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Magnetic monopoles and fractional Witten indices

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The monopole-fermion system is shown to provide an example of a fractional Witten index. The discontinuity exhibited by the massless case is reminiscent of that for the η invariant.

Witten[1] has given a powerful criterion for supersymmetry breaking based on the quantity

$$\Delta_\beta = \mathrm{Tr}(-1)^F f(\beta H) \ , \tag{1}$$

where F is fermion number, H is the Hamiltonian, and f is a suitable regulator with

$$f(0)=1, \ \ f(\infty)=0 \ . \tag{2}$$

If the spectrum of H is discrete, Δ_β is independent of β and is equal to

$$\Delta = n_0(F=\text{even}) - n_0(F=\text{odd}) \ , \tag{3}$$

where n_0 is the number of zero-energy states. In particular, if $\Delta \neq 0$, supersymmetry must be unbroken. The index Δ has a topological significance: In terms of the Nicolai map,[2] it is just the winding number.[3]

If the spectrum of H is continuous, Δ_β is no longer independent of β. However, under suitable conditions $\lim_{\beta\to\infty}\Delta_\beta$ is still equal to Δ, provided n_0 is interpreted as the number of normalizable zero-energy states.[4]

Recently, it has been suggested that there are cases where even this weaker equality fails, and $\lim_{\beta\to\infty}\Delta_\beta$ becomes fractional.[5] Here, we wish to show that a system which has been studied in another context, namely, a massless Dirac particle interacting with an Abelian monopole,[6-9] provides such an example. Specifically, we take

$$Q_1 = \gamma_5 \vec{\sigma}\cdot(-i\vec{\nabla} - e\vec{A}) \ , \tag{4}$$

$$Q_2 = \beta \vec{\sigma}\cdot(-i\vec{\nabla} - e\vec{A}) \ , \tag{5}$$

$$H = (-i\vec{\nabla} - e\vec{A})^2 - e\,\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A}) \ , \tag{6}$$

$$(-1)^F = \rho_2 \equiv i\beta\gamma_5 \ , \tag{7}$$

where $\vec{A}$ is the vector potential for an Abelian monopole with strength $N(>0)$ in Dirac units.[10]

The operators (4)–(7) formally satisfy the supersymmetry algebra[11,12]

$$\{Q_i, Q_j\} = 2\delta_{ij} H \ , \tag{8}$$

$$\{(-1)^F, Q_i\} = 0 \ . \tag{9}$$

However, to make them well defined, it is necessary to impose a boundary condition[6] in the lowest partial wave $j=(N-1)/2$. For Q_1, the requirement is

$$\lim_{r\to 0} r\psi(\vec{x}) \propto \zeta(\theta)\eta_{jm}(\Omega), \ \ \zeta(\theta) \equiv \begin{pmatrix} i\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right) \\ \cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right) \end{pmatrix} \tag{10}$$

in the notation of Ref. 8. The action of ρ_2 preserves the boundary condition only for $\theta = 0, \pi$, and we find that the supersymmetry algebra (8), (9) is actually obeyed only with $Q_2 = -i\rho_2 Q_1$ $(\theta = 0, \pi)$.

For $\theta = 0$, the eigenfunctions of H for the lowest partial wave are

$$\rho_2 = +1: \ \chi_k^B(r) = \frac{1}{\sqrt{2}}[u_{k0}(r) - v_{k0}(r)] = \begin{pmatrix} i \\ -1 \end{pmatrix} \sin kr \ , \tag{11}$$

$$\rho_2 = -1: \ \chi_k^F(r) = \frac{1}{\sqrt{2}}[u_{k0}(r) + v_{k0}(r)] = \begin{pmatrix} i \\ 1 \end{pmatrix} \cos kr \ , \tag{12}$$

$$H\frac{1}{r}\chi_k(r)\eta_{jm}(\Omega) = k^2 \frac{1}{r}\chi_k(r)\eta_{jm}(\Omega) \ \ (-j \leq m \leq j) \ , \tag{13}$$

with

$$\int_0^\infty dr u_{k0}^\dagger(r) u_{k'0}(r) = \int_0^\infty dr v_{k0}^\dagger(r) v_{k'0}(r) = \pi\delta(k-k') \ . \tag{14}$$

There are no normalizable zero modes and hence $\Delta = 0$. On the other hand, the continuous spectrum extends down to zero, with zero itself being included only for $\rho_2 = -1$.[13]

It is now easy to calculate Δ_β, since higher partial waves do not contribute owing to γ_5 conservation. We have

$$\begin{aligned} \Delta_\beta(\theta=0) &= \frac{2j+1}{\pi}\int_0^\infty dr \int_0^\infty dk [\chi_k^{B\dagger}(r)\chi_k^B(r) - \chi_k^{F\dagger}(r)\chi_k^F(r)] f(\beta k^2) \\ &= -\frac{2N}{\pi}\int_0^\infty dr \int_0^\infty dk \cos 2kr f(\beta k^2) \\ &= -\frac{2N}{\sqrt{\beta}}\int_0^\infty dr \delta_f\left(\frac{2r}{\sqrt{\beta}}\right) \ , \end{aligned} \tag{15}$$

where δ_f is a regularized delta function. Evidently,

$$\Delta_\beta(\theta=0) = -\frac{N}{2} \neq \Delta \ . \tag{16}$$

Similarly, we find

$$\Delta_\beta(\theta=\pi) = \frac{N}{2} \neq \Delta \ . \tag{17}$$

The appearance of N suggests that $\lim_{\beta\to\infty}\Delta_\beta$ may have a topological significance, even if it is not equal to Δ. [A simple check on this point is obtained by adding the term $\gamma_5 V(r)$ to Q_1.]

Two other remarks are in order. One is what happens if we replace (4)–(6) by

$$Q_1=\gamma_5\vec{\sigma}\cdot(-i\vec{\nabla}-e\vec{A})+\beta M \ , \quad (18)$$

$$Q_2=\beta\vec{\sigma}\cdot(-i\vec{\nabla}-e\vec{A})-\gamma_5 M \ , \quad (19)$$

$$H=(-i\vec{\nabla}-e\vec{A})^2-e\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})+M^2 \ , \quad (20)$$

corresponding to a massive fermion. For $\theta=0$, the spectrum of H will be continuous starting from M^2, and hence supersymmetry will be broken with $\Delta=0$. On the other hand, for $\theta=\pi$, H will have N normalizable zero modes[7] with $\rho_2=1$, and hence supersymmetry is unbroken with $\Delta=N$. Explicit calculation confirms that

$$\lim_{\beta\to\infty}\Delta_\beta(M\neq 0)=\Delta \quad (21)$$

for both cases, and hence $\lim_{\beta\to\infty}\Delta_\beta$ is discontinuous at $M=0$. The situation is similar to that of the η invariant[8,9]

$$\lim_{\beta\to 0}\mathrm{Tr}f(\beta H)\,\mathrm{sgn}Q_1 \ , \quad (22)$$

which is also discontinuous at $M=0$.

The other remark is that Δ_β will not be continuous at $R=\infty$ if we restrict the system to a sphere of radius R, since Δ_β must be integer for a discrete spectrum. This is not surprising since the analogous restriction in field theory would be a cutoff on the *magnitude* of the fields; such procedures are expected to affect Δ. This is to be contrasted with the original procedure[1] (which has no analog in our example), where the restriction is on the spatial domain on which the fields are defined.

Our example also confirms the necessity of great caution when massless particles are involved. Discontinuities may exist, although they are not necessarily pathological. What happens in the infinite-volume limit for a supersymmetric field theory with massless particles is an interesting question,[14] but it is outside the scope of this brief note.

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