Comment on "Quantization in the temporal gauge"

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We show that the difficulties encountered by Kakudo, Taguchi, Tanaka, and Yamamoto in the canonical quantization of gauge fields in the temporal gauge stem only from insufficient care in their handling of eigenvectors belonging to continuous spectra.

In a recent paper on the quantization of gauge theories,[1] Kakudo, Taguchi, Tanaka, and Yamamoto (KTTY) have questioned the validity of the canonical quantization procedure in the temporal gauge. In this Comment on their paper we show that the inconsistency of the commutation relations found by KTTY is due entirely to forbidden mathematical manipulations of expressions involving non-normalizable eigenvectors. Also, in view of growing interest in the temporal gauge, we set the record straight concerning the history of this important method of quantization.

We start with a brief summary of the "contradictions" found by KTTY. They consider the commutation relations between the generators of gauge transformations $G^a(\vec{x})$ and the vector potential of the gauge field $A_j^b(\vec{y})$,

$$[G^a(\vec{x}),A_j^b(\vec{y})]=i[\delta_{ab}\partial_j+gf_{abc}A_j^c(x)]\delta(\vec{x}-\vec{y}) \quad . \tag{1}$$

Next, they sandwich these relations between two state vectors obeying the subsidiary conditions

$$G^a(\vec{x})|\alpha\rangle=0 \quad . \tag{2}$$

They find that the left-hand side vanishes, whereas the right-hand side is, in general, different from zero. At this point they conclude that the subsidiary conditions (2) are invalid and that "in this way it becomes clear that we have no knowledge of the treatment of the Gauss's law, one of the fundamental equations. Therefore further discussions on the canonical formalism in this gauge seem to be meaningless."

In our view, further discussions are not only meaningful, but also enlightening, since they show how one may easily be led astray by formal manipulations of operator expressions in infinite-dimensional spaces.

There is a natural tendency among the great majority of physicists to disregard warnings voiced by the mathematically sophisticated minority and to treat operators and vectors in the Hilbert space as if they were finite-dimensional arrays subject to the rules of ordinary matrix algebra. While on the average these rules work well, from time to time unprepared researchers fall into a trap. After all, functional analysis is not equivalent to linear algebra.

In order to show where KTTY erred, we do not have to stick to non-Abelian gauge theories. We can find the same "contradiction" in quantum electrodynamics and even in elementary quantum mechanics. We shall explain the whole problem in a simple case of two particles, mutually interacting via some forces, described by the Hamiltonian

$$H=\vec{p}_1^{\,2}/2m_1+\vec{p}_2^{\,2}/2m_2+V(\vec{r}_1-\vec{r}_2) \quad . \tag{3}$$

To set the scene for the appearance of the same difficulty as the one encountered by KTTY, let us write down the commutation relations between the operators of the total momentum of the system and the position of either particle,

$$[P_i,r_j^A]=-i\hbar\delta_{ij}, \quad A=1,2 \quad . \tag{4}$$

Next, let us consider state vectors describing the two-particle system in their rest frame, i.e., the vectors obeying the subsidiary condition

$$\vec{P}|\alpha\rangle=0 \quad . \tag{5}$$

Sandwiching both sides of Eq. (4) between two such state vectors, describing different internal configurations of the system, we obtain

$$0=-i\hbar\delta_{ij}\langle\alpha'|\alpha''\rangle \quad . \tag{6}$$

This is the quantum-mechanical analog of the relation obtained by KTTY, which led them to question the validity of the canonical quantization procedure. Where did we go wrong in the derivation of (6)? Obviously, in treating the eigenvectors $|\alpha\rangle$ as if they were normalizable vectors in the Hilbert space. Of course, they are not. The operator $\vec{P}$ has a purely continuous spectrum. Therefore the subsidiary condition (5) forces all vectors obeying it to have infinite norm and hence to lie outside of the Hilbert space. Wave functions corresponding to such generalized eigenvectors are, however, well defined and are used all the time in quantum mechanics. For example, the wave function describing the bound state of the hydrogen atom belongs to this category. It is a square-integrable function of the relative coordinates, but since it does not depend on the center-of-mass coordinates, its norm is infinite in the space of two-particle states.

We have given such a detailed analysis of these elementary and well-known facts, because every statement made above about the two-particle system has its counterpart in the theory of gauge fields. In order to exhibit fully this analogy, we give in Table I a list of the corresponding notions and properties.

There exists a natural and elegant mathematical construction, which is best suited to describe the structure of the Hilbert space in relation to the operators defining the subsidiary condition. This construction employs the decomposition of the Hilbert space into a direct integral[2] associated with a continuous-spectrum operator. Each component of such a decomposition, called fiber, is a Hilbert space by itself. Fibers are labeled by the eigenvalues of the operators used to define the decomposition.

In our quantum-mechanical example the decomposition of the two-particle Hilbert space is defined by the total

TABLE I. Correspondence between quantum mechanics and gauge field theory.

Quantum mechanics	Gauge field theory
Canonical operators of individual particles $\vec{r}^A, \vec{p}^A$.	Canonical field operators in the temporal gauge $\vec{A}^a, \vec{E}^a$.
Center-of-mass degrees of freedom.	Gauge degrees of freedom.
Hamiltonian does not depend on center-of-mass coordinates and so it commutes with total momentum operator $\vec{P}$.	Hamiltonian does not depend on gauge degrees of freedom and so it commutes with generators of gauge transformations $G^a(\vec{x})$.
Operator $\vec{P}$ has continuous spectrum and its eigenvectors are not normalizable.	Generators G^a have continuous spectra and their eigenvectors are not normalizable.
Wave functions obeying subsidiary condition (5) may be normalized only if their dependence on center-of-mass coordinates is ignored, i.e., when they are viewed as functions of relative coordinates alone.	Wave functionals obeying subsidiary condition (2) may be normalized only if their dependence on gauge degrees of freedom is ignored, i.e., when they are viewed as functionals of physical variables alone.

momentum operator; fibers are labeled by the momentum vector. Each fiber is the Hilbert space of all state vectors describing the motion of the system characterized by a given total momentum. Clearly, such vectors are not normalizable in the original Hilbert space.

In the simplest gauge theory, in sourceless quantum electrodynamics, the decomposition of the Hilbert space is defined by the operator $\nabla \cdot \vec{E}(\vec{x})$. The spectrum of this operator is continuous[3] and the eigenvalues are labeled by a function of $\vec{x}$, say, $q(\vec{x})$, which can be interpreted as a background charge density. The physical states form the fiber corresponding to $q(\vec{x})=0$.

In non-Abelian gauge theories the situation is qualitatively the same, although the noncommutativity of the generators G^a leads to some technical complications. In general, the fibers defined by different G's do not coincide, but there exists one common fiber—the space of physical state vectors—which belongs to the zero eigenvalue of each generator. Wave functionals which describe physical states can be parametrized in such a way that the gauge degrees of freedom do not appear[4] and hence become normalizable in the physical Hilbert space.

Finally, to conclude our Comment we would like to point out the true origins of the temporal gauge. This gauge is almost as old as quantum field theory itself. It was successfully introduced in the second classic paper by Heisenberg and Pauli.[3] For that reason it would be appropriate to call it the Heisenberg-Pauli gauge (HP gauge). In the quoted paper, not only is the HP gauge introduced and used, but also the role of the operators defining the subsidiary condition as the generators of gauge transformations is explicitly stated and the continuous nature of their spectrum is noted. We believe that the present-day authors, writing about the quantization of gauge theories, should properly acknowledge the contribution to this subject by the founding fathers of quantum field theory.

[1]Y. Kakudo, Y. Taguchi, A. Tanaka, and K. Yamamoto, Phys. Rev. D 27, 1954 (1983).
[2]See, for example, M. Reed and B. Simon, *Methods of Modern Mathematical Physics. Analysis of Operators* (Academic, New York, 1978), p. 280.
[3]W. Heisenberg and W. Pauli, Z. Phys. 59, 168 (1930).
[4]J. Goldstone and R. Jackiw, Phys. Lett. 74B, 81 (1978).