm} + m_{\pi^0})/3 = 138$ MeV. We find that the eigenvalues of M^2 can be fit to the observed values of the pseudoscalar mesons—$m_{\pi^0} = 138$ MeV, $m_\eta = 549$ MeV, and $m_{\eta'} = 958$ MeV—with

$$v = (212\text{ MeV})^3,\quad K_0 = (923\text{ MeV})^2,\quad f_{\eta'} = 1.95 f_\pi , \tag{7}$$

where $f_\pi = 93$ MeV, and K_0 is the value of K at zero temperature. The values of v and K_0 are typical of the strong interactions; e.g., $f_\pi^2 K_0 \sim (293\text{ MeV})^4$.

What is surprising about the values of Eq. (7) is that $f_{\eta'}$ turns out to be almost twice as large as f_π.[24] This is easy to understand numerically from the values of M^2: taking v from Eq. (7), $M^2_{\eta\eta} = (567\text{ MeV})^2$. For large K, the greatest mixing is between the η and the η'. The effect of η-η' mixing will be to depress the η mass, so because $M^2_{\eta\eta}$ is already close to m_η^2, the η-η' mixing must be small. We have no fundamental understanding as to why $f_{\eta'}$ should be so much larger than f_π.

At finite temperatures, all of the parameters v, f_π, $f_{\eta'}$, and K will vary with temperature. To illustrate the effects possible, we assume that only K varies. Figure 1 shows the values of the (neutral) pseudoscalar-meson masses as a function of K/K_0. For $K > 0.5K_0$, only $m_{\eta'}$ decreases much, while for $K < 0.5K_0$, m_η starts to decrease as well.

We remark that if K becomes very small, the mass spectrum changes dramatically. For instance, when K vanishes, $m_{\pi^0} = 77$ MeV, $m_\eta = 141$ MeV, and $m_{\eta'} = 600$ MeV. This happens because without the effect of the anomaly, the π^0, η, and η' mix to form, as mass eigenstates, nearly pure states of $\bar{u}u$, $\bar{d}d$, and $\bar{s}s$, respectively.[25] Since the charged pseudoscalar mesons are unaffected by the π^0-η-η' mixing, if $K \approx 0$, $m_{\pi^\pm}$ would remain ~ 138 MeV, and there would

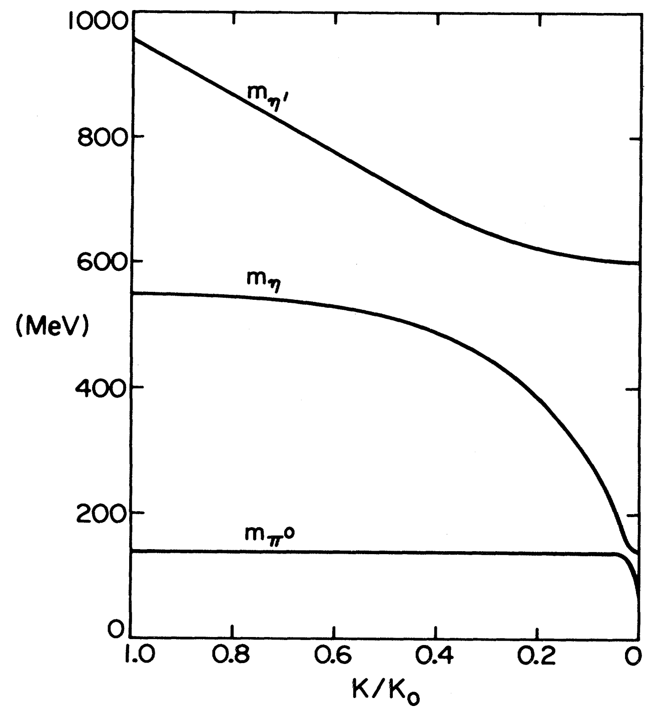

FIG. 1. The masses of the neutral pseudoscalar mesons as a function of K/K_0.

be a phase with massive violations of isospin.

Such a phase appears unlikely, however. From Fig. 1, m_{π^0} does not decrease below 130 MeV until $K < 0.02K_0$. Unless K is very much smaller than K_0 at temperatures below T_{ch} (e.g., $K < 0.01K_0$), there will not be any significant isospin violation.

Perhaps more representative of what might occur at finite temperatures $< T_{ch}$ is a value $K \sim 0.1K_0$. From Fig. 1, at this value of K, $m_{\pi^0} = 137$ MeV, $m_\eta = 283$ MeV, and $m_{\eta'} = 610$ MeV. Thus, while pions are insensitive to changes in K until $K \ll K_0$, the masses of the η, and especially the η', can decrease significantly as K does.

As the temperature approaches T_{ch}, $v \sim \Phi_0$ decreases, and the variation of the other parameters in M^2, beside that of K, becomes important. Our discussion to now has been based on the assumption that $c(T_{ch}) < c(0)$, so $K < K_0$ for $T < T_{ch}$. We remark, though, that even if $c(T_{ch}) \approx c(0)$, K must decrease as Φ_0 does. This is because for three or more flavors, L'_Φ does not provide a mass to the η' unless $\Phi_0 \neq 0$. Simply on dimensional grounds, $K \sim c\Phi_0$ (for N_f flavors, this becomes $K \sim c\Phi_0^{N_f - 2}$). As $\Phi_0 \sim v$, then, $K \sim cv$. When $T \to T_{ch}$, K has to decrease at least as fast as v does.

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[8]This can be seen from the transformation of the Θ vacua, $|\Theta\rangle$ (Ref. 7). $U_A(1)$ rotations are generated by $q \rightarrow \exp(i\alpha\gamma_5/2)q$, $\alpha{:}0 \rightarrow 2\pi$. Note that when $\alpha = 2\pi$, $q \rightarrow -q$; this is not a chiral rotation, but merely a U(1) rotation of total baryon number, and so must be treated as the identity for $U_A(1)$. Under $U_A(1)$ transformations, $|\Theta\rangle \rightarrow |\Theta + \alpha N_f\rangle$. Since $|\Theta\rangle = |\Theta + 2\pi\rangle$, nothing changes if $\alpha = 2\pi m/N_f$, m is an integer. This is an axial symmetry of $Z_A(N_f)$ [and not $Z_A(2N_f)$, as asserted in Ref. 7].

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[14]Contrary to Refs. 12 and 13, the global symmetry of L_Φ is $U_A(1) \times SU(N_f) \times SU(N_f)$, and not $U(N_f) \times U(N_f)$—there is just a single independent U(1) group associated with the rotations of $\underline{\Phi}$, Eq. (2). The other U(1) group corresponds to that for total baryon number, and does not affect the chiral dynamics.

[15]There is the obvious caveat that the chiral symmetry may be restored due to a strongly first-order deconfining transition. For instance, when $N_f = 0$, this happens for $N_c = 3$; our remarks then apply to, e.g., $N_c = 2$, where both transitions appear to be of second order (Ref. 2).

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