Complex-energy solutions of the relativistic Kepler problem at high energies

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In the relativistic Kepler problem the analytic continuation of the principal quantum number n into $i\nu$ describes the scattering states. When ν^2 ranges from $+\infty$ to α^2 the energy ranges from (m_1+m_2) to infinity. In the remaining gap $0<\nu^2<\alpha^2$, the energy becomes complex corresponding to a continuum of resonances or perhaps pair production. This phenomenon also exists in the classical relativistic problem, and we suggest that one should look experimentally for the manifestation of this gap, for example when $Z \geqslant 137$.

We have observed in the mass spectrum of the relativistic Kepler problem, described by a covariant infinite-component wave equation, a well-defined region where the energy becomes complex. This region comes after the discrete and continuous spectrum as a function of the principal quantum number and seems to have been heretofore unnoticed. Then we went back to the classical relativistic theory of Sommerfeld.[1] Indeed, exactly the same region exists also in the classical case. The Sommerfeld energy spectrum is given by

$$E = m_0c^2\left\{1+\frac{\alpha^2Z^2}{n_r+(n_\phi{}^2-\alpha^2Z^2)^{1/2}}\right\}^{-1/2}$$

and the region of complex energy is $0<n_\phi{}^2<\alpha^2Z^2$, when the quantum numbers are analytically continued to the *scattering region.* The Dirac spectrum has the same property in the region $0<\kappa^2<\alpha^2Z^2$, again when κ is analytically continued. The analytic continuation is best seen in the case of infinite-component wave equations, so we shall study these equations in the rest of the paper.

Infinite-component field theory was used extensively in the sixties and early seventies in attempts at cataloging the large number of hadronic resonances found by the experimentalist.[2] The essential feature of such models is to incorporate the mass spectrum from the very beginning by constructing Majorana-type wave equations containing convective probability currents.[3] Such equations yield physically acceptable mass spectra and provide a relativistic formulation of composite models. Realistic wave equations predicting correct electromagnetic form factors for nucleons[4] and pionic decays of baryon resonances,[5] etc., as well as a relativistic covariant equation for the H atom containing corrections for proton recoil were constructed. These equations have the dynamical group structure SO(4,2), incorporating internal relativistic degrees of freedom, from which the representations of the Poincaré group are induced. The theory is covariant and well defined.

In spite of the great number of truths contained in such equations, they have the feature that spacelike solutions are needed to complete the Hilbert space. This is generally seen as a "disease" of infinite-component field theories.[6] Nevertheless, Barut and Nagel[7] have recently given an interpretation of such spacelike states for the equation of the H atom. Whether similar interpretations can be obtained for other realistic wave equations depends on the internal structure of the corresponding composite objects.

The mass spectra of such equations actually contain several branches, namely timelike, discrete, and continuous spectra, and the spacelike and lightlike (continuous) solutions. By far the greatest attention has been paid to the discrete part of the spectrum. In this paper we study a feature of the timelike continuous (scattering) spectrum for the H atom, namely, the appearance of a complex-mass spectrum sector yielding a continuous series of resonances for e^-p^+ scattering. Complex-mass states may be associated with nonunitary representations of the Poincaré group.[8]

The wave equation of interest here has the form[6]

$$(J_\mu P^\mu+\beta S+\gamma)|\tilde{\psi}(P)\rangle=0 \ , \tag{1}$$

where J_μ is the conserved matter current

$$J_\mu=\alpha_1\Gamma_\mu+\alpha_2P_\mu+\alpha_3SP_\mu \ . \tag{2}$$

The operators Γ_μ and S are generators of the dynamical group SO(4,2). In the rest frame where $P_\mu=(M,0,0,0)$ such that $P_\mu P^\mu=M^2>0$, Eq. (1) becomes

$$(\alpha_1\Gamma_0M+\alpha_2M^2+\alpha_3SM^2+\beta S+\gamma)|\tilde{\psi}(M)\rangle=0 \ . \tag{3}$$

Performing the "tilting" transformation to remove the noncompact generator S we obtain

$$\{[(\alpha_1M)^2-(\alpha_3M^2+\beta)^2]^{1/2}\Gamma_0+\alpha_2M^2+\gamma\}|\psi(M)\rangle=0 \ . \tag{4}$$

Taking $|\psi\rangle$ to belong to the most degenerate unitary irreducible representation of SO(4,2) such that $|\psi\rangle=|n\rangle$ and $\Gamma_0|n\rangle=n|n\rangle$, we obtain the mass spectrum

$$(M_n{}^{+-})^2=-\frac{\alpha_2\gamma+\alpha_3\beta n^2-\frac{1}{2}\alpha_1{}^2n^2}{\alpha_2{}^2+\alpha_3{}^2n^2}\pm\frac{[(\alpha_2\gamma+\alpha_3\beta n^2-\frac{1}{2}\alpha_1{}^2n^2)^2-(\alpha_2{}^2+\alpha_3{}^2n^2)(\gamma^2+\beta^2n^2)]^{1/2}}{\alpha_2{}^2+\alpha_3{}^2n^2} \ . \tag{5}$$

With the coefficients chosen as

$$\alpha_1=1,\quad \alpha_2=\frac{e^2}{2m_2},\quad \alpha_3=(2m_2)^{-1}\ ,$$
$$\beta=(m_2^2-m_1^2)/2m_2,\quad \gamma=-e^2(m_1^2+m_2^2)/2m_2\ , \tag{6}$$

Eq. (5) becomes

$$(M_n^{\pm})^2=m_1^2+m_2^2\pm 2m_1m_2\left[1+\frac{e^4}{n^2}\right]^{-1/2}\ , \tag{7}$$

where m_1 and m_2 are taken as the masses of the electron and proton, respectively. As has been shown elsewhere,[6] M_n^+ yields the usual Dirac formula (with spin suppressed) plus the correction for proton recoil.

In a similar fashion Eq. (3) may be diagonalized for the noncompact generator S. Since S has a continuous spectrum we take $|\psi\rangle=|\nu\rangle$ such that $S|\nu\rangle=\nu|\nu\rangle$, where $-\infty<\nu<\infty$. This yields the continuous timelike spectrum

$$(M_\nu^{\pm})^2=m_1^2+m_2^2\pm 2m_1m_2\left[1-\frac{e^4}{\nu^2}\right]^{-1/2}\ . \tag{8}$$

Note that Eq. (8) may be obtained from Eq. (7) by analytic continuation $n\rightarrow i\nu$.

The mass spectra from Eqs. (7) and (8) are displayed in Fig. 1, where it should be noticed that the spectrum for the region $0<\nu^2<e^4$ is missing. This is because $(M_\nu^{\pm})^2$ becomes complex there and we have only included the real part of the spectrum in Fig. 1.

Before we discuss the significance of this complex spectrum, let us first say something about the $\pm$ ambiguity of Eqs. (7) and (8). From Eq. (8) we have that $(M_\nu^-)^2<0$ for $e^4<\nu^2<\nu_0^2$, where

$$\nu_0=\pm e^2\frac{m_2^2+m_1^2}{m_2^2-m_1^2}\ .$$

This implies that $(M_\nu^-)^2$ in this region becomes spacelike, which contradicts the original assumption of timelike four-momentum. Thus we must discard the $(M_\nu^-)^2$ solution.

Now writing $(M_\nu^+)^2$ as

$$(M_\nu^+)^2=m_1^2+m_2^2-2im_1m_2\left[\frac{e^4}{\nu^2}-1\right]^{-1/2}\ , \tag{9}$$

we see that the spectrum is complex for $0<\nu^2<e^4$. With $s=(M_\nu^+)^2$ we obtain $s^{1/2}=m-i\Gamma$, where

$$m\equiv \mathrm{Re}M_\nu^+=[\tfrac{1}{2}|(M_\nu^+)^2|+\tfrac{1}{2}(m_1^2+m_2^2)]^{1/2} \tag{10}$$

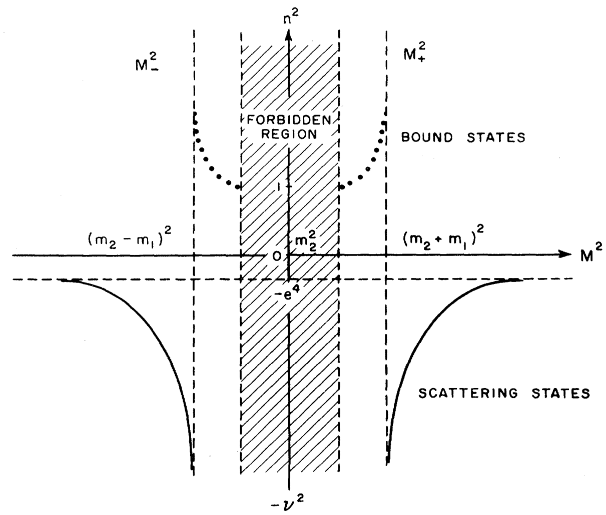

FIG. 1. Different branches of the mass spectrum of Eq. (1) as a function of the square of the principal quantum number. Note the gap $0<\nu^2<e^4$.

and

$$\Gamma=-\mathrm{Im}M_\nu^+=[\tfrac{1}{2}|(M_\nu^+)^2|-\tfrac{1}{2}(m_1^2+m_2^2)]^{1/2}\ . \tag{11}$$

Thus we obtain a continuous series of unstable composite particles, (ep^+) resonances, whose decay times in the rest frame are given by $\tau=[2\Gamma]^{-1}$.

Finally we remark that had we retained the $(M_\nu^-)^2$ branch of the spectrum, we would have obtained $s^{1/2}=m+i\Gamma$, which would have yielded a physically unacceptable exponentially increasing amplitude.

It is interesting to note that a series of *discrete* resonances also occurs in the classical relativistic scattering of two dyons.[9] This implies that our results here may also have relevance for the infinite-component wave equation model of the proton[10] whose physical picture can be as a bound state of two dyons. Such scattering spectra should exist for that equation as for the H-atom equation. In conclusion, it may be possible to test experimentally whether such complex resonance energies may be discerned from the real-energy scattering states.

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[1]A. Sommerfeld, *Atomic Structure and Spectral Lines* (Methuen, London, 1934), p. 256, Eq. (26).

[2]See rapporteur's talk, A. O. Barut, in *Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, U.S.S.R., 1970* (Naukova Dumka, Kiev, 1972), pp. 454–495; Y. Nambu, Phys. Rev. 160, 1171 (1967).

[3]See A. O. Barut, *Dynamical Groups and Generalized Symmetries in Quantum Theory* (University of Canterbury, Christchurch, 1972).

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