Fermions from bosons in 3+1 dimensions through anomalous commutators

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A derivation in the canonical formulation of the Wess-Zumino anomaly is given. This leads naturally to a construction of half-integer-spin, anticommuting operators starting from spin-zero bosonic ones. This fermionic operator creates a state that is a superposition of all half-integer spins. This is related to the effective Lagrangian for QCD.

Skyrme[1] was the first to suggest that the solitons of the chiral model may be fermions. This idea has recently been revived by many authors,[2–5] leading to a synthesis of many aspects of QCD. But an explicit construction of anticommuting operators from those satisfying canonical commutation relations (CCR) has not yet been done. This is one of the aims of this paper. It will be shown that the Wess-Zumino anomaly[6] of the chiral model modifies the equal-time current algebra. The creation operators for bosonic states of soliton number 1, therefore, anticommute.

This construction is modeled on that of Coleman and Mandelstam[7] for the sine-Gordon theory, as interpreted by Segal.[8]

Consider the problem of quantizing sine-Gordon theory. The field variable is an angle so that the configuration space can be thought of as the space Γ of maps $S^1 \to U(1)$. Thus the configuration space is itself a group, under pointwise multiplication of these maps. When the system is quantized one should introduce unitary operators $\hat{U}(g)$ in some Hilbert space which will create the state $|g(x)\rangle$ from the vacuum state $|1\rangle$. These must satisfy

$$|g_1g_2\rangle = \hat{U}(g_1g_2)|1\rangle = \hat{U}(g_1)\hat{U}(g_2)|1\rangle \ . \quad (1)$$

Now, states of a quantum system are represented as *rays* of the Hilbert space. Thus (2) only implies that the $\hat{U}$'s provide a projective representation of the group Γ,

$$\hat{U}(g_1)\hat{U}(g_2) = \omega(g_1,g_2)\hat{U}(g_1g_2) \ . \quad (2)$$

Segal showed that the Mandelstam operators that create fermionic states are part of a nontrivial projective representative of Γ. For, choose

$$U(g) = \exp\left[i\int_{-\infty}^{+\infty}[\hat{\phi}'(x)\chi(x) + \hat{\pi}(x)\chi(x)]dx\right] , \quad (3)$$

where $\hat{\phi}(x)$ and $\hat{\pi}(x)$ satisfy the CCR

$$[\hat{\phi}(x),\hat{\pi}(x)] = i\delta(x-y) \ . \quad (4)$$

These $\hat{U}$'s clearly provide a projective representation of Γ. If we take the limit $\chi(x) \to 2\pi\theta(x-x_0)$ we get

$$\hat{\psi}(x_0) = \exp\left[i\int_{-\infty}^{+\infty}2\pi[-\hat{\phi}'(x)\theta(x-x_0) + \hat{\pi}(x)\theta(x-x_0)]dx\right] , \quad (5)$$

which is Mandelstam's expression for the creation operator for fermions, up to some constant overall factors.

This suggests the generalization to four dimensions. I will show that this also gives a new interpretation of the Wess-Zumino anomaly. Consider a chiral model where the field variable takes values in a compact Lie group G. All finite-energy configurations must satisfy

$$\lim_{|x|\to\infty} g(\hat{x}) \to 1 \ . \quad (6)$$

Thus the configuration space Γ can be thought of as the space of maps $S^3 \to G$. This space has many connected components, labeled by the winding number. In quantum mechanics, as for the sine-Gordon theory, there must be unitary operators providing a projective representation of the group Γ:

$$\hat{U}(g_1)\hat{U}(g_2) = \omega(g_1,g_2)\hat{U}(g_1g_2) \ . \quad (7)$$

Associativity implies that the phase factor ω satisfies the "cocycle" condition

$$\omega(g_1,g_2g_3)\omega(g_2,g_3) = \omega(g_1g_2,g_3)\omega(g_1,g_2) \ . \quad (8)$$

Quite often the phase factor can be factorized,

$$\omega(g_1,g_2) = e^{if(g_1)}e^{if(g_2)}e^{-if(g_1g_2)} , \quad (9)$$

so that it can be transformed away. In a classic paper, Bargmann[9] showed that nontrivial phase factors exist only if $H_2(\Gamma,U(1))$, the secondary cohomology of Γ, is nontrivial. For every element of $H_2(\Gamma)$, there exists a corresponding quantization scheme of the chiral model.

Now, $\Gamma = \{S^3 \to G\}$. Thus we can see that

$$H_2(\Gamma) = H_5(G) \ . \quad (10)$$

The condition for the existence of a nontrivial projective representation is that $H_5(G)$ be nontrivial.

But Witten[4] has shown that this is precisely the condition for the existence of a Wess-Zumino anomaly. I will now show that adding the Wess-Zumino term to the action is equivalent to choosing a nontrivial projective representation for the operators $\hat{U}(g)$.

To see this, remember that the canonical formalism for the chiral model can be done entirely in terms of currents.[2] The equal-time current algebra

$$[I_0^\alpha(x),I_0^\beta(y)] = if_{\alpha\beta\gamma}I_0^\gamma(x)\delta^3(x-y) , \quad (11)$$

$$[I_0^\alpha(x),I_a^\beta(y)]=if_{\alpha\beta\gamma}I_a^\gamma(x)\delta^3(x-y)+i\partial_a\delta^3(x-y)\ , \qquad (12)$$

$$[I_a(x),I_b(y)]=0\ , \qquad (13)$$

along with the Hamiltonian

$$H=\int d^3x[I_0^\alpha(x)I_0^\alpha(x)+I_a^\alpha(x)I_a^\alpha(x)]\ , \qquad (14)$$

give the equations of motion

$$\partial_\mu I_\mu^\alpha=0\ , \qquad (15)$$

$$\partial_\mu I_\nu^\alpha(x)-\partial_\nu I_\mu^\alpha(x)+if_{\alpha\beta\gamma}I_\mu^\beta I_\nu^\gamma=0\ . \qquad (16)$$

I_μ may thus be identified as $g^{-1}\partial_\mu g$. The Wess-Zumino (WZ) anomaly modifies the first of these equations:

$$\partial_\mu I_\mu^\alpha+\lambda\omega_{\alpha\beta\gamma\delta\epsilon}\epsilon_{\mu\nu\rho\sigma}I_\mu^\beta I_\nu^\gamma I_\rho^\delta I_\sigma^\epsilon=0\ , \qquad (17)$$

λ being proportional to the number of colors.[10] Witten has shown how to find an action and a path-integral quantization for this theory. Let us instead look for a canonical formulation. If we leave the Hamiltonian and the commutation relations (12) and (13) invariant, but change (11) to

$$[I_0^\alpha(x),I_0^\beta(y)]=if_{\alpha\beta\gamma}I_0^\gamma(x)\delta^3(x-y)+iN\epsilon_{abc}\omega_{\alpha\beta\gamma\delta\rho}I_a^\gamma I_b^\delta I_c^\rho(x)\delta^3(x-y)\ , \qquad (18)$$

we get the required equation of motion. This is the first result of this paper: the WZ anomaly modifies the current algebra. Such a modification of the current algebra can have experimental consequences. For example, a scattering process $K^+K^-\to\pi^+\pi^-\pi^0$ is predicted.[4] All this is in analogy with the axial anomaly.[11,12]

The modified current algebra $\tilde{\Gamma}$ can be shown to be a central extension of the unmodified one, Γ. For this, let us ignore the Schwinger term for the moment. The Jacobi identity implies that the five-form ω in (18) is closed. But if it had been exact, $\omega=d\xi$, we would have been able to define

$$I_0'^\alpha=I_0^\alpha+N\xi_{\beta\gamma\delta}^\alpha\epsilon_{abc}I_a^\beta I_b^\gamma I_c^\delta\ , \qquad (19)$$

which satisfies nonanomalous commutators. Thus nontrivial modifications of the current algebra are given by closed but not exact five-forms, i.e., by $H_5(G)$.

We can obtain operators that create bosonic states $|g(x)\rangle$ by exponentiating the current algebra,

$$\hat{F}(g)=\exp\left[i\int I_0^\alpha(x)\theta^\alpha(x)dx\right]\ , \qquad (20)$$

where $g(x)=e^{i\lambda^\alpha\theta^\alpha(x)}$.

By the Baker-Campbell-Hausdorff theorem[13]

$$\hat{F}(g_1)\hat{F}(g_2)=\omega(g_1,g_2)\hat{F}(g_1g_2)\ , \qquad (21)$$

where ω is some phase. It is important to remember that I_0, and hence F, will depend implicitly on the point g_0 at which $I_a=g_0{}^{-1}\partial_a g_0$ in (18) is evaluated. I will now find how $\hat{F}(g_0\,|\,g_1)$ changes under a continuous change of the base point g_0 along a curve $g_0(x,t)$:

$$g_0(x,0)=g_0(x)\ ,\qquad g_0(x,1)=g_0'(x)\ . \qquad (22)$$

First one finds how I_0 and F change under an infinitesimal transformation of g_0. The effect of a finite transformation along a curve can be found by integrating

$$F[g_0'\,|\,g_1]=e^{i\chi}F(g_0\,|\,g_1)\ , \qquad (23)$$

where

$$\chi=N\int_0^1 dt\int d^3x\,\omega_{\alpha\beta\gamma\delta\epsilon}\epsilon_{abc}I_a^\gamma I_b^\delta I_c^\epsilon Y^\alpha(\theta_1\,|\,x)\times\left[g_0{}^{-1}\frac{\partial g_0}{\partial t}\right]^\beta\ , \qquad (24)$$

the $Y^\alpha(\theta_1\,|\,x)$ of this equation defined by

$$e^{i\theta^\alpha(x)[\lambda^\alpha-x^\alpha]}=e^{iY^\alpha(\theta\,|\,x)x^\alpha}e^{i\theta^\alpha(x)\lambda^\alpha}\ ,$$

where x^α is some quantity transforming in the adjoint representation,

$$[x^\alpha,x^\beta]=0\ .$$

Thus we find that under smooth changes of g_0, F changes by a phase. Thus the projective representative of Γ provided by F changes "trivially" when g_0 is varied smoothly.[14]

We are now in a position to ask how $F(g_0\,|\,g_1)$ changes under a rotation. Since it is constructed out of spin-zero fields, naively one would expect it to be "invariant,"

$$F[g_1]\to F[\mathrm{Ro}g_1]\ ,$$

where $\mathrm{Ro}g_1$ is the rotated configuration. But under a rotation the base point g_0 also changes so that in the presence of the anomaly

$$F[g_0\,|\,g_1]\to e^{i\chi}F[g_0\,|\,\mathrm{Ro}g_1]\ , \qquad (25)$$

with the path $g_0(x,t)$ representing a rotation. Of special interest are rotations through 2π around any axis.

Equation (24) can be written in a convenient form if g_0 and g_1 can be deformed smoothly into each other. Let $g(x,t,s)$ be defined by

$$g(x,t,-\infty)=g_0(x,t)\ ,\qquad g(x,t,+\infty)=g_1(x)\ , \qquad (26)$$

with $g^{-1}\partial g/\partial s\ll 1$ for all s (i.e., an adiabatic deformation). Then

$$\chi=N\int_{-\infty}^{+\infty}ds\int_0^1 dt\int d^3x\,\omega_{\alpha\beta\gamma\delta\epsilon}\left[g^{-1}\frac{\partial g}{\partial t}\right]^\alpha\left[g^{-1}\frac{\partial g}{\partial t}\right]^\beta\epsilon_{abc}\left[g^{-1}\frac{\partial g}{\partial x^a}\right]^\gamma\left[g^{-1}\frac{\partial g}{\partial x^b}\right]^\delta\left[g^{-1}\frac{\partial g}{\partial x^c}\right]^\epsilon\ . \qquad (27)$$

Now consider a rotation through 2π of F when g is a one-soliton configuration. Witten[4] has evaluated the above integral and has shown that $e^{i\chi}=(-1)^N$. Thus if N, the number of colors, is odd, any state with soliton number 1 changes sign under a 2π rotation, i.e., is of half-integral spin. An operator that creates only spin $\frac{1}{2}$ has to be obtained by projection.

Before doing this, let us show anticommutativity. As will be shown later, it is sufficient to consider configurations that are far separated: i.e., either g_1 or g_2 is equal to one everywhere. Further, let them be such that there is a rotation around some axis through an angle π that interchanges g_1 and g_2. In this case

$$F(g_1)F(g_2)F^{-1}(g_1)F^{-1}(g_2)=e^{i\chi_{\rm I}(g_1)}e^{i\chi_{\rm II}(g_2)}\,, \qquad (28)$$

where $\chi_{\rm I}$ corresponds to a rotation of g_1 to g_2 and $\chi_{\rm II}$ to a rotation of g_2 to g_2. Thus the combined phase factor is again given by the integral (27) to be $(-1)^N$. The creation operators of soliton configurations (satisfying the above conditions) anticommute.

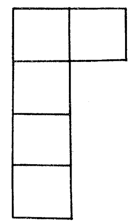
I have thus obtained the analogs of the operators (3) for the chiral model. To get those that create local fermions, we should pick $\theta^\alpha(x)$ to be corresponding to a "point" soliton. Skyrme[1] has given an ansatz for a soliton of size ρ located at some point x_1:

$$g_1(x)=\exp\left[i\lambda^\alpha e_{\alpha a}(x-x_1)2\tan^{-1}\frac{|x-x_1|}{\rho}\right]\,, \qquad (29)$$

where e_α is some rectangular matrix [8×3 if $G=\mathrm{SU}(3)$] of rank at least 3. Since we are interested ultimately in point solitons, we need to consider only configurations (2) with very small ρ. These will be "far separated" in the sense of the last paragraph. Also if g_1 and g_2 are such configurations concentrated around χ_1 and χ_2 there is always a rotation that interchanges them: a rotation through π around the bisector of the line joining them.

Even after substituting (29) into the expressions for F, we will not get the creation operator for a spin-$\frac{1}{2}$ fermion. This is because F creates all possible half-integral spins. Skyrme[1] has shown how to resolve this. We can partial-wave analyze F into a product of operators each creating a particular spin. To write an explicit expression for this is not easy. This and related problems will be discussed in a longer paper.

One can show that the states with spin $\frac{1}{2}$ transform under the



representations of $\mathrm{SU}(N_f)$, when the chiral model is based on a group $G=\mathrm{SU}(N_f)$. [There are (N_c-1) boxed in the first column of the above representation.]

For, consider some fixed SU(2) subgroup with generators T. Under a rotation generated by $\vec{\rm J}$ (angular momentum) of $\vec{\rm T}$ through 2π, the evaluation of the Witten phase factor gives $(-1)^{N_c}$. However, under a rotation through 2π generated by $\vec{\rm J}+\vec{\rm T}$ it is equal to 1. This must be true of every SU(2) subgroup. By arguments basically identical to those of Witten one can then get the representations of the fermions.

What is obtained after the point limit is taken and the projection to spin $\frac{1}{2}$ is done is the "left-handed" part of the fermion operator. By starting from the right-handed isospin current $\partial_\mu g g^{-1}$ (instead of $g^{-1}\partial_\mu g$ as I did) one can get the right-handed components. The right and left components so obtained will commute. Completely anticommuting operators will then have to be obtained by Jordan-Wigner transformation.

This is only a first step toward constructing fermions from bosons. Dynamical questions such as (1) do the ψ's satisfy a Dirac equation and (2) what are their masses and couplings to mesons, are all yet to be understood. Some of these questions will be discussed in forthcoming papers with G. Bhattacharya.

One of the interesting facts to be noted is that the Wess-Zumino anomaly does not exist when the number of flavors is less than 3. Thus the above construction would not work in the case of two flavors. The analogous problem there involves quaternionic projective representations of the group Γ and is therefore much more involved.[2] Thus the case of three of more flavors is simpler to understand than that of two flavors.

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[10]$\omega_{\alpha\beta\gamma\delta\epsilon}=(2/15\pi^2)$ tr $\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta\lambda_\epsilon$. This is a closed but not exact five-form that has been normalized as in Ref. 4. Note that

this is nonzero only for groups $SU(N_f)$, $N_f \geq 3$.

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[12]The Wess-Zumino term makes no contribution to the stress tensor. This is analogous to the particle in a magnetic field. $H=\frac{1}{2}mv^2$ is independent of B. But $[v_i, v_j]=ie\epsilon_{ijk}B_k$. See also S. Deser and J. Ranols, Phys. Rev. 187, 1935 (1969).

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[14]Thus if $g_0(x)$ is of soliton number zero so that it can be continuously deformed to the identity, the projective representation we have is trivial.