# Canonical commutation relations and Gauss's law in the temporal gauge

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An apparent inconsistency of the temporal gauge is shown to be inconsistent.

From time to time there appears in the literature[1] the claim that in the temporal gauge $A_0=0$ there is an inconsistency between the canonical commutation relations[2]

$$[A_l^a(\vec{x}),\dot{A}_j^b(\vec{x}')]=i\delta_{ab}\delta_{lj}\delta(\vec{x}-\vec{x}') \tag{1}$$

and the Gauss's law

$$D_l^{ab}(\vec{A})\dot{A}_l^b(\vec{x})|\psi_{\rm ph}\rangle=[\delta^{ab}\partial_l-gf^{abc}A_l^c(\vec{x})]\dot{A}_l^b(\vec{x})|\psi_{\rm ph}\rangle=0\ . \tag{2}$$

The states which obey (2) are those which represent physical systems. The *wrong* argument looks very simple and goes essentially as follows. One takes the covariant divergence of Eq. (1),

$$[A_l^a(\vec{x}),D_j^{bc}(\vec{A})\dot{A}_j^c(\vec{x}')]=iD_{x_l'}^{ab}(\vec{A})\delta(\vec{x}-\vec{x}')\ , \tag{3}$$

and sandwiches Eq. (3) between physical states:

$$\langle\Phi_{\rm ph}|[A_l^a(\vec{x}),D_j^{bc}(\vec{A})\dot{A}_j^c(\vec{x}')]|\psi_{\rm ph}\rangle=i\langle\Phi_{\rm ph}|D_{x_l'}^{ab}(\vec{A})\delta(\vec{x}-\vec{x}')|\psi_{\rm ph}\rangle\ . \tag{4}$$

Since both $|\Phi_{\rm ph}\rangle$ and $|\psi_{\rm ph}\rangle$ satisfy Eq. (2), the left-hand side (LHS) of (4) is apparently zero in contrast with the right-hand side (RHS).

In this note we want to show that in the framework of the general formulation of the temporal gauge given in Ref. 3 this paradox does not arise.

To start with, let us briefly recall the points of Ref. 3 relevant for this discussion. As is well known the set of states $\mathscr{H}_{\rm ph}$, which satisfy Eq. (2), consists of the states which are invariant under the group $\mathscr{G}_0$ of the local, time-independent gauge transformations (of zero winding number) which go to the identity of spatial infinity. Their infinitesimal generators are precisely the Gauss's law operators, $D_l^{ab}(\vec{A})\dot{A}_l^b(\vec{x})$. $\mathscr{H}_{\rm ph}$ is a subspace of the larger space $\mathscr{H}$ (which describes systems in the presence of arbitrary external sources[3] spanned by the eigenstates of the Hamiltonian

$$H=\int d\vec{x}\left[-\frac{1}{2}\frac{\delta^2}{\delta A_l^a(\vec{x})\delta A_l^a(\vec{x})}+\tfrac{1}{4}F_{lj}^aF_{lj}^a\right]\ . \tag{5}$$

In this representation of $\mathscr{H}$, $A_l^a(\vec{x})$ is a multiplicative operator and

$$\dot{A}_l^a(\vec{x})\to -i\frac{\delta}{\delta A_l^a(\vec{x})}\ . \tag{6}$$

The scalar product in $\mathscr{H}$ is obviously defined by

$$\langle\Phi|\psi\rangle=\int\delta\vec{A}(\vec{x})\Phi^*(\vec{A})\psi(\vec{A})\ . \tag{7}$$

When restricted to states in $\mathscr{H}_{\rm ph}$, Eq. (7) is always infinite due to the gauge invariance of $\Phi(\vec{A})$ and $\psi(\vec{A})$. This simply means that the physical states are not normalizable in $\mathscr{H}$. One has to remember, however, that in every spatial gauge the metric in the space of states is induced by the convolution property of the Feynman propagation kernel

$$K(\vec{A}_2,\vec{A}_1;T_1+T_2)=\int\delta\vec{A}\mu(\vec{A})K(\vec{A}_2,\vec{A};T_2)K(\vec{A},\vec{A}_1;T_1)\ . \tag{8}$$

This follows from the expansion

$$K(\vec{A}_2,\vec{A}_1;T)=\sum_\gamma e^{-E_\gamma T}\psi_\gamma(\vec{A}_2)\psi_\gamma^*(\vec{A}_1)\ , \tag{9}$$

where the $\psi_\gamma$'s are the eigenstates of the Hamiltonian within the particular gauge one is considering and the $E_\gamma$'s are the corresponding eigenvalues.

In the temporal gauge it has been shown in Ref. 3 that the measure $\mu(\vec{A})$ is $\vec{A}$ independent and is precisely given by the inverse of the (infinite) volume of the group $\mathscr{G}_0$:[4]

$$\mu(\vec{A})=\frac{1}{\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})}\ . \tag{10}$$

When restricted to physical states, Eq. (7) then reads

$$\langle\Phi_{\rm ph}|\psi_{\rm ph}\rangle=\int\delta\vec{A}\mu(\vec{A})\Phi_{\rm ph}^*(\vec{A})\psi_{\rm ph}(\vec{A})=\frac{\int\delta\vec{A}\Phi_{\rm ph}^*(\vec{A})\psi_{\rm ph}(\vec{A})}{\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})}\ . \tag{11}$$

In Eq. (11) the ratio of two infinite quantities has to be defined by the usual Faddeev-Popov trick. One must then insert the following into (11):

$$1=\Delta_F(\vec{A})\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})\delta[F(\vec{A}^{U_h})]\ . \tag{12}$$

This gives

$$\langle\Phi_{\rm ph}|\psi_{\rm ph}\rangle=\int\delta\vec{A}\mu_F(\vec{A})\Phi_{\rm ph}^*(\vec{A})\psi_{\rm ph}(\vec{A})\ , \tag{13a}$$

$$\mu_F(\vec{A})\equiv\Delta_F(\vec{A})\delta[F(\vec{A})]\ , \tag{13b}$$

where $F(\vec{A})=0$ is any spatial-gauge condition.[5]

 

After these preliminaries, let us come back to the consistency of Eqs. (1) and (2).

To this end let us consider the matrix element of the gauge-invariant operator:

$$\Gamma \equiv \left[ D_j^{ac}\frac{\delta}{\delta A_j^c(\vec{x}')}, A_i^b(\vec{x}) \right] + g f^{abc} A_i^c(\vec{x})\delta(\vec{x}-\vec{x}') = \delta_{ab}\partial_{x_i'}\delta(\vec{x}-\vec{x}') \tag{14}$$

between two physical states $\psi_{\rm ph}(\vec{A})$ and $\Phi_{\rm ph}(\vec{A}) \in \mathscr{H}_{\rm ph}$. According to (13) one has to write

$$\langle \Phi_{\rm ph}|\Gamma|\psi_{\rm ph}\rangle = \int \delta\vec{A}\mu_F(\vec{A})\Phi^*_{\rm ph}(\vec{A}) \left\{ \left[ D_j^{ac}\frac{\delta}{\delta A_j^c(\vec{x}')}, A_i^b(\vec{x}) \right] + g f^{abc} A_i^c(\vec{x})\delta(\vec{x}-\vec{x}') \right\} \psi_{\rm ph}(\vec{A})$$

$$= \delta_{ab}\partial_{x_i'}\delta(\vec{x}-\vec{x}') \int \delta\vec{A}\mu_F(\vec{A})\Phi^*_{\rm ph}(\vec{A})\psi_{\rm ph}(\vec{A}) \ . \tag{15}$$

Because of gauge invariance, of course, both sides of this equation are independent of the particular condition $F(\vec{A})$ chosen.

Using Eq. (2) and functional integration by parts, Eq. (15) can be transformed into

$$-\int \delta\vec{A}\Phi^*_{\rm ph}(\vec{A}) \left[ D_{x_j'}^{ac}\frac{\delta\mu_F(\vec{A})}{\delta A_j^c(\vec{x}')} \right] A_i^b(\vec{x})\psi_{\rm ph}(\vec{A}) = \int \delta\vec{A}\mu_F(\vec{A})\Phi^*_{\rm ph}(\vec{A}) D_{x_i'}^{ab}(\vec{A})\delta(\vec{x}-\vec{x}')\psi_{\rm ph}(\vec{A}) \ . \tag{16}$$

If the measure $\mu_F(\vec{A})$ were not present in Eq. (16), we would be back to the contradiction between the Gauss's law and the canonical commutation relations which we have sketched before.

One should notice that the problem already arises when $g=0$. In other words, the presence of a nontrivial measure in (16) is necessary even in free QED.

The validity of Eq. (16) can be explicitly checked in perturbation theory.

Let us show, as an example, how it works at the level of free QED taking $F(\vec{A})=\partial_i A_i$ and

$$\psi_{\rm ph}(\vec{A}) = \Phi_{\rm ph}(\vec{A}) = \psi_\Omega(\vec{A}) \ ,$$

where $\psi_\Omega(\vec{A})$ is the free vacuum functional:

$$\psi_\Omega(\vec{A}) = \mathscr{N}\exp\left[ -\tfrac{1}{2}\int d\vec{x}\, d\vec{x}' A_i(\vec{x}) G_{ij}^T(\vec{x}-\vec{x}') A_j(\vec{x}') \right] \ , \tag{17a}$$

$$G_{ij}^T(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^3}\int d\vec{p}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')}(\vec{p}^{\,2})^{1/2}\left[ \delta_{ij} - \frac{p_i p_j}{\vec{p}^{\,2}} \right] \ . \tag{17b}$$

To check the validity of Eq. (16) it is convenient to go one step back in order to be able to use the Fourier representation for $\delta(\partial_i A_i)$. We then start with the formula (15), whose LHS in this case becomes

$$\langle \psi_\Omega|\Gamma|\psi_\Omega\rangle = \int \delta\vec{A}\delta(\partial_i A_i)\psi^*_\Omega(\vec{A}) \left[ \partial_j\frac{\delta}{\delta A_j(\vec{x}')}, A_i(\vec{x}) \right] \psi_\Omega(\vec{A})$$

$$= \int \delta\vec{A}\delta(\partial_i A_i)\psi^*_\Omega(\vec{A})\partial_j\frac{\delta}{\delta A_j(\vec{x}')}[A_i(\vec{x})\psi_\Omega(\vec{A})] \ . \tag{18}$$

We now use the $\delta$ function in order to eliminate in $\psi^*_\Omega(\vec{A})$ the terms proportional to $p_i p_j/\vec{p}^{\,2}$:

$$\delta(\partial_i A_i)\psi^*_\Omega(\vec{A}) = \delta(\partial_i A_i)\mathscr{N}\exp\left[ -\tfrac{1}{2}\int d\vec{x}\, d\vec{x}' A_i(\vec{x}) G(\vec{x}-\vec{x}') A_i(\vec{x}') \right] \ , \tag{19a}$$

$$G(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^3}\int d\vec{p}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')}(\vec{p}^{\,2})^{1/2} \ . \tag{19b}$$

In this way we will explicitly have an exponential damping in the longitudinal part of $\vec{A}$.

Integrating by parts and using the Fourier representation of the $\delta$ function, we get

$$\langle \psi_\Omega|\Gamma|\psi_\Omega\rangle = -\int \delta\vec{A}\delta\lambda \left[ \partial_{x_j'}\frac{\delta}{\delta A_j(\vec{x}')}\exp\left[ i\int \lambda(\vec{y})\partial_i A_i(\vec{y})d\vec{y} - \tfrac{1}{2}\int A_i(\vec{y}) G(\vec{y}-\vec{y}') A_i(\vec{y}') d\vec{y}\, d\vec{y}' \right] \right]$$

$$\times A_i(\vec{x})\exp\left[ -\tfrac{1}{2}\int A_i(\vec{y}) G_{ij}^T(\vec{y}-\vec{y}') A_j(\vec{y}') d\vec{y}\, d\vec{y}' \right]$$

$$= -\int \delta\vec{A}\delta\lambda \left[ -i\nabla_{\vec{x}'}^2\lambda(\vec{x}') - \partial_{x_j'}\int G(\vec{x}'-\vec{y}') A_j(\vec{y}') d\vec{y}' \right] A_i(\vec{x})$$

$$\times \exp\left[ i\int \lambda(\vec{y})\partial_i A_i(\vec{y})d\vec{y} - \tfrac{1}{2}\int A_i(\vec{y})[G(\vec{y}-\vec{y}')\delta_{ij} + G_{ij}^T(\vec{y}-\vec{y}')]A_j(\vec{y}') d\vec{y}\, d\vec{y}' \right] \ . \tag{20}$$

The second term in the second equality of Eq. (20) is zero because of the δ function itself; the first term is a Gaussian integral which is easily checked to give exactly $\partial_{x_i'}\delta(\vec{x}-\vec{x}')$ as expected [see Eq. (15)].

The last thing we want to show is that the introduction of a metric in the definition of the scalar product does not destroy the Hermiticity of physical operators. More precisely we want to show that if an operator $\mathscr{O}$ is Hermitian in the large space $\mathscr{H}$,

$$\int \Phi_2^*(\vec{A})[\mathscr{O}\Phi_1(\vec{A})]\delta\vec{A}=\int [\mathscr{O}\Phi_2(\vec{A})]^*\Phi_1(\vec{A})\delta\vec{A}, \quad \Phi_1,\Phi_2\in\mathscr{H} \tag{21}$$

and gauge invariant (i.e., if it commutes with Gauss's law), then it is Hermitian also with respect to the scalar product in $\mathscr{H}_{\rm ph}$ as defined in Eq. (13a), i.e.,

$$\int \psi_2^*(\vec{A})[\mathscr{O}\psi_1(\vec{A})]\mu_F(\vec{A})\delta\vec{A} =\int [\mathscr{O}\psi_2(\vec{A})]^*\psi_1(\vec{A})\mu_F(\vec{A})\delta\vec{A}, \quad \psi_1,\psi_2\in\mathscr{H}_{\rm ph}\ . \tag{22}$$

This theorem is intuitively obvious. In fact, if $\mathscr{O}$ commutes with the Gauss's law, $\mathscr{O}\psi_1(\vec{A})$ and $\mathscr{O}\psi_2(\vec{A})$ are gauge invariant, if $\psi_1(\vec{A})$ and $\psi_2(\vec{A})$ are.

By the Faddeev-Popov procedure the same infinite volume is extracted from both members of Eq. (21), ending with Eq. (22). More precisely we can always write the following for an $\mathscr{O}$ Hermitian in $\mathscr{H}$:

$$\int \psi_2^*(\vec{A})[\mathscr{O}\psi_1(\vec{A})]\mu_F(\vec{A})\delta\vec{A} =\int \{\mathscr{O}[\psi_2(\vec{A})\mu_F(\vec{A})]\}^*\psi_1(\vec{A})\delta\vec{A}\ . \tag{23}$$

We now introduce the identity (12) in the RHS and obtain

$$\int \{\mathscr{O}[\psi_2(\vec{A})\mu_F(\vec{A})]\}^*\psi_1(\vec{A})\delta\vec{A} \tag{24a}$$

$$=\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})\int \{\mathscr{O}[\psi_2(\vec{A})\mu_F(\vec{A})]\}^*\psi_1(\vec{A})\mu_F(\vec{A}^{U_h})\delta\vec{A} \tag{24b}$$

$$=\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})\{\mathscr{O}[\psi_2(\vec{A}^{U_h^+})\mu_F(\vec{A}^{U_h^+})]\} \times {}^*\psi_1(\vec{A}^{U_h^+})\mu_F(\vec{A})\delta\vec{A} \tag{24c}$$

$$=\int_{\mathscr{G}_0}\mathscr{D}h(\vec{x})\{\mathscr{O}[\psi_2(\vec{A})\mu_F(\vec{A}^{U_h^+})]\}^*\psi_1(\vec{A})\mu_F(\vec{A})\delta\vec{A} \tag{24d}$$

$$=\int [\mathscr{O}\psi_2(\vec{A})]^*\psi_1(\vec{A})\mu_F(\vec{A})\delta\vec{A}\ , \tag{24e}$$

where the change of variables $\vec{A}\to\vec{A}^{U_h^+}$ has been performed to go from (24b) to (24c). The gauge invariance of $\mathscr{O}$, $\psi_1$, and $\psi_2$ has been used from (24c) to (24d). The last equality is obtained using again Eq. (12).

In conclusion, when one takes proper care of the definition of the scalar product in the space of physical states no contradictions arise within the temporal gauge.

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[1]Y. Kakuda, Y. Taguchi, A. Tanaka, and K. Yamamoto, Phys. Rev. D 27, 1954 (1983). This is the last paper on this subject.

[2]We work in the Schrödinger representation at fixed time $t=0$.

[3]G. C. Rossi and M. Testa, Nucl. Phys. B163, 109 (1980); B176, 477 (1980); Nucl. Phys. B (to be published).


[4]We use for the time-independent gauge transformations the notation $U_h=\exp[i\lambda^a h^a(\vec{x})]$, with $\mathrm{Tr}(\lambda^a\lambda^b)=\delta^{ab}/2$. $\mathscr{D}h(\vec{x})$ is the invariant measure over the group $\mathscr{G}_0$.

[5]More precisely (see Ref. 3), $F(\vec{A})$ must be such that the equation $F(\vec{A}^{U_h})=0$, $U_h\in\mathscr{G}_0$, has only one solution apart from possible discrete Gribov-type ambiguities.