PHYSICAL REVIEW D VOLUME 29, NUMBER 12 15 JUNE 1984

Non-Abelian monopoles break color. I. Classical mechanics

A. P. Balachandran
Physics Department, Syracuse University, Syracuse, New York 13210

G. Marmo
Istituto di Fisica Teorica, Universitá di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d'Oltremare Pad. 19, Napoli, Italy

N. Mukunda
Indian Institute of Science, Bangalore 560012, India

J. S. Nilsson
Institute of Theoretical Physics, S-41296 Göteborg, Sweden

E. C. G. Sudarshan
Center for Particle Theory and Department of Physics, The University of Texas at Austin, Austin, Texax 78712

F. Zaccaria
Istituto di Fisica Teorica, Universitá di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d'Oltremare Pad. 19, Napoli, Italy

(Received 31 August 1983)

Monopoles which are sources of non-Abelian magnetic flux are predicted by many models of grand unification. It has been argued elsewhere that a generic transformation of the "unbroken" symmetry group H cannot be globally implemented on such monopoles for reasons of topology. In this paper, we show that similar topological obstructions are encountered in the mechanics of a test particle in the field of these monopoles and that the transformations of H cannot all be globally implemented as canonical transformations. For the SU(5) model, if H is $\mathrm{SU}(3)_C\times\mathrm{U}(1)_{\mathrm{em}}$, a consequence is that color multiplets are not globally defined, while if H is $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_{\mathrm{WS}}\times\mathrm{U}(1)_Y$, the same is the case for both color and electroweak multiplets. There are, however, several subgroups K_T, $K_{T'},\ldots$ of H which *can* be globally implemented, with the transformation laws of the observables differing from group to group in a novel way. For $H=\mathrm{SU}(3)_C\times\mathrm{U}(1)_{\mathrm{em}}$, a choice for K_T is $\mathrm{SU}(2)_C\times\mathrm{U}(1)_{\mathrm{em}}$, while for $H=\mathrm{SU}(3)_C\times\mathrm{SU}(2)_{\mathrm{WS}}\times\mathrm{U}(1)_Y$, a choice is $\mathrm{SU}(2)_C\times\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$. The paper also develops the differential geometry of monopoles in a form convenient for computations.

I. INTRODUCTION

Dirac[1] introduced the magnetic monopole into modern physics. He showed that their study revealed a new richness in concepts entailing the need to generalize quantum mechanics to include nonintegrable phases. He further showed that the existence of monopoles in conjunction with quantum theory gave a natural reason for electric charge quantization; conversely the monopole strength itself was quantized, the minimum value g_0 satisfying the relation

$$e_0 g_0=\tfrac{1}{2}\hbar\ , \tag{1.1}$$

where e_0 is the electric charge unit.

Saha[2] pointed out that the conserved total angular momentum of a charge-monopole system contained a contribution of $\frac{1}{2}\hbar$ which was electromagnetic in origin, in addition to the orbital angular momentum. (This is provided we choose the elementary pole strength g_0 and elementary charge e_0.) This implies that we can make spinorial composites out of spinless constituents.[3] It has also been demonstrated[4] that such systems obey Fermi statistics in accordance with the familiar spin-statistics relation.

Monopoles have become much more interesting in contemporary physics with the advent of non-Abelian gauge theories accompanied by spontaneous symmetry breakdown.[5] Such theories, after the breakdown of the symmetry, exhibit distinct sectors corresponding to topologically inequivalent asymptotic configurations. Some of them appear as sources of long-range non-Abelian magnetic fields very much like the Dirac monopole. However, these excitations have the property that the monopole has acquired a nontrivial internal structure: it is a persistent but soft field configuration with a finite total energy. The singularity of the point monopole of Dirac no longer obtains.

The transition from the old point monopole of Dirac to the monopoles of non-Abelian gauge theory thus involves two independent steps. One is the replacement of the Abelian U(1) gauge symmetry of electromagnetism by a non-Abelian symmetry. The other is the introduction of the notion of topologically nontrivial asymptotic behavior and the classification of configurations according to distinct behaviors. All this is of course apart from the removal of the singularity of the point monopole.

Some aspects of the first step mentioned above can be elucidated by studying the motion of test particles in an externally given non-Abelian gauge field configuration.

Here we do not yet assume that the asymptotic behavior of the gauge field is topologically nontrivial. For simplicity let us consider the test particle to be classical and nonrelativistic and the external field to be purely magnetic and static. Following Wong,[6] we may assume the dynamical equations

$$m\ddot{x}_j = -e\,\mathrm{Tr}(IG_{jk}(\vec{x}))\dot{x}_k\ ,$$
$$\dot{I} + ie\dot{x}_j[W_j(\vec{x}),I] = 0\ . \tag{1.2}$$

The test particle carries a non-Abelian charge I_a ("isospin") which is a vector in the space $\mathscr{V}$ of the adjoint representation of the gauge group H. We have here used the generators T_a of a suitable faithful matrix representation of H to express the components $W_{aj}(\vec{x})$ of the vector potential, $G_{ajk}(\vec{x})$ of the field strength, and I_a of the isospin as matrices in the Lie algebra of H:

$$W_j(\vec{x}) = W_{aj}(\vec{x})T_a\ ,\quad G_{jk}(\vec{x}) = G_{ajk}(\vec{x})T_a\ ,$$
$$I = I_aT_a\ . \tag{1.3}$$

The matrices T_a fulfill the Lie relations of H,

$$[T_a,T_b] = iC_{abc}T_c\ , \tag{1.4}$$

and may be assumed to also satisfy

$$\mathrm{Tr}(T_aT_b) = R\delta_{ab}\ ,\quad R = \text{constant}\ . \tag{1.5}$$

The field strength is obtained from the potential by

$$G_{jk}(\vec{x}) = \partial_j W_k(\vec{x}) - \partial_k W_j(\vec{x}) + ie[W_j(\vec{x}),W_k(\vec{x})]\ . \tag{1.6}$$

The isospin $I_a(t)$ precesses in the external field, subject to the invariants formed from it remaining independent of time. We have the freedom to alter the description of the external field by a (static) gauge transformation: with $A(\vec{x})$ a matrix in the representation of H generated by the T_a,

$$W_j'(\vec{x}) = A(\vec{x})\left[W_j(\vec{x}) - \frac{i}{e}\nabla_j\right]A(\vec{x})^{-1}\ ,$$
$$G_{jk}'(\vec{x}) = A(\vec{x})G_{jk}(\vec{x})A(\vec{x})^{-1}\ . \tag{1.7}$$

Then $I_a(t)$ transforms according to the adjoint representation of H, via the value of $A(\vec{x}(t))$ where $\vec{x}(t)$ is the instantaneous position of the particle:

$$I'(t) = A(\vec{x}(t))I(t)A(\vec{x}(t))^{-1}\ . \tag{1.8}$$

Both a Lagrangian and a canonical Hamiltonian description of this dynamics can be developed.[7] The classical equations of motion suggest the use of $\vec{x}(t)$ and $I(t)$ as configuration variables (the latter obeying first-order equations of motion). It is however known that a Lagrangian description is facilitated by choosing the position $\vec{x}(t)$ and a group element $B(t)\in H$ as generalized coordinates. [$B(t)$ may be realized as a matrix in the representation of H generated by the T_a.] Following Ref. 7, we pick some K in the Lie algebra $\underline{H}$ of H (i.e., $K = K_aT_a$ with real numbers K_a), and write the Lagrangian

$$\mathscr{L}(\vec{x},B,\dot{\vec{x}},\dot{B};\{W\}) = \tfrac{1}{2}m\dot{\vec{x}}^2 + i\,\mathrm{Tr}(KB^{-1}\dot{B}) - e\dot{x}_j\mathrm{Tr}[BKB^{-1}W_j(\vec{x})]\ , \tag{1.9}$$

where $\{W\}$ denotes the collection of W_{aj}'s. This leads to the earlier equations of motion for $\vec{x}$ and I provided the latter is identified as

$$I(t) = B(t)KB(t)^{-1}\ . \tag{1.10}$$

Under the (static) gauge change (1.7), and consistent with (1.8) and (1.10), $B(t)$ must transform as

$$B'(t) = A(\vec{x}(t))B(t)\ . \tag{1.11}$$

The Lagrangian then retains its functional form in the sense that

$$\mathscr{L}(\vec{x},B,\dot{\vec{x}},\dot{B};\{W\}) = \mathscr{L}(\vec{x},B',\dot{\vec{x}},\dot{B}';\{W'\})\ . \tag{1.12}$$

This Lagrangian description of classical test-particle motion in a (topologically trivial) external gauge field induces a corresponding canonical formalism where due care must be taken of the noncommutative nature of the group elements $B(t)$. A canonical momentum $\mathscr{I}_a$ conjugate to B can be set up fulfilling the Poisson-bracket (PB) relations

$$\{B,\mathscr{I}_a\} = -iT_aB\ ,\quad \{\mathscr{I}_a,\mathscr{I}_b\} = C_{abc}\mathscr{I}_c\ . \tag{1.13}$$

(Note that $\mathscr{I}_a$ is *not* a matrix; also that another canonical momentum generating changes of B on the right can be defined. This and its relation to the physically relevant $\mathscr{I}_a$ are described in Appendix A.) To relate $\mathscr{I}_a$ to the Lagrangian variables we consider an infinitesimal variation

$$\delta B = 0\ ,\quad \delta\dot{B} = i\epsilon_aT_aB\ , \tag{1.14}$$

and write the resulting change in $\mathscr{L}$ as

$$\delta\mathscr{L} = -\epsilon_aI_a(B,\dot{B})\ . \tag{1.15}$$

Then $\mathscr{I}_a$ is identified with $I_a(B,\dot{B})$. For the case of $\mathscr{L}$ given in (1.9) we find for $\mathscr{I}_a$, and for the momentum $\vec{p}$ conjugate to $\vec{x}$,

$$\vec{p} = m\dot{\vec{x}} - e\,\mathrm{Tr}[I\vec{W}(\vec{x})]\ ,$$
$$\mathscr{I}_a = \mathrm{Tr}(IT_a)\ ,\quad I = BKB^{-1}\ . \tag{1.16}$$

The latter set of equations, being independent of velocities, are primary constraints:

$$\phi_a \equiv \mathscr{I}_a - I_a \approx 0\ , \tag{1.17}$$

where $I_a = \mathrm{Tr}(IT_a)$. Following Dirac's general rules the Hamiltonian can be taken in either of the two forms

$$H = \frac{1}{2m}[\vec{p} + e\,\mathrm{Tr}BKB^{-1}\vec{W}(\vec{x})]^2 + i\,\mathrm{Tr}(T_a\dot{B}B^{-1})\phi_a$$
$$= \frac{1}{2m}[\vec{p} + e\mathscr{I}_a\vec{W}_a(\vec{x})]^2 + v_a\phi_a\ , \tag{1.18}$$

where v_a are unknown velocities, and the dynamics and constraints analyzed in the canonical framework.

At any level of analysis of classical test-particle motion-equations of motion, Lagrangian or Hamiltonian—it is clear that a given external field describes a condition not invariant under the group H. As a result the isospin I_a of the particle changes with time. *Nevertheless*, at any level of analysis *the action of H on the appropriate particle variables—I_a or B or $\mathscr{I}_a$—is meaningful and can be properly defined.*

The situation changes dramatically when we take the second step mentioned earlier and consider external gauge field configurations which are asymptotically topologically nontrivial,[8,9] *as is the case with fields describing a non-Abelian monopole. In such cases we find that topological obstructions arise in the description of test-particle motion, and as a result the isospin of the particle cannot be properly defined.*

To a certain extent, a somewhat similar problem had already been encountered by Dirac in the context of the Abelian electromagnetic monopole: the vector potential describing the point monopole has a string singularity running from the monopole to infinity. The string can be moved around with the use of a class of singular gauge transformations but never eliminated. Because of the Abelian nature of the monopole, it happens that the classical equations of motion of a charged particle in this field are well defined and show no concern for the string—the monopole field at this level of discussion is just another field. However, both in the Lagrangian and the Hamiltonian the string appears as a topological obstruction standing in the way of a global description.[10]

Going back to the case of a non-Abelian monopole, here the topological obstructions arise already at the level of the equations of motion of classical test particles, making it impossible to define the action of H *even kinematically* on the isospin of the particle. As one can surmise, there are genuine difficulties also for a quantum-mechanical test particle, at the level of implementability of the transformations of a non-Abelian H and hence of the existence of the corresponding noncommuting dynamical generators.[7] The principal purpose of this paper is the elucidation and analysis of these conceptual problems at the level of classical test particles. Such problems at the field-theoretic and quantum-mechanical levels are discussed in Refs. 8 and 9.

Since we are concerned here with configurations of *gauge* fields, the actual field quantities are not definitive: they are subject to change under local gauge transformations. To study such systems the appropriate mathematical language is that of fiber bundles. On the other hand for a non-Abelian monopole configuration we have the added aspect of nontrivial asymptotic topological structure. In the analysis of test-particle motion in such a configuration, as stated earlier, one expects to encounter obstructions in the description of the dynamics in physical space. Previous experience[10] tells us that a proper dynamical description overcoming such topological obstructions is obtained by working in a larger manifold where they are effectively "unfolded." To descend to the standard description we have to project down to local "sections" with the reemergence of the obstructions. The upshot is that the nontrivial topological nature of a non-Abelian monopole configuration is best expressed by and reflected in a suitably constructed corresponding fiber bundle, which is then used to give a global description of test-particle motion. An exposition of the method of construction of the larger manifold and its use is an essential part of this paper.

As already mentioned, in working with gauge fields there is the freedom to make local gauge transformations, resulting in a corresponding freedom in the way the analysis is carried out, i.e., in the choice of gauge. It is appropriate at this point to spell out the framework we will be using. In the description of monopoles in non-Abelian gauge theory one begins with a connected, simply connected Lie group G as the gauge group of a Lagrangian involving the potential W_μ, field strength $G_{\mu\nu}$, and a Higgs multiplet ϕ with a potential energy density $V(\phi)$. [G need not of course act effectively on all the fields. Thus for $G=\mathrm{SU}(2)$, only $\mathrm{SU}(2)/Z_2$ may be faithfully represented.] Sufficiently far away from the location of the monopole ϕ assumes values minimizing $V(\phi)$. The set of such values of ϕ constitutes a manifold $\mathscr{M}_0$. If $H\subset G$ is the stability group of ϕ at some point of $\mathscr{M}_0$, and G acts transitively on $\mathscr{M}_0$, we can identify $\mathscr{M}_0$ with the coset space G/H. In this description all the fields W,G,ϕ are globally defined smooth space-time functions devoid of singularities or discontinuities. Both W_μ and $G_{\mu\nu}$ have values in the Lie algebra $\underline{G}$ of G. We could however make use of the gauge freedom to obtain a new description in which $\phi(x)$ is (asymptotically) a constant. This is the so-called U gauge. Then both W_μ and $G_{\mu\nu}$ assume values in the Lie algebra $\underline{H}$ of H, the stability group of the constant ϕ. As long as we neglect excitations of the gauge fields we may thus ignore the larger group G and work exclusively with H, taking advantage of the ensuing economy of computations. For the test-particle dynamics too, the Higgs field ϕ can be largely ignored. The price we pay for this simplification is of course that there are (in general) topological obstructions to a global definition of W_μ and $G_{\mu\nu}$ as fields on space time: for a monopole configuration there is no globally defined gauge transformation that renders ϕ (asymptotically) constant. By reversing the steps just described we can of course go back to G and thus unfold these obstructions. We shall however work more or less exclusively with H and its structures. Thus for us the unfolding of gauge potentials that cannot be globally defined in space-time must employ H bundles on which globally defined connections are set up. The locally defined gauge potentials result from restricting the connection to local sections in the bundle. It is possible to set up both Lagrangian and Hamiltonian descriptions of the test-particle dynamics on the bundle. *It will then turn out that,* in contrast on the one hand to the topologically trivial case and on the other to the Abelian case, *it is not possible to define an action of the group H on the test particle.*[8] Thus the topological nontriviality of non-Abelian monopole fields has unexpected and striking consequences already at a kinematic level.

The material of this paper is arranged as follows. In Sec. II, the details of the passage to the U gauge are developed. The need for local definitions of gauge potentials tied together by transition rules is made clear. It is shown that a global description, within the framework of the group of H alone, requires the use of suitable principal fiber bundles with structure group H. A general method of construction of such bundles, and their detailed description in some interesting cases, is presented. Section III explores various descriptions of classical test-particle motion in a non-Abelian monopole field: equations of motion, Lagrangian and canonical formalisms. In each case, the topological obstruction arising in attempting to define an action of H on particle variables is clearly brought out. The construction of bundles and connections in Sec. II is used here to develop a global Lagrangian for the test-particle motion. In Sec. IV the use of such global Lagrangians is illustrated in the case of a particularly simple spherically symmetric monopole field. The orbit equation in ordinary space in this example turns out to be identical to that for electric charge motion in a magnetic monopole field. The only sign of the non-Abelian property of the problem is the isospin vector carried by the test particle. Section V takes up systematically the examination of the problem exposed in Sec. III, namely, the impossibility of global definition of the action of the unbroken symmetry group H. The fact that the action of certain subgroups $K_T, K_{T'}, \ldots$ of H can be globally defined, if not that of H itself, emerges. All the discussion up to this point is classical. In the concluding Sec. VI, the relevance of our findings for quantum test particles, as well as for the global action of H on the gauge field itself in the monopole sector, are briefly pointed out. Applications to grand unified theories (GUT's) are also indicated. Appendix A outlines the main points of the canonical formalisms when group elements appear as dynamical coordinates. Appendix B describes some useful properties of the transition functions that appear in the U gauge.

Since the groups $K_T, K_{T'}, \ldots$ have a definite action on the observables, it should be possible to take the product of these actions and determine the full group K_0 of globally definable automorphisms. We postpone the study of K_0 to a later paper. This problem is of obvious physical significance since the representations of K_0 may enter the construction of the quantum states of the system.

II. *U*-GAUGE AND BUNDLE CONSTRUCTION FOR MONOPOLE FIELDS

We begin by recalling briefly the transition to the so-called U gauge in describing monopole solutions in non-Abelian gauge theory.[5] The gauge potentials W_μ and field strengths $G_{\mu\nu}$ are matrices in $\underline{G}$, the Lie algebra of some (simply connected) Lie group G. For simplicity we consider a single scalar Higgs multiplet ϕ belonging to some representation $D(g)$ of G. It has tachyonic mass and a G-invariant self-coupling potential energy density $V(\phi)$ responsible for spontaneous symmetry breakdown.

Let the monopole configuration be centered at the origin of coordinates. Assume the W_0 vanish and the W_j are time independent. Then the electric components G_{0j} vanish. Denote by $\mathscr{M}_0$ the set of all values of ϕ that minimize $V(\phi)$, and by H the stability group of some point of $\mathscr{M}_0$. With G assumed to act transitively on $\mathscr{M}_0$, we can identify $\mathscr{M}_0$ with the coset space G/H. Finiteness of total energy implies that sufficiently far from the origin the values of ϕ lie in $\mathscr{M}_0$; in fact there is some radius r_0 such that for $r \geq r_0$ we have to good approximation

$$\begin{aligned} &\phi(\vec{x}) \in \mathscr{M}_0 , \\ &\mathscr{D}_j \phi(\vec{x}) = \mathscr{D}_j G_{jk}(\vec{x}) = 0 . \end{aligned} \tag{2.1}$$

Here $\mathscr{D}_j$ is the gauge-covariant derivative. Let Σ be a two-dimensional sphere of large enough radius $r_1 \geq r_0$ such that for each direction specified by a unit vector $\hat{x}$, $\phi(\vec{x})$ has to a good approximation attained an r-independent value in $\mathscr{M}_0$ when $r \geq r_1$. (We sometimes denote points on Σ by $\hat{x}$.) Let the previously mentioned subgroup $H \subset G$ be the stability group of ϕ at the "north pole" N of Σ. At any other $\hat{x} \in \Sigma$, the stability group is written $H_{\hat{x}}$; it arises from $H \equiv H_N$ by conjugation with any $g(\hat{x}) \in G$ that transports ϕ at N to $\phi(\vec{x})$:

$$\begin{aligned} &\phi(\vec{x}) = D(g(\hat{x}))\phi_N , \\ &H_{\hat{x}} = g(\hat{x}) H g(\hat{x})^{-1} . \end{aligned} \tag{2.2}$$

(These equations are valid for all $r \geq r_1$.) Whereas for each $\hat{x}$, $g(\hat{x})$ is not unique, $H_{\hat{x}}$ is unique and in fact varies smoothly as $\hat{x}$ goes over Σ. The nontrivial topology of the monopole is expressed in the fact that, notwithstanding the smooth variation of $H_{\hat{x}}$ with $\hat{x}$, we cannot choose $g(\hat{x})$ to vary smoothly with $\hat{x}$ all over Σ. The topological "type" of the monopole consists in the particular mapping $S^2 \to \mathscr{M}_0 = G/H$ provided by following $\phi(\vec{x}) \in \mathscr{M}_0$ as $\vec{x}$ runs over Σ. This is a particular element of $\pi_2(G/H)$. From (2.1) we see that for $r > r_0$

$$D(G_{jk}(\vec{x}))\phi(\vec{x}) = 0 . \tag{2.3}$$

In particular this is true for $r \geq r_1$ where $\phi(\vec{x})$ is r independent. Thus for such $\vec{x}$, $G_{jk}(\vec{x})$ lies within the Lie algebra $\underline{H}_{\hat{x}}$ of $H_{\hat{x}}$; but to the extent that $\phi(\vec{x})$ does vary with $\hat{x}$, $W_j(\vec{x})$ must involve elements of $\underline{G}$ outside $\underline{H}_{\hat{x}}$. More precisely, for $r \geq r_1$, $W_r(\vec{x})$ lies in $\underline{H}_{\hat{x}}$ while $W_\theta(\vec{x})$ and $W_\phi(\vec{x})$ contain terms from $\underline{G}$ outside $\underline{H}_{\hat{x}}$. It can then be seen that, by means of a gauge transformation using at each $\vec{x}$ in the region $r \geq r_1$ an element from $H_{\hat{x}}$, W_r can be transformed to zero in this region. Moreover as a result of this transformation, neither $\phi(\vec{x})$ nor $H_{\hat{x}}$, nor the fact that $G_{jk}(\vec{x})$ lies in $\underline{H}_{\hat{x}}$, is altered in $r \geq r_1$. We assume that this gauge transformation has been carried out, and also that it has been extended in some smooth way down to $r = 0$.

At this stage both $W_j(\vec{x})$ and $G_{jk}(\vec{x})$ are globally defined. We now express Σ as the union of two open contractible subsets Σ_N and Σ_S; the former contains N but definitely excludes the "south pole" S of Σ, and conversely for the latter. [These choices of Σ_N, Σ_S can be imagined to be uniformly extended for all $r \geq r_1$; it is also often convenient to take Σ_N (Σ_S) to be all of Σ minus S (N).] It is then possible to make smooth choices of elements $g_N(\hat{x}), g_S(\hat{x})$ in G, over Σ_N and Σ_S, respectively, so that for all $r \geq r_1$ we have

$$\phi(\vec{x})=D(g_N(\hat{x}))\phi_N\ ,\quad \vec{x}\in\Sigma_N$$
$$=D(g_S(\hat{x}))\phi_N\ ,\quad \vec{x}\in\Sigma_S\ . \tag{2.4}$$

Again because of the nontrivial topology of the monopole, no single smooth global choice of $g(\hat{x})$ exists to achieve (2.2), but a pair $g_N(\hat{x}),g_S(\hat{x})$ can be found. We may imagine $g_N(\hat{x}),g_S(\hat{x})$ being smoothly extended with suitable r dependences into $0\le r<r_1$. Thus $g_N(\vec{x})$ is an element of G, defined smoothly for all $\vec{x}$ such that $\hat{x}$ points into Σ_N, and correspondingly for $g_S(\vec{x})$. Both $g_N(\vec{x})$ and $g_S(\vec{x})$ become r independent when $r\ge r_1$ and then obey (2.4). By convention we assume $g_N(\vec{x})=e$, the identity of G, when $\hat{x}=N$ and $r\ge r_1$.

With the help of $g_N(\vec{x}),g_S(\vec{x})$ so constructed, we carry out two separate gauge transformations, each over its appropriate angular domain, and thus reduce $\phi(\vec{x})$ to a constant value ϕ_N for all $r\ge r_1$. It will be assumed that a test particle in the monopole field does not probe the interior region $0\le r<r_1$; and that in the exterior region $r\ge r_1$ it couples to the gauge fields minimally and not to ϕ. After the above gauge changes are carried out, what the test particle "sees" in the exterior region are gauge potentials $W_{Nj}(\vec{x}), W_{Sj}(\vec{x})$ defined, respectively, over Σ_N and Σ_S as follows:

$$W_{Nj}(\vec{x})=g_N(\vec{x})^{-1}\left[W_j(\vec{x})-\frac{i}{e}\nabla_j\right]g_N(\vec{x})\ ,\quad \hat{x}\in\Sigma_N\ ,$$
$$W_{Sj}(\vec{x})=g_S(\vec{x})^{-1}\left[W_j(\vec{x})-\frac{i}{e}\nabla_j\right]g_S(\vec{x})\ ,\quad \hat{x}\in\Sigma_S\ . \tag{2.5}$$

(These definitions of W_N and W_S hold good for *all* $r\ge 0$.) For $r\ge r_1$ it is clear that the radial components of W_N and W_S are zero. It is also clear from the gauge covariance of Eqs. (2.1) that for $r\ge r_1$ both $W_N(\vec{x})$ and $W_S(\vec{x})$ lie in $\underline{H}$. In the overlap of Σ_N and Σ_S, the two potentials are connected by a gauge transformation corresponding to a "transition group element" $h_T(\hat{x})\in H$: $r\ge r_1$, $\hat{x}\in\Sigma_N\cap\Sigma_S$:

$$h_T(\hat{x})=g_S(\hat{x})^{-1}g_N(\hat{x})\in H\ ,$$
$$W_{Nj}(\vec{x})=h_T(\hat{x})^{-1}\left[W_{Sj}(\vec{x})-\frac{i}{e}\nabla_j\right]h_T(\hat{x})\ , \tag{2.6}$$
$$G_{Njk}(\vec{x})=h_T(\hat{x})^{-1}G_{Sjk}(\vec{x})h_T(\hat{x})\ .$$

We shall not need the extensions of these formulas into the interior region $0\le r<r_1$; for this reason we have, correctly, used $\hat{x}$ and not $\vec{x}$ as argument in h_T, g_N, and g_S.

The set of quantities W_N,W_S,h_T taken together describes the monopole in the U gauge. It is a structure defined totally in the framework of the group H, since there is no reference any longer to G. Thus we have here an H monopole and our problem is to describe it appropriately and then examine test-particle motion in its field. Originally the topological type of the monopole was indicated by a particular element of $\pi_2(G/H)$. In the U gauge, this type is indicated by a particular element of $\pi_1(H)$, i.e., essentially by the kind of closed curve in H that $h_T(\hat{x})$ describes as $\hat{x}$ goes round a closed curve in $\Sigma_N\cap\Sigma_S$. This is consistent with the known fact that $\pi_2(G/H)\simeq\pi_1(H)$ when G is simply connected. In the rest of this section we examine the first aspect mentioned above, namely, that of describing an H monopole appropriately.

We recall again that both $\vec{W}_N$ and $\vec{W}_S$ have no radial components but only angular ones which, however, have radial as well as angular dependences. The proper global way to describe the above situation, namely, the existence of well-defined $\vec{W}_N,\vec{W}_S$ over Σ_N,Σ_S, respectively, related through $h_T(\hat{x})$ in the overlap, is to say that we have a connection Ω on a principal H-bundle $\mathscr{B}$ over $S^2=\Sigma$, employing h_T as transition function. The radial variable r appears as a parameter in Ω; the bundle itself is an r-independent construct since h_T has no r dependence (recall $r\ge r_1$). The locally defined potentials $\vec{W}_N,\vec{W}_S$ emerge on restricting Ω, which is a global object, to local sections over Σ_N,Σ_S, respectively.

In order to realize the bundle $\mathscr{B}$ in an explicit way, permitting easy computations, we take the following results from Appendix B. (1) We can exploit the freedom in choices of $g_N(\hat{x}),g_S(\hat{x})$ permitted by Eqs. (2.4) to assume without loss of generality that $h_T(\hat{x})$ depends only on the azimuth ϕ:

$$h_T(\hat{x})=c(\phi)\in H\ ,$$
$$c(0)=c(2\pi)\ . \tag{2.7}$$

(2) Every continuous closed curve $c(\phi),0\le\phi\le 2\pi$, in H is homotopic to a one-parameter subgroup $e^{i\phi T},0\le\phi\le 2\pi,T\in\underline{H}$, with the generator T obeying

$$e^{2\pi iT}=e=\text{identity in } H\ . \tag{2.8}$$

(3) By exploiting again the freedom in $g_N(\hat{x}),g_S(\hat{x})$, we can thus assume that

$$h_T(\hat{x})=c(\phi)=e^{i\phi T}\ . \tag{2.9}$$

Essentially this means that for any element of $\pi_1(H)$, i.e., in any class of homotopically equivalent closed continuous curves in H, one can find a representative in the form of a one-parameter subgroup. [However, as will be clarified later, this does not imply that the generator T is characteristic of a given class in $\pi_1(H)$; two different generators $T,T'\in\underline{H}$, both obeying (2.8) but, for example, with quite different spectra, can lead to homotopically equivalent curves $e^{i\phi T},e^{i\phi T'}$.]

With the transition function $h_T(\hat{x})$ in the form (2.9), the construction of $\mathscr{B}$ is quite straightforward. We use the fact that[11] SU(2) as a manifold is known to be a U(1) bundle over the two-sphere base S^2, which is also the base for $\mathscr{B}$. Write the elements of H in some faithful matrix representation, for instance, the one in Sec. I with generators T_a, as $B,B',\ldots$, and introduce the Cartesian product of H and SU(2):

$$\overline{\mathscr{B}}=H\times\text{SU}(2)=\{(B,u)\,|\,B\in H,u\in\text{SU}(2)\}\ . \tag{2.10}$$

On $\overline{\mathscr{B}}$ introduce the equivalence relation

$$(B,u)\sim(e^{-i\alpha T}B,ue^{i\alpha\sigma_3})\ ,\quad 0\le\alpha\le 2\pi\ , \tag{2.11}$$

leading to equivalence classes $\langle B,u\rangle$:

$$\langle B,u\rangle=\langle e^{-i\alpha T}B,ue^{i\alpha\sigma_3}\rangle\ . \tag{2.12}$$

The required bundle $\mathscr{B}$ is the quotient of $\overline{\mathscr{B}}$ with respect to this relation of equivalence:

$$\mathscr{B}=\overline{\mathscr{B}}/\sim=\{\langle B,u\rangle \mid B\in H,u\in \mathrm{SU}(2)\}\ . \tag{2.13}$$

The projection $\pi:\mathscr{B}\to S^2$ is defined by

$$\pi(\langle B,u\rangle)=\tfrac{1}{2}\,\mathrm{Tr}(\vec{\sigma}u\sigma_3u^{-1})=\hat{x}\in S^2\ . \tag{2.14}$$

To verify this construction we specifically assume that Σ_N (Σ_S) is all of Σ minus S (N). For $\hat{x}\in\Sigma_N$ (Σ_S), define $u_N(\hat{x})(u_S(\hat{x}))\in\mathrm{SU}(2)$ as

$$\begin{aligned}&0\le\theta<\pi:\ u_N(\hat{x})=e^{-i\phi\sigma_3/2}e^{-i\theta\sigma_2/2}e^{i\phi\sigma_3/2}\ ,\\ &0<\theta\le\pi:\ u_S(\hat{x})=e^{-i\phi\sigma_3/2}e^{-i\theta\sigma_2/2}e^{-i\phi\sigma_3/2}\ .\end{aligned} \tag{2.15}$$

These are well defined in their respective domains where they obey

$$u_{N,S}(\hat{x})\sigma_3u_{N,S}(\hat{x})^{-1}=\vec{\sigma}\cdot\hat{x}\ . \tag{2.16}$$

Furthermore, in the overlap we have

$$0<\theta<\pi:\ u_N(\hat{x})=u_S(\hat{x})e^{i\phi\sigma_3}\ . \tag{2.17}$$

In case $\langle B,u\rangle\in\mathscr{B}$ is such that $\pi(\langle B,u\rangle)=\hat{x}\neq S$, the south pole of S^2, there is a definite α for which

$$u=u_N(\hat{x})e^{-i\alpha\sigma_3}\ . \tag{2.18}$$

By (2.12) this element in $\mathscr{B}$, or the corresponding equivalence class in $\overline{\mathscr{B}}$, can be represented by the element

$$(e^{-i\alpha T}B,u_N(\hat{x}))\in\overline{\mathscr{B}}\ . \tag{2.19}$$

If we write

$$B_N=e^{-i\alpha T}B\ , \tag{2.20}$$

each such $\langle B,u\rangle\in\mathscr{B}$ corresponds uniquely to the pair $(B_N,\hat{x})$; in this way $\pi^{-1}(\Sigma_N)$ is exhibited as (being homeomorphic to) the Cartesian product $H\times\Sigma_N$. Similarly if $\langle B,u\rangle\in\mathscr{B}$ is such that $\pi(\langle B,u\rangle)=\hat{x}\neq N$, the north pole of S^2, there is a definite β for which

$$u=u_S(\hat{x})e^{-i\beta\sigma_3}\ . \tag{2.21}$$

Again by (2.12), such an element in $\mathscr{B}$ can be represented by the pair $(B_S,\hat{x})$ where

$$B_S=e^{-i\beta T}B\ . \tag{2.22}$$

That exhibits $\pi^{-1}(\Sigma_S)$ as (being homeomorphic to) the Cartesian product $H\times\Sigma_S$. If, finally, $\langle B,u\rangle$ lies in both $\pi^{-1}(\Sigma_N)$ and $\pi^{-1}(\Sigma_S)$, it follows from (2.17), (2.18), and (2.21) that

$$\alpha-\beta=\phi=\text{azimuth of }\hat{x}\ . \tag{2.23}$$

The elements B_N,B_S of H which appear in the two local trivializations of $\mathscr{B}$ are related by a left action of H: from (2.20), (2.22), and (2.23),

$$B_N=e^{-i\phi T}B_S=h_T(\hat{x})^{-1}B_S\ . \tag{2.24}$$

This proves that[11] $\mathscr{B}$ is a principal H bundle over S^2 with transition function $e^{i\phi T}$.

There is a global action of H on $\mathscr{B}$ which happens to be a *right* action because in the equivalence relation (2.11) we have $e^{-i\alpha T}$ appearing on the *left:*

$$B'\in H:\ \langle B,u\rangle\in\mathscr{B}\to\langle BB',u\rangle\in\mathscr{B}\ . \tag{2.25}$$

This global action is the one always available in a principal fiber bundle, and is used below in characterizing a connection.

The global description of the pair of gauge potentials $\vec{\mathrm{W}}_N(x),\vec{\mathrm{W}}_S(\vec{x})$ defined for $\hat{x}\in\Sigma_N,\Sigma_S$, respectively, and related in the overlap by

$$\vec{\mathrm{W}}_N(\vec{x})=e^{-i\phi T}\left[\vec{\mathrm{W}}_S(\vec{x})-\frac{i}{e}\vec{\nabla}\right]e^{i\phi T} \tag{2.26}$$

[which is (2.6) given (2.9)] is through a connection Ω on $\mathscr{B}$. We write it as $\Omega(B,u)$ and demand two principal properties:

$$\text{(i)}\ \ \Omega(e^{-i\alpha T}B,ue^{i\alpha\sigma_3})=\Omega(B,u)\ ,\ \text{ any }\alpha\ , \tag{2.27}$$

$$\text{(ii)}\ \ \Omega(BB',u)=B'^{-1}\Omega(B,u)B'-\frac{i}{e}B'^{-1}dB'\ ,\ \ B'\in H\ . \tag{2.28}$$

[These equations are written somewhat loosely. Thus (2.27) is supposed to mean that the pullback of Ω by the map $(B,u)\to(e^{-i\alpha T}B,ue^{i\alpha\sigma_3})$ must be equal to Ω. There is an analogous interpretation for (2.28) in terms of mappings of sections. See also (4.2).] Property (i) ensures that Ω is defined on $\mathscr{B}$, not on $\overline{\mathscr{B}}$; and property (ii), specifying the behavior of Ω under the (right) action of H on $\mathscr{B}$ (2.25), ensures that it is a connection form. To recover $\vec{\mathrm{W}}_N(\vec{x})$, we consider the local section

$$\Sigma_N\to\pi^{-1}(\Sigma_N)\subset\mathscr{B}$$

given by

$$\hat{x}\in\Sigma_N\to\langle \mathbb{1},u_N(\hat{x})\rangle\ . \tag{2.29}$$

The pullback of Ω to this section gives $\vec{\mathrm{W}}_N(\vec{x})$:

$$\Omega(\mathbb{1},u_N(\hat{x}))=\vec{\mathrm{W}}_N(\vec{x})\cdot d\vec{x}\ . \tag{2.30}$$

In a similar manner, by using the local section

$$\hat{x}\in\Sigma_S\to\langle \mathbb{1},u_S(\hat{x})\rangle\in\pi^{-1}(\Sigma_S)\subset\mathscr{B}\ , \tag{2.31}$$

we have

$$\Omega(\mathbb{1},u_S(\hat{x}))=\vec{\mathrm{W}}_S(\vec{x})\cdot d\vec{x}\ . \tag{2.32}$$

(Here it is important to remember that $W_{N,S}$ have only "surface" components, and that Ω depends parametrically on r.) From (2.17), (2.27), (2.28), (2.30), and (2.32) we immediately recover (2.26). Thus while the nontrivial topological nature of the H monopole forces us to give separate local descriptions of the gauge potentials in ordinary space, glued together by (2.26) in the overlap, this obstruction gets "unfolded" if we use the global connection Ω on $\mathscr{B}$. This Ω will be used in subsequent sections to describe test-particle motion.

In the preceding construction, the principal bundle $\mathscr{B}$

was realized as the quotient of a larger space $\overline{\mathscr{B}}$ with respect to an equivalence relation. In some cases, it is possible to give a more direct description of the manifold structure of $\mathscr{B}$. Because of its intrinsic interest, we conclude this section by illustrating this method in four cases, with a suitable choice of T in each: (i) $H=\mathrm{U}(2)$,[12] (ii) $H=\mathrm{U}(3)$,[13] (iii) $H=[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)]/Z_6$,[13] and (iv) $H=\mathrm{SO}(3)$.[14]

(i) $H=\underline{U}(2)$.

The set $\overline{\mathscr{B}}=\mathrm{U}(2)\times\mathrm{SU}(2)$ is

$$\overline{\mathscr{B}}=\{(h,u)\mid h\in\mathrm{U}(2),u\in\mathrm{SU}(2)\}\ . \tag{2.33}$$

We choose T in $\underline{U}(2)$ to be

$$T=\tfrac{1}{2}(1+\sigma_3) \tag{2.34}$$

so that $e^{i\phi T},0\le\phi\le 2\pi$, is a nontrivial closed loop in U(2).[12] The equivalence relation (2.11) is

$$(h,u)\sim(e^{-i\alpha(1+\sigma_3)/2}h,u\,e^{i\alpha\sigma_3})\ . \tag{2.35}$$

Thus under the transformations defining this equivalence the rows of h and the columns of u behave as

$$\begin{aligned}&h_{1a}\to e^{-i\alpha}h_{1a}\ ,\quad h_{2a}\to h_{2a}\ ,\\ &u_{a1}\to e^{i\alpha}u_{a1}\ ,\quad u_{a2}\to e^{-i\alpha}u_{a2}\ .\end{aligned} \tag{2.36}$$

With the help of $\Delta=\det h$ which under (2.36) changes as

$$\Delta\to e^{-i\alpha}\Delta\ , \tag{2.37}$$

we can easily construct new matrices out of the rows of h and columns of u with the property of being invariant under (2.36):

$$\begin{aligned}&V=\begin{bmatrix}h_{1a}/\Delta\\ h_{2a}\end{bmatrix}\in\mathrm{SU}(2)\ ,\\ &U=(u_{a1}\Delta,u_{a2}/\Delta)\in\mathrm{SU}(2)\ .\end{aligned} \tag{2.38}$$

It is easy to check that all equivalence classes $\langle h,u\rangle\in\mathscr{B}$ are obtained faithfully, each one just once, if V and U run independently over SU(2). Thus (V,U) form coordinates for $\mathscr{B}$ and we may write

$$\langle h,u\rangle\equiv(V,U)\ . \tag{2.39}$$

The manifold structure of the U(2) bundle $\mathscr{B}$ over S^2, for the choice (2.34) of T, has turned out to be SU(2)×SU(2).

The global (right) action (2.25) of the structure group on $\mathscr{B}$ can be expressed as an action on V and U. We find that

$$\langle h,u\rangle\in\mathscr{B}\ \overset{h'\in U(2)}{\longrightarrow}\ \langle hh',u\rangle\in\mathscr{B} \tag{2.40}$$

translates into

$$(V,U)\ \overset{h'\in U(2)}{\longrightarrow}\ ((\det h')^{-(1+\sigma_3)/2}Vh',U(\det h')^{\sigma_3})\ . \tag{2.41}$$

The appearance of only integral powers of $\det h'$ here ensures that we have indeed an action of U(2), and not of SU(2)×U(1), on $\mathscr{B}$.

(ii) $H=U(3)$.

Such a choice of H leads to the description of monopoles produced in GUT models when a (simply connected) GUT group breaks to $[\mathrm{SU}(3)_C\times\mathrm{U}(1)_{\mathrm{em}}]/Z_3\ \equiv\mathrm{U}(3)$.[13] The set $\overline{\mathscr{B}}=\mathrm{U}(3)\times\mathrm{SU}(2)$ is

$$\overline{\mathscr{B}}=\{(h,u)\mid h\in\mathrm{U}(3),u\in\mathrm{SU}(2)\}\ . \tag{2.42}$$

We choose $T\in\underline{U}(3)$ to lie partly in the "SU(3) part" and partly in the "U(1) part" of U(3), so that $e^{i\phi T},0\le\phi\le 2\pi$, is a nontrivial closed loop:

$$T=\frac{1}{3}-\frac{\lambda_8}{\sqrt{3}}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\ . \tag{2.43}$$

The equivalence relation on $\overline{\mathscr{B}}$ is

$$(h,u)\sim(e^{-i\alpha/3+i\alpha\lambda_8/\sqrt{3}}h,ue^{i\alpha\sigma_3})\ . \tag{2.44}$$

Under this transformation the rows of h, determinant of h, and columns of u change as

$$\begin{aligned}&h_{1r}\to h_{1r}\ ,\quad h_{2r}\to h_{2r}\ ,\quad h_{3r}\to e^{-i\alpha}h_{3r}\ ,\\ &\Delta=\det h\to e^{-i\alpha}\Delta\ ,\\ &u_{a1}\to e^{i\alpha}u_{a1},u_{a2}\to e^{-i\alpha}u_{a2}\ .\end{aligned} \tag{2.45}$$

We then define new matrices V,U out of h,u invariant under (2.45):

$$\begin{aligned}&V=\begin{bmatrix}h_{1r}\\ h_{2r}\\ h_{3r}/\Delta\end{bmatrix}\in\mathrm{SU}(3)\ ,\\ &U=(u_{a1}\Delta,u_{a2}/\Delta)\in\mathrm{SU}(2)\ .\end{aligned} \tag{2.46}$$

One can check that these are coordinates for $\mathscr{B}$; as V runs over SU(3) and U over SU(2), we recover each element of $\mathscr{B}$ exactly once, so we may write (2.39) again. This exhibits the manifold structure of this U(3) bundle over S^2 to be SU(3)×SU(2).

The global (right) action of U(3) on $\mathscr{B}$ can be worked out. We find

$$\langle h,u\rangle\in\mathscr{B}\ \overset{h'\in\mathrm{U}(3)}{\longrightarrow}\ \langle hh',u\rangle\in\mathscr{B} \tag{2.47}$$

goes into

$$(V,U)\ \overset{h'\in U(3)}{\longrightarrow}\ ((\det h')^{-1/3+\lambda_8/\sqrt{3}}Vh',U(\det h')^{\sigma_3})\ . \tag{2.48}$$

Again, only integral powers of $\det h'$ appear, as is proper.

(iii) $H=[SU(3)\times SU(2)\times U(1)]/Z_6$.

This example is also chosen because of its obvious physical relevance for GUT models.[13] At first we settle the precise definition of H. We begin with the larger group

$$\begin{aligned}\overline{H}&=\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\\ &=\{(M,v,e^{i\chi})\mid M\in\mathrm{SU}(3),v\in\mathrm{SU}(2),e^{i\chi}\in U(1)\}\ .\end{aligned} \tag{2.49}$$

Let $b\in\overline{H}$ be the central element

$$b=(e^{2\pi i/3}\mathbb{1},e^{i\pi}\mathbb{1},e^{i\pi/3}) \tag{2.50}$$

which, because $b^6=e$, generates a cyclic discrete invariant subgroup $Z_6\subset\overline{H}$. We then define H to be the factor group

$$H = \bar{H}/Z_6 \,. \tag{2.51}$$

A faithful matrix realization of H is given by the Kronecker products of SU(3) matrices by U(2) matrices:

$$H = \{ M \otimes s \mid M \in \mathrm{SU}(3), s \in \mathrm{U}(2) \} \,. \tag{2.52}$$

As before, we now set up the space $\overline{\mathscr{B}} = H \times \mathrm{SU}(2)$,

$$\overline{\mathscr{B}} = \{ (M \otimes s, u) \mid M \in \mathrm{SU}(3), s \in \mathrm{U}(2), u \in \mathrm{SU}(2) \} \,. \tag{2.53}$$

For the bundle construction, we pick the generator $T \in \underline{H}$ in such a way that the curve $e^{i\phi T}, 0 \le \phi \le 2\pi$, would have run in $\bar{H}$ from the identity at $\phi = 0$ to b at $\phi = 2\pi$, so that in H we do get a nontrivial closed loop:

$$T = \frac{1}{\sqrt{3}} \lambda_8 + \tfrac{1}{2}\sigma_3 + \tfrac{1}{6}$$
$$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \otimes \mathbb{1}_{2\times 2} + \mathbb{1}_{3\times 3} \otimes \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} . \tag{2.54}$$

The equivalence relation (2.11) on $\overline{\mathscr{B}}$ is

$$(M \otimes s, u) \sim (e^{-i\alpha\lambda_8/\sqrt{3}} M \otimes e^{-i\alpha/2(\sigma_3 + 1/3)} s, u e^{i\alpha\sigma_3}) \,. \tag{2.55}$$

The rows of M and s, the columns of u, and the determinant of $M \otimes s$ experience

$$\begin{aligned} &M_{1r} \to e^{-i\alpha/3} M_{1r} \,, \quad M_{2r} \to e^{-i\alpha/3} M_{2r} \,, \quad M_{3r} \to e^{2i\alpha/3} M_{3r} \,, \\ &s_{1a} \to e^{-2i\alpha/3} s_{1a} \,, \quad s_{2a} \to e^{i\alpha/3} s_{2a} \,, \\ &\Delta = \det(M \otimes s) \to e^{-i\alpha} \Delta \,, \\ &u_{a1} \to e^{i\alpha} u_{a1} \,, \quad u_{a2} \to e^{-i\alpha} u_{a2} \,. \end{aligned} \tag{2.56}$$

The appropriate definitions of invariant matrices V, U in this case are

$$\begin{aligned} V &= \begin{bmatrix} M_{1r} \\ M_{2r} \\ M_{3r}\Delta \end{bmatrix} \otimes \begin{bmatrix} s_{1a}/\Delta \\ s_{2a} \end{bmatrix} , \\ U &= (u_{a1}\Delta, u_{a2}/\Delta) \in \mathrm{SU}(2) \,. \end{aligned} \tag{2.57}$$

One sees that V runs over the set of 6×6 Kronecker product matrices SU(3)⊗SU(2). In fact, the manifold structure of the present bundle $\mathscr{B}$ is (SU(3)⊗SU(2))×SU(2), with (V, U) as faithful coordinates:

$$\langle M \otimes s, u \rangle \equiv (V, U) \,. \tag{2.58}$$

The global (right) action of H on $\mathscr{B}$,

$$\langle M \otimes s, u \rangle \overset{M' \otimes s' \in H}{\longrightarrow} \langle MM' \otimes ss', u \rangle \tag{2.59}$$

alters V and U as

$$\begin{aligned} &V \to (\det[M' \otimes s'])^{-1/6 - \sigma_3/2 - \lambda_8\sqrt{3}} V(M' \otimes s') \,, \\ &U \to U(\det[M' \otimes s'])^{\sigma_3} \,. \end{aligned} \tag{2.60}$$

Needless to say, only integral powers of $\det[M' \otimes s']$ appear here.

In all the three cases considered so far, the projection $\pi: \mathscr{B} \to S^2$ can be simply expressed in the (V, U) description of $\mathscr{B}$, because in all these cases U is related to u by

$$U = u \Delta^{\sigma_3} \,, \tag{2.61}$$

where only the phase factor Δ varies from case to case. Therefore the general rule (2.14) here takes the form

$$\begin{aligned} &\pi((V, U)) = \hat{x} \in S^2 \,, \\ &\vec{\sigma} \cdot \hat{x} = U \sigma_3 U^{-1} \,. \end{aligned} \tag{2.62}$$

(iv) $H = SO(3)$.

This last example is chosen partly because it does not follow the same pattern as the previous three. Further SO(3) monopoles are predicted by the model of Slansky, Goldman, and Shaw,[15] designed to explain the alleged observation of fractional charges.[16] We begin with the definition of $\overline{\mathscr{B}}$ as

$$\begin{aligned} \overline{\mathscr{B}} &= \mathrm{SO}(3) \times \mathrm{SU}(2) \\ &= \{ (B, u) \mid B \in \mathrm{SO}(3), u \in \mathrm{SU}(2) \} \,. \end{aligned} \tag{2.63}$$

The three antisymmetric Hermitian SO(3) generators S_j are taken as

$$(S_j)_{kl} = i\epsilon_{jkl} \,, \tag{2.64}$$

and the 2×2 matrices u, σ_j are enlarged to 3×3 ones by defining

$$\begin{aligned} U &= \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} , \\ \Sigma_j &= \begin{bmatrix} \sigma_j & 0 \\ 0 & 0 \end{bmatrix} . \end{aligned} \tag{2.65}$$

In particular, Σ_2 coincides with $-S_3$. The generator $T \in \underline{SO}(3)$ is taken to be S_3: the loop $e^{i\phi T}, 0 \le \phi \le 2\pi$ is then nontrivial. The equivalence relation on $\overline{\mathscr{B}}$ reads

$$(B, u) \sim (e^{-i\alpha S_3} B, u\, e^{i\alpha\sigma_3}) \,, \tag{2.66}$$

leading to equivalence classes $\langle B, u \rangle$ and the bundle $\mathscr{B}$. A (matrix) combination of B and u that is constant over an equivalence class $\langle B, u \rangle$ turns out to be

$$\begin{aligned} \mathscr{E} &= U \mathscr{C} B \,, \\ \mathscr{C} &= \begin{bmatrix} e^{i\pi\sigma_1/4} & 0 \\ 0 & 1 \end{bmatrix} . \end{aligned} \tag{2.67}$$

To verify this fact we need the equality of Σ_2 and $-S_3$, and

$$e^{-i\pi\sigma_1/4} \sigma_3 e^{i\pi\sigma_1/4} = -\sigma_2 \,. \tag{2.68}$$

The matrices $\mathscr{E}$ correspond one-to-one to the elements of $\mathscr{B}$, and replace the pairs (V, U) of the previous examples. They form a subset of matrices of SU(3), namely, just those that can be put into the form (2.67).

The (right) action of SO(3) in $\mathscr{B}$ is very simple in terms of $\mathscr{E}$,

$$\langle B,u\rangle \overset{B'\in SO(3)}{\rightarrow} \langle BB',u\rangle \Longrightarrow \mathscr{E}\rightarrow\mathscr{E}B' . \quad (2.69)$$

The projection $\pi:\mathscr{B}\rightarrow S^2$ appears slightly complicated. Using standard properties of SU(2) matrices one finds one can write

$$\pi(\mathscr{E})=\hat{x}\in S^2 , \quad \hat{x}=-\tfrac{1}{2}\,\mathrm{Tr}\mathscr{E}\mathscr{E}^T\Sigma_2\vec{\Sigma} . \quad (2.70)$$

Thus in this slightly atypical case too, we have found an intrinsic description of the manifold structure of $\mathscr{B}$ [a certain subset of SU(3)], and expressed both the global SO(3) action and the projection to S^2 in these terms.

III. TEST-PARTICLE CONFIGURATION SPACE: LOSS OF H ACTION

In Sec. I we have recounted briefly three levels of description of classical test-particle motion in a given topologically trivial gauge field. At the (Newtonian) equations of motion level, the configuration space may be taken as the Cartesian product of $\mathscr{V}$, the space of the adjoint representation of H, with the relevant portion of three-dimensional physical space. We shall write R_+ for the appropriate range of the radial variable. (For a point monopole the origin must be excluded and R_+ then consists of $0<r<\infty$, while in the present case we must take R_+ to be $r_1<r<\infty$.) The appropriate configuration space at this level is then $(\mathscr{V}\times S^2)\times\mathsf{R}_+$, and on this space the group H acts naturally and unambiguously: the "isospin" I transforms under the adjoint representation of H acting on $\mathscr{V}$. At the Lagrangian level, the appropriate configuration space is clearly the Cartesian product $(H\times S^2)\times\mathsf{R}_+$. Here we have two possible actions on H available, either by left or by right translation; but it is clear that it is the left action that is physically relevant, in view of Eq. (1.10). In the Hamiltonian formulation, finally, the left action on the configuration space is lifted to a corresponding action on phase space. The infinitesimal generators of this left action of H are the canonical momenta $\mathscr{I}_a$ appearing in (1.13), (1.16), (1.17), and (1.18), which are weakly equal to the isospin components I_a defined already at the Lagrangian level in (1.10). Note that the generators of right action of H are not the physically relevant quantities.

In summary, at each level of description there is a definite and physically appropriate action of H on the configuration space. We now examine the situation when the test particle is placed in a monopole field of nontrivial topological character.

Let the gauge potentials $\vec{W}_N(\vec{x}),\vec{W}_S(\vec{x})$ be specified over Σ_N,Σ_S, respectively, obeying (2.26) in the overlap. The natural generalization of the equations of motion of Sec. I is as follows. When the test particle is located in Σ_N, it must be possible to ascribe an isospin I_N to it and the motion must be governed by

$$\vec{x}\in\Sigma_N:\ m\ddot{x}_j=-e\,\mathrm{Tr}[I_N G_{Njk}(\vec{x})]\dot{x}_k , \quad \dot{I}_N+ie\dot{x}_j[W_{Nj}(\vec{x}),I_N]=0 . \quad (3.1)$$

When the particle is located in Σ_S, we must be able to ascribe an isospin I_S to it and must have

$$\vec{x}\in\Sigma_S:\ m\ddot{x}_j=-e\,\mathrm{Tr}[I_S G_{Sjk}(\vec{x})]\dot{x}_k , \quad \dot{I}_S+ie\dot{x}_j[W_{Sj}(\vec{x}),I_S]=0 . \quad (3.2)$$

In the overlap, to be consistent with the relation (2.26) and the implied relation for G_N and G_S, we must have

$$\vec{x}\in\Sigma_N\cap\Sigma_S:\ I_N(t)=e^{-i\phi T}I_S(t)e^{i\phi T} , \quad (3.3)$$

$\phi=\phi(t)$ being the azimuth of $\vec{x}(t)$. [Thus, $\mathrm{Tr}(I_N G_N)=\mathrm{Tr}(I_S G_S)$ and $m\ddot{x}_j$ in (3.1) and (3.2) are the same.] The proper configuration space in which to embed these equations of motion is obviously the following: we must use a vector bundle over S^2, with fiber $\mathscr{V}$, transition function $h_T(\hat{x})=e^{i\phi T}$, and structure group H, apart from R_+ for the radial variable. The technique of construction is patterned after the work of the last section. We start with the set $\bar{Q}$ which is the Cartesian product of $\mathscr{V}$ and SU(2):

$$\bar{Q}=\mathscr{V}\times SU(2)=\{(I,u)\,|\,I\in\mathscr{V},u\in SU(2)\} . \quad (3.4)$$

On $\bar{Q}$ introduce the equivalence relation [cf. (2.11)]

$$(I,u)\sim(e^{-i\alpha T}I\,e^{i\alpha T},u\,e^{i\alpha\sigma_3}) \quad (3.5)$$

leading to equivalence classes $\langle I,u\rangle$:

$$\langle I,u\rangle=\langle e^{-i\alpha T}I\,e^{i\alpha T},u\,e^{i\alpha\sigma_3}\rangle . \quad (3.6)$$

Pass to the quotient of $\bar{Q}$ with respect to (3.5), namely,

$$Q=\bar{Q}/\sim=\{\langle I,u\rangle\,|\,I\in\mathscr{V},u\in SU(2)\} . \quad (3.7)$$

The projection π: $Q\rightarrow S^2$ is defined by

$$\pi(\langle I,u\rangle)=\hat{x}\in S^2 , \quad \vec{\sigma}\cdot\hat{x}=u\sigma_3u^{-1} . \quad (3.8)$$

That this Q is just the bundle structure required for accommodating (3.1)–(3.3) is easily verified. As in the previous section, for $\pi(\langle I,u\rangle)\in\Sigma_N$ we express u as in (2.18) and then identify

$$I_N=e^{-i\alpha T}I\,e^{i\alpha T} . \quad (3.9)$$

Similarly for $\pi(\langle I,u\rangle)\in\Sigma_S$ we express u as in (2.21) and write

$$I_S=e^{-i\beta T}I\,e^{i\beta T} . \quad (3.10)$$

In this way $\pi^{-1}(\Sigma_N)$ $[\pi^{-1}(\Sigma_S)]$ is exhibited as being homeomorphic to the Cartesian product $\mathscr{V}\times\Sigma_N$ $[\mathscr{V}\times\Sigma_S]$, which is just what (3.1) and (3.2) require. And because of (2.23), in the overlap these two trivializations of Q are connected by (3.3).

This makes it clear that the global configuration space in which to view the equations of motion (3.1) and (3.2) is the Cartesian product $Q\times\mathsf{R}_+$, where Q is the $\mathscr{V}$ bundle over S^2 just constructed. For a nontrivial non-Abelian monopole Q is quite different from $\mathscr{V}\times S^2$, and in particular *there is in general no natural action of H on Q at all.* The problem is that any such action on $\overline{Q}$ will not as a rule commute with T, and in view of (3.6), cannot be viewed as an action on Q. (The problem of course disappears if H is Abelian.)

The situation is qualitatively similar at the Lagrangian level, but it is instructive to give the details, and especially to find the form of the global Lagrangian. Generalizing (1.9), when $\vec{x}\in\Sigma_N$ we assume generalized coordinates $\vec{x},B_N\in H$ and take the Lagrangian to be

$$\mathscr{L}_N(\vec{x},B_N,\dot{\vec{x}},\dot{B}_N;\{W_N\})=\tfrac{1}{2}m\dot{\vec{x}}^2+i\,\mathrm{Tr}(KB_N{}^{-1}\dot{B}_N)-e\,\dot{x}_j\mathrm{Tr}[B_NKB_N{}^{-1}W_{Nj}(\vec{x})],\quad K\in\underline{H}\ . \tag{3.11}$$

This will lead to the equations of motion (3.1) provided we identify

$$I_N=B_NKB_N{}^{-1}\ . \tag{3.12}$$

When $\vec{x}\in\Sigma_S$, we use as Lagrangian coordinates $\vec{x}$ and $B_S\in H$, and take

$$\mathscr{L}_S(\vec{x},B_s,\dot{\vec{x}},\dot{B}_S;\{W_S\})=\tfrac{1}{2}m\dot{\vec{x}}^2+i\,\mathrm{Tr}(KB_S{}^{-1}\dot{B}_S)-e\,\dot{x}_j\mathrm{Tr}[B_SKB_S{}^{-1}W_{Sj}(\vec{x})]\ . \tag{3.13}$$

Notice that the same $K\in\underline{H}$ is used in $\mathscr{L}_N$ and in $\mathscr{L}_S$. We now recover (3.2) provided

$$I_S=B_SKB_S{}^{-1}\ . \tag{3.14}$$

Obviously in the overlap the generalized coordinates B_N and B_S must be connected by

$$\vec{x}\in\Sigma_N\cap\Sigma_S:\ B_N(t)=e^{-i\phi T}B_S(t)\ , \tag{3.15}$$

so as to ensure (3.3).

A comparison of the transition rule (3.15) with the rule verified as holding in the bundle $\mathscr{B}$ constructed in Sec. II [see Eq. (2.24) and the arguments leading thereto] tells us the following: the global configuration space on which to base the Lagrangian description of motion in a non-Abelian monopole field is $\mathscr{B}\times\mathrm{R}_+$. The bundle $\mathscr{B}$ is a principal H bundle over S^2 consisting of the equivalence classes $\langle B,u\rangle$, and on these, *there is in general no global left action of H at all.* The problem is the transition formula (3.15) which will not as a rule be preserved by such an action.

We can now ask: what is the global Lagrangian, defined on (the tangent bundle of) $\mathscr{B}\times\mathrm{R}_+$, which leads to the two local Lagrangians $\mathscr{L}_N$ and $\mathscr{L}_S$ of (3.11) and (3.13) when we use the trivializations of $\pi^{-1}(\Sigma_N)$ and $\pi^{-1}(\Sigma_S)$ and exhibit them as (essentially) $H\times\Sigma_N$ and $H\times\Sigma_S$, respectively? The answer is that this Lagrangian has to be constructed using the connection form $\Omega(B,u)$ which obeys (2.27), (2.28), (2.30), and (3.32). We take as Lagrangian

$$\mathscr{L}=\tfrac{1}{2}m\dot{\vec{x}}^2+i\,\mathrm{Tr}KB^{-1}\left[\dot{B}+ie\frac{\Omega(\mathbb{1},u)}{dt}B\right] \tag{3.16}$$

and now show explicitly that it has all the required properties.

We note first that $\mathscr{L}$ is indeed a function of r, B, and u and their velocities, since $\hat{x}$ is determined by u. We must make sure that $\mathscr{L}$ does not depend individually on B,u (and their velocities) but only on the equivalence class $\langle B,u\rangle$ (and *its* velocity). Only then can we claim that we are working on [the tangent bundle $T(\mathscr{B}\times\mathrm{R}_+)$ of] $\mathscr{B}\times\mathrm{R}_+$. The leading kinetic energy term in $\mathscr{L}$ causes no problem in this regard, since $\hat{x}$ is invariant under the change $u\to u\,e^{i\alpha\sigma_3}$ in u. The rest of $\mathscr{L}$ can be explicitly checked to be unchanged if we make the replacements

$$B\to e^{-i\alpha T}B,u\to u\,e^{i\alpha\sigma_3} \tag{3.17}$$

in it, with arbitrary (time dependent) α; here we use the properties (2.27) and (2.28) of $\Omega(B,u)$. This establishes that $\mathscr{L}$ is well defined on $T(\mathscr{B}\times\mathrm{R}_+)$. Now we check that in each of the two regions Σ_N,Σ_S for $\hat{x}$, this global Lagrangian reduces to the locally valid $\mathscr{L}_N$ and $\mathscr{L}_S$ of (3.11) and (3.13). If it be given that $\hat{x}\in\Sigma_N$, the u in (3.16) can be written in terms of $u_N(\hat{x})$ and some α as in (2.18). Using the property (2.27) followed by (2.28), the factor $e^{-i\alpha\sigma_3}$ can be shifted so as to be attached to B and it then produces $B_N=e^{-i\alpha T}B$ [cf. (2.20)]. Since $\Omega(\mathbb{1},u_N(\hat{x}))$ reduces to the gauge potential $\vec{W}_N(\vec{x})$ according to (2.30), we then find easily that $\mathscr{L}$ becomes $\mathscr{L}_N$ of (3.11). In a similar way, for $\hat{x}\in\Sigma_S$ we find $\mathscr{L}$ reducing to $\mathscr{L}_S$ of (3.13). Incidentally, this shows that in the overlap region, $\mathscr{L}_N$ and $\mathscr{L}_S$ are (numerically) equal—a fact that could have been directly verified using (2.26) and (3.15). Thus we have established the correctness of our choice (3.16) for $\mathscr{L}$ in all respects.

It is clear that the topological nontriviality of the monopole field has led to a "twist" in the way in which the internal space of the test particle $\mathscr{V}$ or H combines with the S^2 of ordinary space to produce its true configuration spaces. These spaces $Q\times\mathrm{R}_+$ or $\mathscr{B}\times\mathrm{R}_+$ are nontrivial bundles over S^2 and in general do not permit a (physically relevant) global action of H. We shall examine in Sec. V the question of what subgroup (or subgroups) of H can in fact be globally defined, once the topological "type" of the monopole [element of $\pi_1(H)$] is specified. In the rest of this section, we examine briefly the phase-space structure resulting from the Lagrangian $\mathscr{L}$ of (3.16), and see again at this level the breakdown of the action of H.

Since the radial variable $r\in\mathrm{R}_+$ and its canonical conjugate p_r are not involved in the following arguments, we sometimes ignore them, and refer mainly to $\mathscr{B}$ and its associated objects. All the needed properties of $\mathscr{B}$ have been assembled in Sec. II. We can regard it as the union of $\mathscr{B}_N$ and $\mathscr{B}_S$ as follows:

$$\mathscr{B}=\mathscr{B}_N\cup\mathscr{B}_S\ ,$$
$$\mathscr{B}_N=\pi^{-1}(\Sigma_N),\quad \mathscr{B}_S=\pi^{-1}(\Sigma_S)\ , \tag{3.18}$$
$$S^2=\Sigma_N\cup\Sigma_S\ .$$

Local coordinates over $\mathscr{B}_N$ are $(B_N,\hat{x})$ with $\hat{x}\in\Sigma_N$; and similar coordinates over $\mathscr{B}_S$ are $(B_S,\hat{x})$ with $\hat{x}\in\Sigma_S$. In the overlap, the gluing map (3.15) must be read as a point transformation relating the local systems. The phase space associated with $\mathscr{B}\times\mathsf{R}_+$ as configuration space is, as always, the cotangent bundle

$$T^*(\mathscr{B}\times\mathsf{R}_+)\simeq T^*\mathscr{B}\times T^*\mathsf{R}_+\ .$$

It is the union of two parts, $T^*\mathscr{B}_N\times T^*\mathsf{R}_+$ and $T^*\mathscr{B}_S\times T^*\mathsf{R}_+$: we call these the N and S parts of the phase space. Over the N part of phase space, we have a coordinate system made up of variables $\vec{x},\vec{p}_N,B_N,\mathscr{I}_{Na}$, among which the only nonzero PB's are

$$\{x_j,p_{Nk}\}=\delta_{jk}\ ,$$
$$\{B_N,\mathscr{I}_{Na}\}=-iT_aB_N\ , \tag{3.19}$$
$$\{\mathscr{I}_{Na},\mathscr{I}_{Nb}\}=C_{abc}\mathscr{I}_{Nc}\ .$$

Over the S part of phase space, i.e., over $T^*\mathscr{B}_S\times T^*\mathsf{R}_+$, we use variables $\vec{x}$, $\vec{p}_S$, B_S, and $\mathscr{I}_{Sa}$, with exactly similar PB's. Note that $\mathscr{I}_{Na}$ are the generators of the left action of H on B_N, and correspondingly for $\mathscr{I}_{Sa}$ and B_S. Now the relation between these two local coordinate systems on phase space is fully determined by the fact that it is the "lift" of the point transformation (3.15) acting on configuration space. Using some results from Appendix A, we find the complete transition rules ($\vec{x}$ omitted):

$$B_N=e^{-i\phi T}B_S\ , \tag{3.20a}$$

$$\vec{p}_N=\vec{p}_S+\frac{i}{R}\operatorname{Tr}e^{-i\phi T}\mathscr{I}_S\vec{\nabla}\,e^{i\phi T}\ , \tag{3.20b}$$

$$\mathscr{I}_N=e^{-i\phi T}\mathscr{I}_Se^{i\phi T}\ . \tag{3.20c}$$

Here R is the constant appearing in the trace orthogonality property (1.5) among the generators T_a, and $\mathscr{I}_N,\mathscr{I}_S$ are the matrix combinations $\mathscr{I}_{Na}T_a,\mathscr{I}_{Sa}T_a$, respectively. These transition formulas from one local phase-space coordinate system to the other are consistent with the PB relations holding in the two systems. Thus (3.19) combined with (3.20) do imply the S version of (3.19). In the canonical formalism, the basic problem appears in the following guise: while in the overlap (3.19) holds, we find in general that

$$\{B_N,\mathscr{I}_{Na}\}\neq e^{-i\phi T}\{B_S,\mathscr{I}_{Sa}\}\ , \tag{3.21}$$

because $[T_a,T]\neq 0$. The interpretation is the following: while over the N part of phase space $\mathscr{I}_{Na}$ generate a (left) action of H on B_N, and over the S part $\mathscr{I}_{Sa}$ generate a corresponding action of H on B_S, we are unable to say that taken together $\mathscr{I}_{Na}$ and $\mathscr{I}_{Sa}$ are local descriptions of a globally existing set of generators of a global (left) action of H on the whole phase space. This is but a "lifting" to phase space of the problem already recognized at the level of the configuration space $\mathscr{B}\times\mathsf{R}_+$. But the importance of stating it in phase-space terms is its significance for a quantum-mechanical test particle.

Relating $\mathscr{I}_N$ and $\mathscr{I}_S$ to the Lagrangian variables is straightforward. For the former we use $\mathscr{L}_N$ of (3.11) and find, as in (1.16), that $\mathscr{I}_N$ is weakly equal to I_N of (3.12). Similarly, $\mathscr{I}_S$ is weakly equal to I_S. [So the transition rules (3.3) and (3.20c) are consistent.] But the basic problem of nonexistence of H action is recognized prior to obtaining these weak equalities.

IV. SIMPLE EXAMPLE OF GLOBAL LAGRANGIAN

As an application of the global Lagrangian developed in the preceding section, we consider the equations of motion for a particularly simple monopole field in this section. The two basic conditions on the connection $\Omega(B,u)$ have been listed in (2.27) and (2.28). To these we may, if we so choose, add a third condition expressing the idea of spherical symmetry in its simplest form. Because of the projection rule (2.14): $u\in\mathrm{SU}(2)\rightarrow\hat{x}\in S^2$, it is seen that left multiplication of u by some (fixed) element of SU(2) produces a corresponding rigid rotation on $\hat{x}$ in three-dimensional space. Spherical symmetry is therefore the requirement

$$\Omega(B,u)=\Omega(B,u'u),u'\in\mathrm{SU}(2)\ , \tag{4.1}$$

it being understood that u' represents a rigid rotation. A solution to all three conditions (2.27), (2.28), and (4.1) is

$$\Omega(B,u)=-\frac{i}{e}B^{-1}dB-\frac{i}{e}B^{-1}TB\tfrac{1}{2}\operatorname{Tr}\sigma_3u^{-1}du\ . \tag{4.2}$$

It is this solution that we shall use in the global Lagrangian in this section.

We note that the possible parametric dependence of Ω on the radial variable r is in fact absent here. Remembering that W_{Nr} and W_{Sr} are both zero in our gauge, this means that when we compute $W_{N\theta},W_{N\phi}$ and $W_{S\theta},W_{S\phi}$, they will all turn out to be proportional to $1/r$. It is known that in such a case the monopole is basically similar in structure to the Dirac monopole of U(1) theory. Specifically, while $\vec{W}_N$ and $\vec{W}_S$ are only locally defined potentials, the field strengths $\vec{G}_N$ and $\vec{G}_S$ coincide in the overlap and hence G_{jk} is globally defined. This is because all the W_j's and G_{jk} are proportional to one common $T\in\underline{H}$. This similarity to the U(1) case will be evident also when we obtain the equations of motion for the test particle.

Setting $B=\mathbb{1}$ in (4.2) and using it in (3.16), the Lagrangian is

$$\begin{aligned}\mathscr{L}&=\tfrac{1}{2}m\dot{\vec{x}}^2+i\operatorname{Tr}KB^{-1}\dot{B}\\&\quad+i\operatorname{Tr}KB^{-1}TB\times\tfrac{1}{2}\operatorname{Tr}\sigma_3u^{-1}\dot{u}\\&=\tfrac{1}{2}m\dot{r}^2+\tfrac{1}{2}mr^2\dot{\hat{x}}^2\\&\quad+i\operatorname{Tr}KB^{-1}\dot{B}+i\operatorname{Tr}KB^{-1}TB\tfrac{1}{2}\operatorname{Tr}\sigma_3u^{-1}\dot{u}\ .\end{aligned} \tag{4.3}$$

The Euler-Lagrange equation for r yields the same result as for a free particle:

$$m\ddot{r}=mr\dot{\hat{x}}^2\ . \tag{4.4}$$

A variation of B, say on the right, leads to the equation

$$[K,B^{-1}\dot{B}]+[K,B^{-1}TB]\tfrac{1}{2}\,\mathrm{Tr}\sigma_3 u^{-1}\dot{u}=0\ , \qquad (4.5)$$

which is the same as

$$\frac{d}{dt}(BKB^{-1})+\tfrac{1}{2}(\mathrm{Tr}\sigma_3 u^{-1}\dot{u})[T,BKB^{-1}]=0\ . \qquad (4.6)$$

From either of the above two equations we get a conservation law:

$$\mathrm{Tr}(BKB^{-1}T)=\text{constant}\ . \qquad (4.7)$$

Turning to the Euler-Lagrange equation for u or $\hat{x}$: an infinitesimal variation of u on the left induces a small rotation of $\hat{x}$,

$$\delta u=i\vec{\sigma}\cdot\vec{\theta}u,\quad |\vec{\theta}|\ll 1 \Longrightarrow$$
$$\delta\hat{x}=-2\vec{\theta}\times\hat{x}\ , \qquad (4.8)$$

and this produces the change $\delta\mathscr{L}$ in $\mathscr{L}$ of amount

$$\delta\mathscr{L}=(-2m\vec{x}\times\dot{\vec{x}}-\hat{x}\,\mathrm{Tr}BKB^{-1}T)\cdot\dot{\vec{\theta}}\ . \qquad (4.9)$$

So the corresponding equation of motion is just the law of conservation of an augmented angular momentum:

$$\frac{d}{dt}\left[m\vec{x}\times\dot{\vec{x}}+\frac{\hat{x}}{2}\,\mathrm{Tr}BKB^{-1}T\right]=0\ . \qquad (4.10)$$

Combining this with (4.4) gives the vector equation of motion for determining the orbit in ordinary space:

$$m\ddot{\vec{x}}=\frac{1}{r^3}\vec{x}\times\dot{\vec{x}}\tfrac{1}{2}\,\mathrm{Tr}BKB^{-1}T\ . \qquad (4.11)$$

This is identical to the equation of motion for an electric charge in the field of a magnetic monopole. The only difference here is that the test particle carries with it an isospin vector $I(t)$ which precesses according to (4.6), while maintaining a constant projection along T.

V. GLOBAL ACTION OF H AND ITS SUBGROUPS

The analysis of Sec. III has shown that in general there are topological obstructions standing in the way of a global action of the "surviving symmetry group" H on the variables of a test particle moving in the field of an H monopole. In this section we analyze this problem in a more systematic way.

To begin with let the monopole configuration be described in the framework of the gauge group G, symmetry under which is spontaneously broken to H. As mentioned in Sec. II, we then have (asymptotically) a globally defined Higgs field $\phi(\vec{x})\in\mathscr{M}_0$, and globally defined gauge fields $\vec{W}(\vec{x}),\vec{G}(\vec{x})$ as well. The stability group $H_{\hat{x}}$ of $\phi(\vec{x})$ is conjugate to $H=H_N$, as is seen in (2.2). In this set-up, a "global definition of the action of H" evidently means the following: for each $h\in H$ and each $\hat{x}\in S^2$, we must define an element $k(\hat{x};h)\in H_{\hat{x}}$ such that we achieve an isomorphism $H\to H_{\hat{x}}$:

$$k(\hat{x};h')k(\hat{x};h)=k(\hat{x};h'h)\ . \qquad (5.1)$$

Of course we demand that $k(\hat{x};h)$ vary smoothly with $\hat{x}$; the smooth dependence of $H_{\hat{x}}$ on $\hat{x}$ does not guarantee that such $k(\hat{x};h)$ can be found. As a convention we assume that

$$k(N;h)=h\in H\ . \qquad (5.2)$$

We now make some choices of $g_N(\hat{x}),g_S(\hat{x})$ in G according to (2.4), pass to the U gauge in which $\phi(\vec{x})$ becomes asymptotically constant with stability group H throughout, and then ask for the way the above problem must be posed. This is clearly the following: for $h\in H$ and $\hat{x}\in\Sigma_N$, we need an element

$$k_N(\hat{x};h)=g_N(\hat{x})^{-1}k(\hat{x};h)g_N(\hat{x})\in H \qquad (5.3)$$

such that for fixed $\hat{x}$,

$$h\to k_N(\hat{x};h) \qquad (5.4)$$

is an automorphism of H. Similarly for $h\in H$ and $\hat{x}\in\Sigma_S$, we need

$$k_S(\hat{x};h)=g_S(\hat{x})^{-1}k(\hat{x};h)g_S(\hat{x})\in H \qquad (5.5)$$

such that for fixed $\hat{x}$,

$$h\to k_S(\hat{x};h) \qquad (5.6)$$

is also an automorphism of H. Moreover, in the overlap we must have

$$\hat{x}\in\Sigma_N\cap\Sigma_S:\ k_N(\hat{x};h)=h_T(\hat{x})^{-1}k_S(\hat{x};h)h_T(\hat{x})\ , \qquad (5.7)$$

where $h_T(\hat{x})$ is given by (2.6). Thus in the U gauge, for given $h_T(\hat{x})$, the problem of defining H globally is the problem of finding smooth functions $k_N(\hat{x};h),k_S(\hat{x};h)$ with the above properties.

The pair $g_N(\hat{x}),g_S(\hat{x})$ carrying us to the U gauge is nonunique to the following extent: we may use instead

$$\hat{x}\in\Sigma_N:\ g_N'(\hat{x})=g_N(\hat{x})h_N(\hat{x}),\quad h_N(\hat{x})\in H\ ,$$
$$\hat{x}\in\Sigma_S:\ g_S'(\hat{x})=g_S(\hat{x})h_S(\hat{x}),\quad h_S(\hat{x})\in H\ . \qquad (5.8)$$

Here $h_{N,S}(\hat{x})$ are smooth functions of $\hat{x}$ in the respective domains, and $h_N(N)=e$ to maintain $g_N'(N)=e$. This change causes the transition function $h_T(\hat{x})$ to be replaced by

$$h_T'(\hat{x})=h_S(\hat{x})^{-1}h_T(\hat{x})h_N(\hat{x})\ , \qquad (5.9)$$

while k_N and k_S (if they exist) go into

$$k_N'(\hat{x};h)=h_N(\hat{x})^{-1}k_N(\hat{x};h)h_N(\hat{x})\ ,$$
$$k_S'(\hat{x};h)=h_S(\hat{x})^{-1}k_S(\hat{x};h)h_S(\hat{x})\ . \qquad (5.10)$$

Now make the reasonable assumption that the *outer* automorphisms of H form a *discrete* set.[17] Then since $k_N(\hat{x};h)$ is to vary smoothly with $\hat{x}$ over Σ_N, and since by (5.2), (5.3), and $g_N(N)=e$ we have

$$k_N(N;h)=h \qquad (5.11)$$

at $\hat{x}=N$, it follows that all over Σ_N there must be an *inner* automorphism carrying h to $k_N(\hat{x};h)$:

$$k_N(\hat{x};h)=h_N(\hat{x})hh_N(\hat{x})^{-1} \qquad (5.12)$$

for some $h_N(\hat{x})\in H$. We can use this $h_N(\hat{x})$ in a change of U gauge (5.8) and (5.9), supplemented by, say, $h_S(\hat{x})=e$, to arrive at the result

$$\hat{x}\in\Sigma_N:\ k'_N(\hat{x};h)=h\ . \tag{5.13}$$

With this simplification, what remains is to find a function $k'_S(\hat{x};h)$ over Σ_S, depending smoothly on $\hat{x}$, so that we have both the automorphism property and the correct transition rule:

$$\hat{x}\in\Sigma_S, h\in H:\ k'_S(\hat{x};h)\in H$$

$$k'_S(\hat{x};h')k'_S(\hat{x};h)=k'_S(\hat{x};h'h)\ , \tag{5.14a}$$

$$\hat{x}\in\Sigma_N\cap\Sigma_S:\ k'_S(\hat{x};h)=h'_T(\hat{x})hh'_T(\hat{x})^{-1}\ . \tag{5.14b}$$

At this stage we may assume that $\Sigma_N(\Sigma_S)$ is all of Σ minus $S(N)$, so that $\Sigma_N\cap\Sigma_S$ is all of Σ except for the two poles. In that case, the transition rule (5.14b) in fact determines $k'_S(\hat{x};h)$ all over Σ_S except at the one point S, and this determination is consistent with (5.14a). For any nontrivial monopole, the transition function $h'_T(\hat{x})$ must fail to be well defined as $\hat{x}$ approaches either pole, N or S. The problem of defining H globally can then be solved if and only if for each $h, k'_S(\hat{x};h)$ as determined by (5.14b) tends to an azimuth independent value in H as $\hat{x}$ tends to S, in spite of there being no such limit for $h'_T(\hat{x})$.

Let us write

$$h'_T\underset{\hat{x}\to S}{\longrightarrow} c'(\phi)\in H\ , \tag{5.15}$$

where ϕ is the azimuth. Then $c'(\phi)$ for $0\le\phi\le 2\pi$ is a closed curve in H lying in some class of $\pi_1(H)$ which determines the "type" of the monopole. Then the condition that $k'_S(\hat{x};h)$ be uniquely determined as $\hat{x}\to S$ reads

$$c'(\phi)hc'(\phi)^{-1}=\text{independent of } \phi \text{ for each } h =c'(0)hc'(0)^{-1}\ ,$$

i.e.,

$$c''(\phi)=h=hc''(\phi) \text{ for all } h\ ,\qquad c''(\phi)=c'(0)^{-1}c'(\phi)\ . \tag{5.16}$$

Now assuming H is connected, $c''(\phi)$ for $0\le\phi\le 2\pi$ is a closed curve in H starting and ending at e, homotopic to the curve $c'(\phi)$. Thus $c''(\phi)$ describes the monopole type as well as $c'(\phi)$. The condition (5.16) demands that for all ϕ, $c''(\phi)$ be in the center of H. Thus *the necessary (and sufficient) condition for the action of H to be globally defined is that the monopole type be represented by a closed curve lying entirely in the center of H.*

We can draw two useful conclusions: (i) if the monopole is nontrivial, H is connected and the center of H is discrete, the action of H cannot be globally defined; (ii) if H has (one or more) U(1) factors, which then belong to the center of H, and the monopole type can be represented by a closed curve in such a U(1) factor, then the action of H can be globally defined. But in that case we are dealing *exactly* with a monopole of the original Abelian Dirac variety.

Having seen that in general the action of the whole of H cannot be globally defined, the following question seems reasonable: for a given type of monopole determined by some class in $\pi_1(H)$, what is the corresponding subgroup of H whose action can be globally defined? We shall give a partial answer to this problem now. We plan to treat it in detail in another paper. (See below, Sec. VI, and Refs. 8 and 9 for such subgroups of H in GUT's.)

Let us assume that $h_T(\hat{x})$ has been put into the form (2.9). It is then obvious that the commutant of T in H can certainly be globally defined. Call this subgroup K_T:

$$K_T=\{k\in H\mid ke^{i\phi T}=e^{i\phi T}k \text{ for all } \phi\}\ . \tag{5.17}$$

Then both $k_N(\hat{x};k)$ and $k_S(\hat{x};k)$ in (5.4) and (5.6) can be taken equal to $k\in K_T$, and the transition rule (5.7) is trivially satisfied.

We can ask for the form this global realization of K_T takes if by means of a transformation (5.9) the transition function is changed to some $h'_T(\hat{x})$. Here if we allow $h_N(\hat{x})$ and $h_S(\hat{x})$ to be chosen freely and independently from H, we arrive at the following picture: for each $\hat{x}\in\Sigma_N$, a subgroup $K_{T,N,\hat{x}}$ in H is determined:

$$K_{T,N,\hat{x}}=h_N(\hat{x})^{-1}K_Th_N(\hat{x})\ . \tag{5.18}$$

Similarly for each $\hat{x}\in\Sigma_S$, a subgroup $K_{T,S,\hat{x}}$ in H is defined by

$$K_{T,S,\hat{x}}=h_S(\hat{x})^{-1}K_Th_S(\hat{x})\ . \tag{5.19}$$

For $\hat{x}=N$, $K_{T,N,N}$ coincides with the original K_T. Then the "image" of each $k\in K_T$ at each $\hat{x}\in\Sigma_N$ is the element

$$k'_N(\hat{x};k)=h_N(\hat{x})^{-1}kh_N(\hat{x})\in K_{T,N,\hat{x}}\ , \tag{5.20}$$

and for fixed $\hat{x}$, $k\to k'_N(\hat{x};k)$ is an isomorphism $K_T\to K_{T,N,\hat{x}}$. Similarly for $\hat{x}\in\Sigma_S$, we have an isomorphism $K_T\to K_{T,S,\hat{x}}$ by

$$k\in K\to k'_S(\hat{x};k)=h_S(\hat{x})^{-1}kh_S(\hat{x})\in K_{T,S,\hat{x}}\ . \tag{5.21}$$

In the overlap, we have two significant relationships:

$$\hat{x}\in\Sigma_N\cap\Sigma_S:\ K_{T,N,\hat{x}}=h'_T(\hat{x})^{-1}K_{T,S,\hat{x}}h'_T(\hat{x})\ ,\qquad k'_N(\hat{x};k)=h'_T(\hat{x})^{-1}k'_S(\hat{x};k)h'_T(\hat{x})\ . \tag{5.22}$$

This degree of complexity seems unavoidable because of the freedom available in the choice of U gauge. It shows that the global definition of the commutant of $T, K_T\subset H$, which appears extremely simple when $h_T(\hat{x})=e^{i\phi T}$, does not even utilize a definite subgroup of H at each $\hat{x}$, but two subgroups $K_{T,N,\hat{x}}$ and $K_{T,S,\hat{x}}$, when we switch to a different U gauge. Not only the representatives at $\hat{x}$ of each $k\in K_T$, but even these subgroups themselves, have to be transformed by $h'_T(\hat{x})$ in the overlap.

Finally, suppose $h'_T(\hat{x})$ had been so chosen as to be another one-parameter subgroup in H, say,

$$h'_T(\hat{x})=e^{i\phi T'}, T'\in\underline{H}\ ,\qquad e^{2\pi iT'}=e\ . \tag{5.23}$$

Thus the closed curves $e^{i\phi T}$ and $e^{i\phi T'}$ both represent the same class in $\pi_1(H)$. In this gauge, the subgroup $K_{T'} \subset H$ can be globally defined in a simple way, *while the subgroup $K_T \subset H$ can be also globally defined in the way outlined in the previous paragraph. However, the actions of K_T and $K_{T'}$ are strikingly different.*

The situation is therefore the following: given the monopole "type" by specifying an element of $\pi_1(H)$, we can search for all $T, T', T'', \ldots \in \underline{H}$ such that $e^{i\phi T}, e^{i\phi T'}, e^{i\phi T''}, \ldots, 0 \le \phi \le 2\pi$, are closed one-parameter subgroups belonging to $\pi_1(H)$. Then each of $K_T, K_{T'}, K_{T''}, \ldots$ can individually be globally defined, and their actions can be exhibited in any choice of U gauge. We can now envisage taking arbitrary products of these actions and determining the group $K_T, K_{T'}, K_{T''}, \ldots$ generate. We hope to study this question elsewhere.

We conclude this section by illustrating these remarks for the physically interesting choice

$$[\mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}]/Z_3 = \mathrm{U}(3)$$

for H. A possible transition function for the elementary monopole is $e^{i\phi T}$ where T is as in (2.43). The commutant K_T of T is (if we ignore taking quotients with discrete subgroups) $\mathrm{SU}(2)_C \times \mathrm{U}(1)_{Y_C} \times \mathrm{U}(1)_{\mathrm{em}}$ where $\mathrm{SU}(2)_C$ acts on the first two colors and $\mathrm{U}(1)_{Y_C}$ is generated by the color hypercharge. Now let us change T to

$$T' = T + \sqrt{3}\,\lambda_8' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\lambda_8' = \frac{1}{\sqrt{3}} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.24}$$

The curve $e^{i\sqrt{3}\lambda_8'\phi}$ $(0 \le \phi \le 2\pi)$ is closed in the simply connected group $\mathrm{SU}(3)_C$ and is therefore homotopically trivial. Thus both the transition functions $h_T = e^{i\phi T}$ and $h_T' = e^{i\phi T'}$ represent the same monopole. However, the commutant of T' is not the group

$$\mathrm{SU}(2)_C \times \mathrm{U}(1)_{Y_C} \times \mathrm{U}(1)_{\mathrm{em}},$$

but rather the Abelian group

$$K_{T'} = \mathrm{U}(1)_{\lambda_3} \times \mathrm{U}(1)_{Y_C} \times \mathrm{U}(1)_{\mathrm{em}},$$

where $\mathrm{U}(1)_{\lambda_3}$ has the generator

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.25}$$

Thus we see explicitly that the groups $K_T, K_{T'}, \ldots$ need not even be isomorphic. We shall not exhibit the analogs of formulas (5.20) and (5.21) for this example.

VI. CONCLUDING REMARKS

In this paper, we have examined the classical mechanics of a test particle in the field of a monopole which emits non-Abelian (and Abelian) magnetic flux (the "non-Abelian" monopole). The background field is not invariant under general transformations of the "unbroken" subgroup H even locally, so that these transformations are not symmetries of the dynamics. Such a property by itself does not however prevent the implementability of transformations: there are after all many canonical transformations which are not symmetries. Further, the global canonical implementability of H is essential to give a global meaning to H multiplets on passage to quantum mechanics. Our discussion shows that it is not in fact possible to realize H globally. In the SU(5) grand unification model, if $H = \mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}$, we thus see that color transformations and color multiplets are not globally defined, while if $H = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_{\mathrm{WS}} \times \mathrm{U}(1)$, the same is seen to be the case for both color and electroweak transformations and multiplets. These effects are consequences of topology and are not associated with any energy scale.

Elsewhere[8,9] it has been demonstrated that the transformations of H cannot all be globally implemented on the non-Abelian monopole. Any such illegal transformation creates a string singularity in the gauge field and maps the monopole solution to an infinite energy configuration. The results of this paper complement the preceding conclusions and show that similar features are encountered in the mechanics of a test particle coupled to a non-Abelian monopole field.

From the physical as well as the formal point of view, it is important to know the transformations of H which *can* be globally implemented. We have discovered that there are several subgroups $K_T, K_{T'}, K_{T''}, \ldots$ which enjoy this property and that the action of these subgroups is not always what we may expect at first sight. We postpone to another publication the completion of all these different actions into a single group of implementable symmetries. Incidentally, it has been shown elsewhere[8,9] (see also the conclusion of Sec. V) that if $H = \mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}$, one of these subgroups is $\mathrm{SU}(2)_C \times \mathrm{U}(1)_{Y_C} \times \mathrm{U}(1)_{\mathrm{em}}$ while if $H = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_{\mathrm{WS}} \times \mathrm{U}(1)_Y$, one of these subgroups is $\mathrm{SU}(2)_C \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$. Here $\mathrm{SU}(2)_C$ acts on the first two colors (say) and $\mathrm{U}(1)_{Y_C}$ is the group generated by the color hypercharge, while the three U(1)'s in $\mathrm{SU}(2)_C \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ along with the rotations around the third axis of $\mathrm{SU}(2)_C$ are generated by the Cartan subalgebra of SU(5).

In a paper under preparation,[8] we will discuss the classical field theory of the non-Abelian monopole as well as the *quantum* mechanics of a test particle in such a monopole field in full detail (see also Balachandran[8]). There we shall see that the lack of continuity of a generic H transformation is transmuted in quantum mechanics into a domain problem: the domain of the Hamiltonian is not invariant under such transformations and these transformations can map a state with finite mean energy into a state with infinite mean energy.

All these results suggest that color and electroweak symmetries are broken in many grand unified models in a novel way. Such a conclusion can be avoided if non-Abelian monopoles do not exist: it may be that such monopoles are confined for dynamical reasons and only Abelian monopoles are observable.[18] Investigation of such issues, however, is beyond the scope of this paper.

ACKNOWLEDGMENTS

This work was supported by the Department of Energy, under Contracts Nos. DE-AC-2-76ER03533 and DE-AS05-76ERO3992, by the Swedish National Research Council under Contract No. F310, and by Istituto Nazionale di Fisica Nucleare, Italy.

APPENDIX A: CANONICAL FORMALISM FOR GROUP ELEMENTS AS COORDINATES

For the convenience of the reader we assemble here a few key formulas that are useful when an element B of a Lie group H is used as a generalized coordinate in a Lagrangian. As in the main text, H is realized via some faithful matrix representation with generators T_a, and B is a matrix in this representation.

Let the configuration space of some system consist of H and other variables which will be suppressed. The intrinsic description of the phase space T^*H, the cotangent bundle over H is as follows: we have both *left* and *right* canonical momenta conjugate to the coordinate $B \in H$, which we write as $\mathscr{I}_a$ and $\mathscr{J}_a$, respectively. The essential Poisson-bracket relations are

$$\{B,\mathscr{I}_a\} = -iT_aB, \quad \{\mathscr{I}_a,\mathscr{I}_b\} = C_{abc}\mathscr{I}_c \ , \tag{A1a}$$

$$\{B,\mathscr{J}_a\} = iBT_a, \quad \{\mathscr{J}_a,\mathscr{J}_b\} = C_{abc}\mathscr{J}_c \ . \tag{A1b}$$

These two kinds of canonical momenta are related through the adjoint representation of H. The neatest way to write the relation is

$$\mathscr{J}_aT_a = -B^{-1}\mathscr{I}_aT_aB \ . \tag{A2}$$

From here in principle one can compute the PB's $\{\mathscr{I}_a,\mathscr{J}_b\}$.

The analogs of the important expressions "$p\delta q$" and "$p\dot{q}$" that appear in canonical mechanics are

$$\text{“}p\delta q\text{”} \rightarrow \frac{i}{R}\,\mathrm{Tr}\mathscr{I}_aT_a\delta BB^{-1} = -\frac{i}{R}\,\mathrm{Tr}\mathscr{J}_aT_aB^{-1}\delta B \ ,$$
$$\text{“}p\dot{q}\text{”} \rightarrow \frac{i}{R}\,\mathrm{Tr}\mathscr{I}_aT_a\dot{B}B^{-1} = -\frac{i}{R}\,\mathrm{Tr}\mathscr{J}_aT_aB^{-1}\dot{B} \ . \tag{A3}$$

Given a Lagrangian $\mathscr{L}(B,\dot{B};\dots)$ the Legendre map from TH to T^*H, i.e., in physical terms the definition of momenta in terms of velocities, is set up as follows: we first imagine making the infinitesimal changes

$$\delta B = 0, \quad \delta\dot{B} = i\epsilon_aT_aB, \quad |\epsilon_a| \ll 1 \ , \tag{A4}$$

and write the resulting change in $\mathscr{L}$ in the form

$$\delta\mathscr{L} = -\epsilon_aI_a(B,\dot{B}) \ . \tag{A5}$$

Then we identify

$$\mathscr{I}_a = I_a(B,\dot{B}) \ . \tag{A6}$$

For the right canonical momenta, we consider

$$\delta B = 0, \quad \delta\dot{B} = iB\epsilon_aT_a \tag{A7}$$

and write

$$\delta\mathscr{L} = \epsilon_aJ_a(B,\dot{B}) \ . \tag{A8}$$

Then the rule is

$$\mathscr{J}_a = J_a(B,\dot{B}) \ . \tag{A9}$$

Directly from their definitions, I_a and J_a may be shown to be related by

$$J_aT_a = -B^{-1}I_aT_aB \ , \tag{A10}$$

so the identifications (A6) and (A9) are consistent with (A2).

The general rule for constructing the Hamiltonian $\mathscr{H}$ is clear from (A3): it reads

$$\mathscr{H} = \frac{i}{R}\,\mathrm{Tr}\mathscr{I}_aT_a\dot{B}B^{-1} + \text{other } p\dot{q} - \mathscr{L}$$
$$= \frac{-i}{R}\,\mathrm{Tr}\mathscr{J}_aT_aB^{-1}\dot{B} + \text{other } p\dot{q} - \mathscr{L} \ . \tag{A11}$$

Two special forms of $\mathscr{L}$ are of interest. If for some fixed $K \in \underline{H}$,

$$\mathscr{L} = i\,\mathrm{Tr}KB^{-1}\dot{B} + \text{terms independent of } \dot{B} \ , \tag{A12}$$

we get

$$I_a = \mathrm{Tr}BKB^{-1}T_a \ ,$$
$$J_a = -K_a \ . \tag{A13}$$

All the test particle Lagrangians in Secs. I, III, and IV have this form, which is why in all of them the left canonical momentum $\mathscr{I}_a$ is physically relevant, while $\mathscr{J}_a$ is uninteresting. If, on the other hand,

$$\mathscr{L} = -i\,\mathrm{Tr}K\dot{B}B^{-1} + \text{terms independent of } \dot{B} \ , \tag{A14}$$

the roles of $\mathscr{I}_a$ and $\mathscr{J}_a$ get interchanged:

$$I_a = -K_a \ ,$$
$$J_a = \mathrm{Tr}B^{-1}KBT_a \ . \tag{A15}$$

Finally we note that the phase-space transition formulas (3.20) arise from the general theorem than "$p\delta q$" is an invariant under a point transformation.

APPENDIX B: PROPERTIES OF $h_T(\hat{x})$

Here we make the definite assumption that Σ_N (Σ_S) consists of all of Σ except $S(N)$. In any U gauge, the transition function $h_T(\hat{x})$ is defined over $\Sigma_N \cap \Sigma_S$, i.e., over all of Σ except both poles, and in its domain of definition it varies smoothly with $\hat{x}$. By its very nature, however, it does not approach a definite limit as $\hat{x}$ approaches either N or S: this is a sign of the topological nontriviality of the monopole. We may define the azimuth-dependent limits of $h_T(\hat{x})$ by taking the limits of the polar angle θ with fixed azimuthal angle ϕ:

$$h_T(\hat{x}) \underset{\hat{x}\to N}{\longrightarrow} C_N(\phi) \in H \ ,$$
$$h_T(\hat{x}) \underset{\hat{x}\to S}{\longrightarrow} C_S(\phi) \in H \ . \tag{B1}$$

As ϕ varies from 0 to 2π, $C_N(\phi)$ and $C_S(\phi)$ describe two closed curves in H which are clearly homotopic to one another. This is because $h_T(\hat{x})=h_T(\theta,\phi)$ interpolates between them smoothly as θ goes from 0 to π. Either one of these two curves determines an element of $\pi_1(H)$ which is the "type" of the monopole.

For all purposes of the theory, a transition function $h_T(\hat{x})$ is equivalent to another $h'_T(\hat{x})$ if we can define two smooth functions $h_N(\hat{x}),h_S(\hat{x})$ over Σ_N,Σ_S, respectively, such that

$$h'_T(\hat{x})=h_S(\hat{x})^{-1}h_T(\hat{x})h_N(\hat{x}) \ . \tag{B2}$$

This is (5.9), and need only hold in $\Sigma_N \cap \Sigma_S$.

First we prove that by a change of the form (B2), we can make $h'_T(\hat{x})$ a function of ϕ alone. For instance, if we let

$$\begin{aligned} &h_N(\hat{x})=h_T(\hat{x})^{-1}C_N(\phi) \ , \\ &h_S(\hat{x})=e \ , \end{aligned} \tag{B3}$$

we see from (B1) that $h_N(\hat{x})$ is well defined over the whole of Σ_N including the north pole; while $h_S(\hat{x})$ is trivially well defined over Σ_S. Using (B3) in (B2) we have produced

$$h'_T(\hat{x})=C_N(\phi) \ . \tag{B4}$$

Similarly the acceptable choices

$$\begin{aligned} &h_N(\hat{x})=e \ , \\ &h_S(\hat{x})=h_T(x)C_S(\phi)^{-1} \end{aligned} \tag{B5}$$

would have rendered

$$h'_T(\hat{x})=C_S(\phi) \ . \tag{B6}$$

Let us assume that any closed curve $C(\phi),0\leq\phi\leq 2\pi$, in H is homotopic to one of the form $e^{i\phi T},T\in \underline{H}$ (we will prove this in a moment). In particular let this be true for $C_N(\phi)$ obtained from $h_T(\hat{x})$ in (B1). Then we can interpolate between $C_N(\phi),0\leq\phi\leq 2\pi$ and $e^{i\phi T},0\leq\phi\leq 2\pi$. That is, we can find a smooth function $a(\theta,\phi)\in H$, defined for $0\leq\theta\leq\pi,0\leq\phi\leq 2\pi$, such that

$$\begin{aligned} &a(0,\phi)=C_N(\phi) \ , \\ &a(\pi,\phi)=e^{i\phi T} \ . \end{aligned} \tag{B7}$$

Now make the choices

$$\begin{aligned} &h_N(\hat{x})=h_T(\hat{x})^{-1}a(\hat{x}) \ , \\ &h_S(\hat{x})=a(\hat{x})e^{-i\phi T} \ . \end{aligned} \tag{B8}$$

Here we have written $a(\theta,\phi)\equiv a(\hat{x})$. From (B1) and (B7), we verify that $h_N(\hat{x})$ is well defined all over Σ_N, and $h_S(\hat{x})$ all over Σ_S. Using (B8) and (B2) we find that we have achieved

$$h'_T(\hat{x})=e^{i\phi T} \ . \tag{B9}$$

In all the above, it is assumed that T obeys

$$e^{2\pi i T}=\text{identity } e \ . \tag{B10}$$

We now prove that every class in $\pi_1(H)$ contains (at least) one representative curve in the form of a one-parameter subgroup. In the process we define somewhat precisely the kind of group H we are dealing with—it covers all cases of physical interest.

Let $\mathscr{G}$ be a compact, simple, simply connected Lie group with center $\{e,z^{(1)},z^{(2)},\ldots,z^{(N)}\}$. Since every element $g\in\mathscr{G}$ lies on some one-parameter subgroup, for any z in the center there is a generator $\mathscr{T}\in\underline{\mathscr{G}}$ such that

$$z=e^{2\pi i\mathscr{T}} \ . \tag{B11}$$

(Strictly speaking, it is $i\mathscr{T}$ that is in $\underline{\mathscr{G}}$.) Further, because $\mathscr{G}$ is simply connected, any curve $\mathscr{C}$ from e to any z is deformable to the curve $e^{i\phi\mathscr{T}},0\leq\phi\leq 2\pi$.

The groups H we are concerned with are all of the form

$$H=[\mathscr{G}_1\times\mathscr{G}_2\times\cdots\times\mathscr{G}_{N_1}\times\underbrace{\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)}_{N_2\text{ factors}}]/D \ , \tag{B12}$$

where each $\mathscr{G}_\alpha$, $\alpha=1,\ldots,N_1$, is a compact simple simply connected Lie group, and D is discrete. We think of H concretely in terms of a faithful matrix representation. Since H is connected (though not simply connected), every closed curve $c(\phi),0\leq\phi\leq 2\pi$, in H is homotopic to one starting and ending at the identity, for instance, to $c(0)^{-1}c(\phi)$. Let a class in $\pi_1(H)$ be then represented by a closed curve $C=\{c(\phi)\in H\text{: } 0\leq\phi\leq 2\pi\}$ in H starting and ending at the identity. This C is the image of a curve $\mathscr{C}$ in $\mathscr{G}_1\times\mathscr{G}_2\times\cdots\times\mathscr{G}_{N_1}\times\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$ which may be represented in each factor in this way:

$$\begin{aligned} &\mathscr{C}=\{\mathscr{C}(\phi)\text{:}0\leq\phi\leq 2\pi\} \ , \\ &\mathscr{C}(\phi)=\mathscr{C}_1(\phi)\times\mathscr{C}_2(\phi)\times\ldots \\ &\qquad\times\mathscr{C}_{N_1}(\phi)\times\mathscr{C}^{(1)}(\phi)\times\cdots\times\mathscr{C}^{(N_2)}(\phi) \ . \end{aligned} \tag{B13}$$

The factors $\mathscr{C}^{(\beta)}(\phi)$ in the various U(1)'s are either trivial curves, so that they can be taken to be the identity, or they can be assumed to have the form

$$\mathscr{C}^{(\beta)}(\phi)=e^{i\phi Q_\beta} \ , \tag{B14}$$

where Q_β is a suitably normalized generator of the βth U(1). The factor $\mathscr{C}_\alpha(\phi)$ in $\mathscr{G}_\alpha$ either (a) runs from the identity in $\mathscr{G}_\alpha$ back to the identity or (b) runs from the identity of $\mathscr{G}_\alpha$ to a central element z_α of $\mathscr{G}_\alpha$. [Without loss of generality we can assume that each $\mathscr{C}_\alpha(0)$ and each $\mathscr{C}^{(\beta)}(0)$ is the identity element.] If case (a) is not true, then case (b) is true because, C being closed in H, the end point $\mathscr{C}(2\pi)$ of $\mathscr{C}(\phi)$ in $\mathscr{G}_1\times\cdots\times\mathscr{G}_{N_1}\times\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$ must be in D. But D is an invariant discrete subgroup of the product $\mathscr{G}_1\times\cdots\times\mathrm{U}(1)$, so each $\mathscr{C}_\alpha(2\pi)$ commutes with all of $\mathscr{G}_\alpha$, i.e., it is in the center of $\mathscr{G}_\alpha$. In case (a), $\mathscr{C}_\alpha(\phi)$ is homotopic to the constant identity path or loop in $\mathscr{G}_\alpha$ and hence is ignorable. In case (b), it can be deformed to a one-parameter subgroup (or a portion thereof) of the form $\{e^{i\phi\mathscr{T}_\alpha}\text{: } 0\leq\phi\leq 2\pi\}$, by (B11). Thus $\mathscr{C}$ in $\mathscr{G}_1\times\cdots\times\mathrm{U}(1)$ is homotopic to (a portion of) the one-parameter subgroup

$$\hat{\mathscr{C}} = \{ \hat{\mathscr{C}}(\phi) : 0 \le \phi \le 2\pi \} ,$$
$$\hat{\mathscr{C}}(\phi) = e^{i\phi\mathscr{T}_1} \times e^{i\phi\mathscr{T}_2} \times \cdots \times e^{i\phi\mathscr{T}_{N_1}} \times e^{i\phi Q_1} \times \cdots \times e^{i\phi Q_{N_2}} . \tag{B15}$$

Here some of the $\mathscr{T}$'s and some of the Q's may be zero. This proves our theorem: if T_K, q_β represent $\mathscr{T}_\alpha, Q_\beta$ in $\underline{H}$, the closed loop C in H is homotopic to the one-parameter subgroup $\hat{C}$ in H with generator $\sum_\alpha T_\alpha + \sum_\beta q_\beta$.

[1]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931); Phys. Rev. 74, 817 (1948).

[2]M. N. Saha, Phys. Rev. 75, 1968 (1949).

[3]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1116 (1976); P. Hasenfratz and G. 't Hooft, *ibid.* 36, 1119 (1976).

[4]A. S. Goldhaber, Phys. Rev. Lett. 36, 1122 (1976); J. L. Friedman and R. D. Sorkin, Commun. Math. Phys. 89, 501 (1983); R. D. Sorkin, Phys. Rev. D 27, 1787 (1983).

[5]For review, see P. Goddard and D. I. Olive, Rep. Prog. Phys. 41, 1357 (1978); P. Langacker, Phys. Rep. 72, 185 (1981).

[6]S. K. Wong, Nuovo Cimento 65A, 689 (1970).

[7]A. P. Balachandran, S. Borchardt, and A. Stern, Phys. Rev. D 17, 3247 (1978); A. P. Balachandran, P. Salomonson, B. S. Skagerstam, and J-O. Winnberg, *ibid.* 15, 2308 (1977); A. Barducci, R. Casalbuoni, and L. Lusanna, Nucl. Phys. B124, 93 (1977). Here we follow the approach of the first paper. See also A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, *Gauge Symmetries and Fibre Bundles, Applications to Particle Dynamics,* Lecture Notes in Physics 188 (Springer, Berlin, 1983).

[8]A. P. Balachandran, G. Marmo, N. Mukunda, J. S. Nilsson, E. C. G. Sudarshan, and F. Zaccaria, Phys. Rev. Lett. 50, 1553 (1983) and paper under preparation; A. P. Balachandran, in Proceedings of the Advanced Winter Institute on 25 Years of Weak Interaction and the Current Status of Gauge Theories, 1982, Indian Institute of Science, Bangalore (unpublished).

[9]P. Nelson and A. Manohar, Phys. Rev. Lett. 50, 943 (1983); A. Abouelsaood, Phys. Lett. 125B, 467 (1983); Nucl. Phys. B226, 309 (1984).

[10]A global description can however be found, see A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, Nucl. Phys. B162, 385 (1980); J. L. Friedman and R. D. Sorkin, Phys. Rev. D 20, 2511 (1979); F. Zaccaria, E. C. G. Sudarshan, J. S. Nilsson, N. Mukunda, G. Marmo, and A. P. Balachandran, *ibid.* 27, 2327 (1983).

[11]For details of fiber-bundle theory, see N. Steenrod, *The Topology of Fibre Bundles* (Princeton Unversity Press, Princeton, New Jersey, 1951).

[12]For U(2) monopoles, see E. Corrigan, D. I. Olive, D. B. Fairlie, and J. Nuyts, Nucl. Phys. B106, 475 (1976); E. Corrigan and D. Olive, *ibid.* B110, 237 (1976). The loop $e^{i\phi T}$ [with T as in (2.34)] leads to an elementary monpole while $e^{-i\phi T}$ leads to an elementary antimonopole.

[13]For $H=[SU(3)_C \times U(1)_{em}]/Z_3 \equiv U(3)$ or $H=[SU(3)_C \times SU(2)_{WS} \times U(1)_Y]/Z_6$ monopoles, see C. Dokos and T. Tomaras, Phys. Rev. D 21, 2940 (1980). In GUT models where the GUT group breaks to one of these H's, the loop $e^{i\phi T}$ with T as in (2.42) or (2.54) describes an elementary monopole with color flux. The loop $e^{-i\phi T}$ describes the corresponding antimonopole.

[14]For SO(3) monopoles produced in the breakdown $SU(3)_C \to SO(3)$, see A. P. Balachandran, V. P. Nair, and C. G. Trahern, Nucl. Phys. B196, 413 (1982).

[15]R. Slansky, T. Goldman, and G. L. Shaw, Phys. Rev. Lett. 47, 887 (1981).

[16]G. S. La Rue, W. M. Fairbank, and A. F. Hebard, Phys. Rev. Lett. 38, 1011 (1977); G. S. La Rue, J. D. Phillips, and W. M. Fairbank, *ibid.* 42, 142 (1979); 42, 1019(E) (1979); M. Martinelli and G. Morpurgo, Phys. Rep. 85, 161 (1982).

[17]For physically interesting cases, H locally decomposes into semisimple and U(1) factors and the group of outer automorphisms do form a discrete set.

[18]For a discussion of bag models for non-Abelian monopoles and their confinement, see F. Lizzi, Phys. Rev. D (to be published).